

## A COMPLEX MAP WITH COMPLEX TOPOLOGY

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**ABSTRACT.** We construct a simple example of a complex algebraic map, which (even locally) has an uncountable number of local topological types at points of the source space.

Let  $f : X \rightarrow Y$  be a morphism of complex algebraic varieties. We say that  $f$  has the same **local topological type** at two points  $a$  and  $b$  of  $X$ , if there exist homeomorphisms of germs (in the strong topology)  $\phi : X_a \rightarrow X_b$  and  $\psi : Y_{f(a)} \rightarrow Y_{f(b)}$ , such that  $\psi \circ f_a = f_b \circ \phi$ .

The aim of this note is to provide a simple example of a complex algebraic map  $f : X \rightarrow Y$  which (even locally) has an infinite number of different local topological types at points of  $X$ . This contrasts with the topology of complex spaces, where by the theory of Whitney stratifications one has a locally finite number of local topological types. Also, if  $Y$  is a smooth curve, then  $f$  can be stratified to verify Thom's  $a_f$  condition [2] which implies local finiteness.

There are related examples of Thom [6] and Nakai [4]. The difference with ours is that they treat the varying of topological type in parametrized families of maps. Nevertheless, our example is inspired by Thom's (which is real and global).

First, we introduce a tool, which is a concrete realization of an idea of Thom.<sup>1</sup> On each fibre  $f^{-1}(y)$ , we define an **invariant relation**  $\mathcal{R}(y)$  (a subset of  $f^{-1}(y) \times f^{-1}(y)$ ). Let  $U = \{x \in X : \dim_x f^{-1}(f(x)) = \dim_x X - \dim_{f(x)} Y\}$ .

**Definition 1.** For  $y \in Y$ , let  $\mathcal{R}(y) = f^{-1}(y) \times f^{-1}(y) \cap \text{closure of } (U \times_Y U)$ , where the closure is taken in the fibred product  $X \times_X X$  induced by the map  $f$ .

**Definition 2.** Let  $A$  and  $B$  be topological spaces,  $a \in A$ ,  $b \in B$ . Let  $\rho$  and  $\delta$  be relations on  $A$  and  $B$ , respectively. We say that  $(\rho, a)$  and  $(\delta, b)$  are **topologically equivalent** if there exists a homeomorphism of germs  $g : A_a \rightarrow B_b$ , such that in the induced product homeomorphism  $g \times g : (A \times A)_{(a,a)} \rightarrow (B \times B)_{(b,b)}$  we have  $(g \times g)^{-1}(\delta_{(b,b)}) = \rho_{(a,a)}$ .

**Proposition 1.** *If  $f$  has the same local topological type at two points  $a$  and  $b$  of  $X$ , then  $(\mathcal{R}(f(a)), a)$  and  $(\mathcal{R}(f(b)), b)$  are topologically equivalent. (Here the relations are considered on the fibres  $A = f^{-1}(f(a))$  and  $B = f^{-1}(f(b))$ ).*

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<sup>1</sup>In [6] Thom writes "...dans un voisinage d'une strate élatée, l'application opère des identifications qui se traduisent par une correspondance dans la strate (S) elle-même;...".

Since Definition 1 involves only local topological objects, Proposition 1 is obvious.

**Example.** Let  $X$  be the hypersurface  $x_1x_2 = 0$  in  $\mathbb{C}^4$  with variables  $(x_1, x_2, z, t)$ . Let  $Y = \mathbb{C}^3$ , and define the map  $f : X \rightarrow Y$  by

$$f(x_1, x_2, z, t) = (x_1 + x_2, (x_1 + tx_2)z, t) .$$

**Claim.** *In any neighbourhood of any point  $(0, 0, 0, t) \in X$ , with  $|t| = 1$ , the map  $f$  has infinitely many different local topological types. More precisely, suppose that  $\alpha_1, \alpha_2 \in [0, 1)$ ,  $\alpha_2$  is irrational,  $\alpha_1 \neq \alpha_2$  and  $\alpha_1 \neq 1 - \alpha_2$ . Let  $t_1 = e^{2\pi\alpha_1 i}$  and  $t_2 = e^{2\pi\alpha_2 i}$ . Then  $f$  has different local topological types at  $(0, 0, 0, t_1)$  and at  $(0, 0, 0, t_2)$ .*

*Proof of the claim.* For  $t \in \mathbb{C}$ , write  $a_t = (0, 0, 0, t)$ . We first determine the invariant relation  $\mathcal{R}(f(a_t))$ . Notice that  $f^{-1}(f(a_t)) \cong \mathbb{C}$ , with coordinate  $z$ . Let  $(z, z')$  be corresponding coordinates in  $f^{-1}(f(a_t)) \times f^{-1}(f(a_t)) \cong \mathbb{C}^2$ . Then, the relation  $\mathcal{R}(f(a_t))$  is the hypersurface  $H_t(z, z') = 0$ , with

$$H_t(z, z') = (z - z')(z - tz')(tz - z') .$$

To establish this, remark that the set  $U$  from Definition 1 is in this case the complement of  $\{x_1 = x_2 = 0\} \cup \{t = x_1 = 0\}$ . Then, one can use computer algebra to calculate the appropriate closure or do it directly (look at limits of pairs of points of  $U$  with the same values of  $f$ ). We leave these simple details to the reader.

By Proposition 1 it is enough to show that  $(\mathcal{R}(f(a_{t_1})), 0)$  and  $(\mathcal{R}(f(a_{t_2})), 0)$  are not topologically equivalent. Assume the contrary. This means that there exists a homeomorphism of germs  $g : \mathbb{C}_0 \rightarrow \mathbb{C}_0$ , such that the product homeomorphism  $g \times g : \mathbb{C}_0^2 \rightarrow \mathbb{C}_0^2$  maps the germ of the hypersurface  $H_{t_1}(z, z') = 0$  to the germ of  $H_{t_2}(z, z') = 0$ . Of course, the diagonal is mapped to the diagonal and every other irreducible component into an irreducible component (since after removing the origin they become connected components). Suppose that the component  $z' = t_1z$  is mapped to  $z' = t_2z$  (the other case being similar). This means that  $g$  satisfies the following identity on  $\mathbb{C}_0$  :

$$e^{2\pi\alpha_1 i} z = g^{-1}(e^{2\pi\alpha_2 i} g(z)) .$$

Replace  $g$  by a representative and let  $S$  be a small circle centered at 0. Now  $S$  is closed under multiplication by  $t_1$  and, therefore,  $g(S)$  is closed under multiplication by  $t_2$ . Since  $\alpha_2$  is irrational,  $g(S)$  is also a circle. Thus, the above identity holds after restricting  $g$  and  $g^{-1}$  to circles. This implies that rotations of the circle by  $2\pi\alpha_1$  and by  $2\pi\alpha_2$  have the same *rotation number* (see e.g. [1],[3],[5]) and contradicts the assumption that  $\alpha_1 \neq \alpha_2$ .  $\square$

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