SOME RAPIDLY CONVERGING SERIES FOR $\zeta(2n + 1)$

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Abstract. For a natural number $n$, the author derives several families of series representations for the Riemann Zeta function $\zeta(2n + 1)$. Each of these series representing $\zeta(2n + 1)$ converges remarkably rapidly with its general term having the order estimate:

$$O(k^{-2n-1} \cdot m^{-2k}) \quad (k \to \infty; \ m = 2, 3, 4, 6).$$

Relevant connections of the results presented here with many other known series representations for $\zeta(2n + 1)$ are also pointed out.

1. Introduction and preliminaries

The Riemann Zeta function $\zeta(s)$ and the (Hurwitz’s) generalized Zeta function $\zeta(s, a)$, defined usually by (see, e.g., Titchmarsh [26])

$$(1.1) \quad \zeta(s) := \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n - 1)^s} \quad (\Re(s) > 1), \\ \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \quad (\Re(s) > 0; \ s \neq 1) \end{cases}$$

and

$$(1.2) \quad \zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} \quad (\Re(s) > 1; \ a \neq 0, -1, -2, \cdots),$$

so that

$$(1.3) \quad \zeta(s, 1) = \zeta(s) \quad \text{and} \quad \zeta(s, 2) = \zeta(s) - 1,$$

are known to be meromorphic (that is, analytic everywhere in the complex $s$-plane except for a simple pole at $s = 1$ with residue 1). Making use of the binomial theorem and the Pochhammer symbol $(\lambda)_n$ defined by

$$(1.4) \quad (\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 \quad (n = 0), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) \quad (n \in \mathbb{N} := \{1, 2, 3, \cdots\}) \end{cases},$$

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it is easily seen from the definition (1.2) that

\[ \sum_{k=0}^{\infty} \frac{(s)_k}{k!} \zeta(s + k, a) t^k = \zeta(s, a - t) \quad (|t| < |a|), \]

which immediately yields the familiar identities (cf., e.g., Hansen [16, p. 359] where other references are also cited):

\[ \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \zeta(s + 2k, a) t^{2k} = \frac{1}{2} \left[ \zeta(s, a - t) + \zeta(s, a + t) \right] \quad (|t| < |a|) \]

and

\[ \sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!} \zeta(s + 2k + 1, a) t^{2k+1} = \frac{1}{2} \left[ \zeta(s, a - t) - \zeta(s, a + t) \right] \quad (|t| < |a|) \]

or, equivalently,

\[ \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k+1)!} \zeta(s + 2k, a) t^{2k+1} \]

\[ = \frac{1}{2(s-1)} \left[ \zeta(s-1, a - t) - \zeta(s-1, a + t) \right] \quad (|t| < |a|). \]

Since

\[ \zeta(s) = \frac{1}{m^{s-1}} \sum_{j=1}^{m-1} \zeta \left( s, \frac{j}{m} \right) \quad (m \in \mathbb{N} \setminus \{1\}), \]

which follows readily from the definitions (1.1) and (1.2), the special case of the identity (1.6) when \( a = 1 \) and \( t = 1/m \) can be rewritten in the form:

\[ \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k)}{m^{2k}} \]

\[ = \begin{cases} \frac{1}{2} \left[ (2s - 1) \zeta(s) - 2^{s-1} - (m^s - 1) \zeta(s) - m^s - \sum_{j=2}^{m-2} \zeta \left( s, \frac{j}{m} \right) \right] & (m = 2), \\
\frac{1}{2} \left[ (m^s - 1) \zeta(s) - m^s - \sum_{j=2}^{m-2} \zeta \left( s, \frac{j}{m} \right) \right] & (m \in \mathbb{N} \setminus \{1, 2\}), \end{cases} \]

where (and throughout this paper) an empty sum is to be interpreted as nil. In addition to the case \( m = 2 \), the formula (1.10) simplifies also in the cases when \( m = 3, 4, \) and \( 6 \), and we thus obtain the identities:

\[ \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k)}{3^{2k}} = \frac{1}{2} \left[ (3^s - 1) \zeta(s) - 3^s \right], \]

\[ \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k)}{4^{2k}} = \frac{1}{2} \left[ (4^s - 2^s) \zeta(s) - 4^s \right], \]
and

\[
\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k)}{6^{2k}} = \frac{1}{2} \left[ (6^s - 3^s - 2^s + 1) \zeta(s) - 6^s \right],
\]

(1.13)

respectively.

Identities of this kind seem to have first appeared in the work of Ramaswami [22], who actually proved the cases \( m = 2, 3, \) and 6 of the general result in (1.10). Each of these three identities of Ramaswami [22] can also be found in the work of Hansen [16, p. 357], who referred to Apostol [1] as his source for the identities (1.11) and (1.13) only. As a matter of fact, Apostol [1] reproduced the identities (1.11) and (1.13) from Ramaswami’s work [22] and then proved an interesting arithmetical generalization of these identities (see also Klusch [17, p. 520]).

In its slightly variant form:

\[
\sum_{k=1}^{\infty} \frac{(s + 1)_{2k}}{(2k)!} \frac{\zeta(s + 2k)}{2^{2k}} = (2^s - 2) \zeta(s),
\]

(1.14)

which can indeed be proven directly from the known special cases of (1.6) and (1.7) when \( a = 1 \) and \( t = \frac{1}{2} \), the case \( m = 2 \) of the general result (1.10) was applied by Zhang and Williams [29] (and, more recently, by Cvijović and Klinowski [8]) with a view to finding two seemingly different series representations for \( \zeta(2n + 1) \) \((n \in \mathbb{N})\). The main object of this paper is to obtain much more rapidly converging series representations for \( \zeta(2n + 1) \) \((n \in \mathbb{N})\) chiefly by appealing appropriately to each of the aforementioned cases \((m = 2, 3, 4, \) and 6\) of the general result (1.10).

The following properties of the Riemann \( \zeta \)-function will be required in our investigation:

\[
\zeta(0) = -\frac{1}{2}; \quad \zeta(-2n) = 0 \quad (n \in \mathbb{N}); \quad \zeta'(0) = -\frac{1}{2} \log(2\pi),
\]

and (in general)

\[
\zeta'(-2n) = \lim_{\epsilon \to 0} \frac{\zeta(-2n + \epsilon)}{\epsilon} = \frac{(-1)^n}{2(2\pi)^{2n}} (2n)! \zeta(2n + 1) \quad (n \in \mathbb{N}),
\]

(1.15)

where use is made of the familiar functional equation:

\[
2^s \Gamma(1-s) \zeta(1-s) \sin \left( \frac{1}{2} \pi s \right) = \pi^{1-s} \zeta(s).
\]

(1.16)

Furthermore, by l’Hôpital’s rule, we have

\[
\lim_{s \to -2n} \left\{ \sin \left( \frac{1}{2} \pi s \right) \right\} = (-1)^n \frac{\pi}{2n} \quad (n \in \mathbb{N})
\]

(1.17)

and

\[
\lim_{s \to -2n} \left\{ \frac{\zeta(s + 2k)}{s + 2n} \right\} = \frac{(-1)^{n-k}}{2(2\pi)^{2(n-k)}} (2n - 2k)! \zeta(2n - 2k + 1) \quad (k = 1, \ldots, n - 1; \quad n \in \mathbb{N} \setminus \{1\}).
\]

(1.18)
2. A SET OF SERIES REPRESENTATIONS

We begin with the case \( m = 2 \) of the general result (1.10). Upon separating the first \( n + 1 \) terms of the series occurring on the left-hand side, if we transpose the terms for \( k = 0 \) and \( k = n \) to the right-hand side, we obtain

\[
\sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k)}{2^{2k}} + \sum_{k=n+1}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k)}{2^{2k}}
\]

(2.1)

\[
= (2^s - 2)\zeta(s) - 2^{s-1} \frac{(s)_{2n}}{(2n)!} \frac{\zeta(s + 2n)}{2^{2n}},
\]

which readily yields the identity:

\[
\sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} 2^{2(n-k)} \zeta(s + 2k) + \sum_{k=1}^{\infty} \frac{(s)_{2n+2k}}{(2n + 2k)!} \frac{\zeta(s + 2n + 2k)}{2^{2k}}
\]

(2.2)

\[
= 2^{2n}(2^s - 2)\zeta(s) - 2^{s+2n-1} \frac{(s)_{2n}}{(2n)!} \frac{\zeta(s + 2n)}{2^{2n}} \quad (n \in \mathbb{N}),
\]

it being understood, as before, that an empty sum is to be interpreted as nil.

Now we apply the functional equation (1.16) in the first term on the right-hand side of (2.2) and divide both sides by \( s + 2n \). We thus find that

\[
\sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} 2^{2(n-k)} \left\{ \frac{\zeta(s + 2k)}{s + 2n} \right\} + \sum_{k=1}^{\infty} \frac{(s)_{2n}(s + 2n + 1)_{2k-1}}{(2n + 2k)!} \frac{\zeta(s + 2n + 2k)}{2^{2k}}
\]

\[
= 2^{s+2n}(2^s - 2) \pi^{s-1} \Gamma(1 - s) \zeta(1 - s) \left\{ \frac{\sin \left( \frac{1}{2} \pi s \right)}{s + 2n} \right\}
\]

(2.3)

\[
- \left\{ \frac{2^{s+2n-1} + (s)_{2n}}{(2n)!} \frac{\zeta(s + 2n)}{s + 2n} \right\} \quad (s \neq -2n; \ n \in \mathbb{N}).
\]

Since

\[
(-n)_k = (-1)^k \frac{n!}{(n - k)!} \quad (k = 0, 1, \cdots, n; \ n \in \mathbb{N}),
\]

so that, obviously,

(2.4)

\[
(-n)_n = (-1)^n \ n! \quad (n \in \mathbb{N}),
\]

in view of the definition (1.4), it is easily seen by logarithmic differentiation that

(2.5)

\[
\frac{d}{ds} \{ (s)_n \} = (s)_n \sum_{j=0}^{n-1} \frac{1}{s + j} \quad (n \in \mathbb{N}),
\]

so that

(2.6)

\[
\frac{d}{ds} \{ (s)_{2n} \} \bigg|_{s=-2n} = -(2n)! \ H_{2n} \quad (n \in \mathbb{N}),
\]
where \( H_n \) denotes the familiar harmonic numbers defined by

\[
H_n := \sum_{j=1}^{n} \frac{1}{j} \quad (n \in \mathbb{N}).
\]

We observe also that the limit formula (1.18) is needed in the first sum on the left-hand side of (2.3) only when this sum is nonzero (that is, only when \( n \in \mathbb{N} \setminus \{1\} \)). Furthermore, by l'Hôpital's rule, we have

\[
\lim_{s \to -2n} \left\{ \frac{2^{s+2n-1} + \frac{(s)_{2n}}{(2n)!}}{s+2n} \zeta(s+2n) \right\} = \left[ 2^{s+2n-1} \log 2 + \frac{d}{ds} \left\{ \frac{(s)_{2n}}{(2n)!} \zeta'(s+2n) \right\} \right]_{s=-2n}
\]

\[
= \frac{1}{2} (H_{2n} - \log \pi) \quad (n \in \mathbb{N}).
\]

Finally, letting \( s \to -2n \) in (2.3), and making use of the limit relationships (1.17), (1.18), and (2.8), we obtain our first series representation for \( \zeta(2n+1) \):

\[
\zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2^{2n+1} - 1} \left[ \frac{H_{2n} - \log \left( \frac{3}{2} \pi \right)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \right] + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{2^{2k}} \quad (n \in \mathbb{N}).
\]

In precisely the same manner, we can apply the identities (1.11), (1.12), and (1.13) in order to prove the following additional series representations for \( \zeta(2n+1) \):

\[
\zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n+1} + 2^{2n} - 1} \left[ \frac{H_{2n} - \log \left( \frac{1}{2} \pi \right)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{(2\pi)^{2k}} \right] + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{4^{2k}} \quad (n \in \mathbb{N});
\]

\[
\zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{2^{4n+1} + 2^{2n} - 1} \left[ \frac{H_{2n} - \log \left( \frac{1}{2} \pi \right)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{(1/2)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{4^{2k}} \right] \quad (n \in \mathbb{N});
\]

\[
\zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n}(2^{2n+1} + 2^{2n} - 1) + 2^{2n} - 1} \left[ \frac{H_{2n} - \log \left( \frac{3}{2} \pi \right)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!} \frac{\zeta(2k+1)}{(2\pi)^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{6^{2k}} \right] \quad (n \in \mathbb{N}).
\]
3. Remarks and observations

Our series representation (2.9) is markedly different from each of the series representations for $\zeta(2n + 1)$, which were given earlier by Zhang and Williams [29, p. 1591, Equation (3.16)] and (more recently) by Cvijović and Klinowski [8, p. 1265, Theorem A]. Since $\zeta(2k) \to 1$ as $k \to \infty$, the general term in our series representation (2.9) has the order estimate:

$$O \left( 2^{-2k} \cdot k^{-2n-1} \right) \quad (k \to \infty; \ n \in \mathbb{N}),$$

whereas the general term in each of these earlier series representations has the order estimate:

$$O \left( 2^{-2k} \cdot k^{-2n} \right) \quad (k \to \infty; \ n \in \mathbb{N}).$$

By suitably combining (2.9) and (2.11), it is fairly straightforward to obtain the series representation:

$$\zeta(2n + 1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n} - 1)(2^{2n+1} - 1)} \left[ \log \frac{2}{(2n)!} \right]$$

$$+ \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k} - 1) \zeta(2k + 1)}{(2n - 2k)! \pi^{2k}}$$

$$- 2 \sum_{k=1}^{\infty} \frac{(2k - 1)! (2^{2k} - 1) \zeta(2k) \pi^{2k}}{(2n + 2k)!} \right] \quad (n \in \mathbb{N}).$$

(3.1)

Now, in terms of the Bernoulli numbers $B_n$ and the Euler polynomials $E_n(x)$ defined by the generating functions:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi)$$

(3.2)

and

$$\frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} \quad (|z| < \pi),$$

(3.3)

respectively, it is known that (cf., e.g., Magnus et al. [20, p. 29])

$$E_n(0) = (-1)^n E_n(1) = \frac{2(1 - 2^{n+1})}{n+1} B_{n+1} \quad (n \in \mathbb{N})$$

(3.4)

and [20, p. 19]

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \quad (n \in \mathbb{N}),$$

(3.5)

which, together, imply that

$$E_{2n-1}(0) = \frac{4(-1)^n}{(2\pi)^{2n}} (2n - 1) (2^{2n} - 1) \zeta(2n) \quad (n \in \mathbb{N}).$$

(3.6)
Making use of this last relationship (3.6), the series representation (3.1) can immediately be put in the form:

\[
\zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n+1} - 1)(2^{2n+1} + 1)} \left[ \log \frac{2}{(2n)!} \right] + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k+1} - 1)}{(2n - 2k)!} \frac{\zeta(2k + 1)}{\pi^{2k}} \left( \frac{\pi}{2} \right)^{2k} E_{2k-1}(0) \]  

(3.7) \begin{align*}
\ \ \ \ \ + \left( -1 \right)^{k-1} \frac{\zeta(2k + 1)}{\pi^{2k}} \pi^2 \left( \frac{\pi}{2} \right)^{2k} E_{2k-1}(0) \end{align*}  

\ \ \ \ \ (n \in \mathbb{N}),

which is a slightly modified (and corrected) version of a result proven in a significantly different way by Tsumura [27, p. 383, Theorem B].

Another interesting combination of our series representations (2.9) and (2.11) leads us to the following variant of Tsumura’s result (3.1) or (3.7):

\[
\zeta(2n+1) = (-1)^{n-1} \frac{\pi^{2n}}{2^{2n+1} - 1} \left[ H_{2n} - \log \left( \frac{\pi}{2} \right) \right] + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k+1} - 1)}{(2n - 2k)!} \frac{\zeta(2k + 1)}{\pi^{2k}} \left( \frac{\pi}{2} \right)^{2k} E_{2k-1}(0) \]  

(3.8) \begin{align*}
\ \ \ \ \ -4 \left( \frac{2k - 1)! (2^{2k-1} - 1)}{(2n + 2k)!} \frac{\zeta(2k)}{24k} \right) \]  

\ \ \ \ \ (n \in \mathbb{N}),

which is essentially the same as the determinantal expression for \( \zeta(2n+1) \) derived recently by Ewell [12, p. 1010, Corollary 3] by employing an entirely different technique from ours.

Other similar combinations of our series representations (2.9) to (2.12) would yield the following (presumably new) companions of Ewell’s result (3.8):

\[
\zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n+1} - 1)(3^{2n+1} + 1)} \left[ H_{2n} - \log \left( \frac{\pi}{6} \right) \right] + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k+1} - 1)}{(2n - 2k)!} \frac{\zeta(2k + 1)}{2^{2k}} \left( \frac{\pi}{3} \right)^{2k} \]  

(3.9) \begin{align*}
\ \ \ \ \ -4 \left( \frac{2k - 1)! (2^{2k-1} - 1)}{(2n + 2k)!} \frac{\zeta(2k)}{6^{2k}} \right) \]  

\ \ \ \ \ (n \in \mathbb{N});

\[
\zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n+1} + 1)(3^{2n+1} + 1)} \left[ 2H_{2n} - \log \left( \frac{\pi^2}{12} \right) \right] + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k+1} - 1)}{(2n - 2k)!} \frac{\zeta(2k + 1)}{\pi^{2k}} \]  

(3.10) \begin{align*}
\ \ \ \ \ -6 \left( \frac{2k - 1)! (3^{2k-1} - 1)}{(2n + 2k)!} \frac{\zeta(2k)}{6^{2k}} \right) \]  

\ \ \ \ \ (n \in \mathbb{N});
\( \zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n+2} - 2^{2n+3} + 1} \left[ \frac{H_{2n} - \log \left( \frac{8n}{\pi} \right)}{(2n)!} \right] \\
+ \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1} - 2^{2k+1})}{(2n - 2k)!} \zeta(2k + 1) \left( \frac{(2k - 1)!}{(2\pi)^{2k}} \right) \left( \frac{\zeta(2k)}{6^{2k}} \right) \right] \quad (n \in \mathbb{N});

(3.11)

\( \zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{2^{4n+3} + 2^{2n+2} - 3^{2n+2} - 1} \left[ \frac{H_{2n} - \log \left( \frac{6\pi}{27} \right)}{(2n)!} \right] \\
+ \sum_{k=1}^{n-1} \frac{(-1)^k (4^{2k+1} - 3^{2k+1})}{(2n - 2k)!} \zeta(2k + 1) \left( \frac{(2k - 1)!}{(2\pi)^{2k}} \right) \left( \frac{\zeta(2k)}{12^{2k}} \right) \right] \quad (n \in \mathbb{N}),

(3.12)

and

\( \zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n+1}(2^{2n+1} + 2^{2n} - 1)} \left[ \frac{H_{2n} - \log \left( \frac{4\pi}{27} \right)}{(2n)!} \right] \\
+ \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1} - 2^{2k+1})}{(2n - 2k)!} \pi^{2k} \zeta(2k + 1) \left( \frac{(2k - 1)!}{(2\pi)^{2k}} \right) \left( \frac{\zeta(2k)}{12^{2k}} \right) \right] \quad (n \in \mathbb{N}).

(3.13)

Next we turn to the identity (1.7). By setting \( t = 1/m \) and differentiating both sides with respect to \( s \), we find from (1.7) that

\[
\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!m^{2k}} \left[ \zeta'(s + 2k + 1, a) + \zeta(s + 2k + 1, a) \sum_{j=0}^{2k} \frac{1}{s+j} \right] = \frac{m}{2} \frac{d}{ds} \left\{ \zeta \left( s, a - \frac{1}{m} \right) - \zeta \left( s, a + \frac{1}{m} \right) \right\} \quad (m \in \mathbb{N} \setminus \{1\}),
\]

(3.14)

where we have made use of the derivative formula (2.5). In particular, when \( m = 2 \), (3.14) immediately yields

\[
\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!2^{2k}} \left[ \zeta'(s + 2k + 1, a) + \zeta(s + 2k + 1, a) \sum_{j=0}^{2k} \frac{1}{s+j} \right] = - \left( a - \frac{1}{2} \right)^{-s} \log \left( a - \frac{1}{2} \right).
\]

(3.15)

By letting \( s \to -2n - 1 \ (n \in \mathbb{N}) \) in the further special of this last identity (3.15) when \( a = 1 \), Wilton [28, p. 92] obtained the following series representation for
\[ \zeta(2n + 1) = (-1)^{n-1} \pi^{2n} \left[ H_{2n+1} - \log \pi \frac{1}{(2n+1)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} \right] + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \] (n \in \mathbb{N}),

which may be compared with our first series representation (2.9). As a matter of fact, since

\[ \frac{(2k)!}{(2n+2k)!} = \frac{(2k-1)!}{(2n+2k-1)!} - 2n \frac{(2k-1)!}{(2n+2k)!} (n, k \in \mathbb{N}), \]

it is not difficult to deduce from (2.9) and (3.16) (with \( n \) replaced by \( n-1 \)) that

\[ \zeta(2n + 1) = (-1)^n \frac{(2\pi)^{2n}}{n(2n+1)!} \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k)!} \frac{\zeta(2k)}{\pi^{2k}} + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k)!} \frac{\zeta(2k)}{2^{2k}} \] (n \in \mathbb{N}),

which is precisely the aforementioned main result of Cvijović and Klinowski [8, p. 1265, Theorem A] (see also Zhang and Williams [29, p. 1591, Equation (3.16)] where an obviously more complicated version of (3.18) was proven by applying the same identity (1.14) above).

Observe also that

\[ \frac{(2k)!}{(2n+2k+1)!} = \frac{(2k-1)!}{(2n+2k)!} - (2n + 1) \frac{(2k-1)!}{(2n+2k+1)!} \] (n, k \in \mathbb{N}),

we obtain yet another series representation for \( \zeta(2n + 1) \) by applying (2.9) and (3.16):

\[ \zeta(2n + 1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n-1)! 2^{2n+1}} \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \] (n \in \mathbb{N}),

which provides a significantly simpler (and much more rapidly convergent) version of the other main result of Cvijović and Klinowski [8, p. 1265, Theorem B]:

\[ \zeta(2n + 1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n)!} \sum_{k=0}^{\infty} \Omega_{n,k} \frac{\zeta(2k)}{2^{2k}} \] (n \in \mathbb{N}),

where the coefficients \( \Omega_{n,k} \) are given explicitly by

\[ \Omega_{n,k} := \sum_{j=0}^{2n} \binom{2n}{j} \frac{B_{2n-j}}{(j + 2k + 1)(j + 1)!} \frac{1}{2^j} \] \( (n \in \mathbb{N}; k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}) \),

in terms of the Bernoulli numbers defined by (3.2). Since [20, pp. 27 and 28]

\[ B_1 = -\frac{1}{2} \quad \text{and} \quad B_{2n+1} = 0 \quad (n \in \mathbb{N}), \]
the definition (3.22) can be rewritten at once in the form:

\[ \Omega_{n,k} = \sum_{j=0}^{\infty} \binom{2n}{2j} \frac{B_{2n-2j}}{(2j + 2k + 1)(2j + 1) 2^{2j}} - \frac{1}{(2n + 2k) 2^{2n}} \quad (n \in \mathbb{N}; \ k \in \mathbb{N}_0), \]  

or, equivalently,

\[ \Omega_{n,k} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{j=0}^{n} \frac{(-\pi^2)^j \zeta(2n - 2j)}{(2j + 1)! (2j + 2k + 1)} - \frac{1}{(2n + 2k) 2^{2n}} \quad (n \in \mathbb{N}; \ k \in \mathbb{N}_0), \]  

by virtue of the relationship (3.5). Combining the partial fractions occurring in (3.23) or (3.24), it is easily seen that

\[ \Omega_{n,k} = \prod_{\ell=0}^{n} \left\{ \frac{(2k + 2\ell + 1)^{-1}}{(2n + 2k) 2^{2n}} \right\} \sum_{j=0}^{n} \binom{2n}{2j} \frac{2n + 2k}{2j + 1} 2^{2n-2j} B_{2n-2j} \]

\[ \times \prod_{\ell=0}^{n} (2k + 2\ell + 1) - \prod_{\ell=0}^{n} (2k + 2\ell + 1) \quad (n \in \mathbb{N}; \ k \in \mathbb{N}_0). \]

In view of the identity:

\[ \sum_{j=0}^{n} \binom{2n}{2j} \frac{2^{2n-2j} B_{2n-2j}}{2j + 1} = 1 = \sum_{j=0}^{n} \binom{2n}{2j} \frac{2^{2j} B_{2j}}{2n - 2j + 1}, \]

which is due essentially to Euler (cf., e.g., Riordan [23, p. 123, Problem 12]), the expression inside brackets in (3.25) is a polynomial in \( k \) of degree \( n \) (not \( n + 1 \)), and therefore

\[ \Omega_{n,k} = O(k^{-2}) \quad (k \to \infty; \ n \in \mathbb{N}). \]

It follows from (3.27) that the general term in (3.21) has the order estimate:

\[ O(2^{-2k} \cdot k^{-2}) \quad (k \to \infty), \]

whereas the general term in our series representation (3.20) has precisely the same order estimate:

\[ O(2^{-2k} \cdot k^{-2n-1}) \quad (k \to \infty; \ n \in \mathbb{N}), \]

as that in (2.9). Thus, even in the special case when \( n = 1 \), the series representing \( \zeta(3) \) converges faster in (3.20) than in (3.21).

Various known series representations for \( \zeta(2n + 1) \ (n \in \mathbb{N}) \) of other types include those given (for example) by Ramanujan [21] (see also Berndt [3]), Glaisher [13] (see also Hansen [16, p. 359]), Koshliakov [18], Leshchiner [19], Grosswald ([14] and [15]), Terras [25], Cohen [7], Butzer et al. ([5] and [6]), Dąbrowski [9], and others (see, e.g., Berndt [4, pp. 275 and 276]).

We conclude this paper by remarking that a particular case of our series representation (2.12) when \( n = 1 \) was proven, by an entirely different method, by Zhang...
and Williams [30, p. 707, Theorem 9]. Furthermore, the following particular case of (3.18) when \( n = 1 \):

\[
\zeta(3) = -\frac{4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)2^{2k}}, \tag{3.30}
\]

which is contained in a 1772 paper entitled *Exercitationes Analyticae* by Euler (see, e.g., Ayoub [2, pp. 1084–1085]), was rediscovered by Ramaswami [22] and (more recently) by Ewell [10]. In fact, Euler’s formula (3.30) was reproduced by Srivastava [24, p. 7, Equation (2.23)] from the work of Ramaswami [22]. In the current mathematical literature, however, Euler’s formula (3.30) is being attributed to Ewell [10].

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**References**

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