SOME RAPIDLY CONVERGING SERIES FOR $\zeta(2n+1)$

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Abstract. For a natural number $n$, the author derives several families of series representations for the Riemann Zeta function $\zeta(2n+1)$. Each of these series representing $\zeta(2n+1)$ converges remarkably rapidly with its general term having the order estimate:

$$O(k^{-2n-1} \cdot m^{-2k}) \quad (k \to \infty; \quad m = 2, 3, 4, 6).$$

Relevant connections of the results presented here with many other known series representations for $\zeta(2n+1)$ are also pointed out.

1. Introduction and preliminaries

The Riemann Zeta function $\zeta(s)$ and the (Hurwitz’s) generalized Zeta function $\zeta(s, a)$, defined usually by (see, e.g., Titchmarsh [26])

$$\zeta(s) := \begin{cases} \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1 - 2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1), \\ \frac{1}{1 - 2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; \quad s \neq 1) \end{cases}$$

and

$$\zeta(s, a) := \sum_{n=0}^{\infty} \frac{1}{(n + a)^s} \quad (\Re(s) > 1; \quad a \neq 0, -1, -2, \cdots),$$

so that

$$\zeta(s, 1) = \zeta(s) \quad \text{and} \quad \zeta(s, 2) = \zeta(s) - 1,$$

are known to be meromorphic (that is, analytic everywhere in the complex $s$-plane except for a simple pole at $s = 1$ with residue 1). Making use of the binomial theorem and the Pochhammer symbol $(\lambda)_n$ defined by

$$(\lambda)_n := \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0), \\ \lambda(\lambda + 1) \cdots (\lambda + n - 1) & (n \in \mathbb{N} := \{1, 2, 3, \cdots\}) \end{cases},$$

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it is easily seen from the definition (1.2) that

\[
\sum_{k=0}^{\infty} \frac{(s)_{k}}{k!} \zeta(s + k, a) t^{k} = \zeta(s, a - t) \quad (|t| < |a|),
\]

which immediately yields the familiar identities (cf., e.g., Hansen [16, p. 359] where other references are also cited):

\[
\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \zeta(s + 2k, a) t^{2k} = \frac{1}{2} \left[ \zeta(s, a - t) + \zeta(s, a + t) \right] \quad (|t| < |a|)
\]

and

\[
\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!} \zeta(s + 2k + 1, a) t^{2k+1} = \frac{1}{2} \left[ \zeta(s, a - t) - \zeta(s, a + t) \right] \quad (|t| < |a|)
\]

or, equivalently,

\[
\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k+1)!} \zeta(s + 2k, a) t^{2k+1} = \frac{1}{2(s - 1)} \left[ (s - 1) \zeta(s - 1, a - t) - \zeta(s - 1, a + t) \right] \quad (|t| < |a|).
\]

Since

\[
\zeta(s) = \frac{1}{m^{s} - 1} \sum_{j=1}^{m-1} \zeta(s, \frac{j}{m}) \quad (m \in \mathbb{N} \setminus \{1\}),
\]

which follows readily from the definitions (1.1) and (1.2), the special case of the identity (1.6) when \(a = 1\) and \(t = 1/m\) can be rewritten in the form:

\[
\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k, a)}{m^{2k}} = \frac{1}{2} \left[ (2^s - 1) \zeta(s) - 2^{s-1} \right] \quad (m = 2),
\]

\[
\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k, a)}{m^{2k}} = \frac{1}{2} \left[ (m^s - 1) \zeta(s) - m^s - \sum_{j=2}^{m-2} \zeta \left( s, \frac{j}{m} \right) \right] \quad (m \in \mathbb{N} \setminus \{1, 2\}),
\]

where (and throughout this paper) an empty sum is to be interpreted as nil. In addition to the case \(m = 2\), the formula (1.10) simplifies also in the cases when \(m = 3, 4, 6\), and we thus obtain the identities:

\[
\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k, a)}{3^{2k}} = \frac{1}{2} [(3^s - 1) \zeta(s) - 3^s],
\]

\[
\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k, a)}{4^{2k}} = \frac{1}{2} [(4^s - 2^s) \zeta(s) - 4^s],
\]
and

\[ \sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k)}{6^{2k}} = \frac{1}{2} \left[ (6^s - 3^s - 2^s + 1) \zeta(s) - 6^s \right], \]

(1.13)

respectively.

Identities of this kind seem to have first appeared in the work of Ramaswami [22], who actually proved the cases \( m = 2, 3, \) and \( 6 \) of the general result in (1.10).

Each of these three identities of Ramaswami [22] can also be found in the work of Hansen [16, p. 357], who referred to Apostol [1] as his source for the identities (1.11) and (1.13) only. As a matter of fact, Apostol [1] reproduced the identities (1.11) and (1.13) from Ramaswami’s work [22] and then proved an interesting arithmetical generalization of these identities (see also Klusch [17, p. 520]).

In its slightly variant form:

\[ \sum_{k=1}^{\infty} \frac{(s + 1)_{2k}}{(2k)!} \frac{\zeta(s + 2k)}{2^{2k}} = (2^s - 2) \zeta(s), \]

(1.14)

which can indeed be proven directly from the known special cases of (1.6) and (1.7) when \( a = 1 \) and \( t = \frac{1}{2} \), the case \( m = 2 \) of the general result (1.10) was applied by Zhang and Williams [29] (and, more recently, by Cvijović and Klinowski [8]) with a view to finding two seemingly different series representations for \( \zeta(2n + 1) \) \((n \in \mathbb{N})\). The main object of this paper is to obtain much more rapidly converging series representations for \( \zeta(2n + 1) \) \((n \in \mathbb{N})\) chiefly by appealing appropriately to each of the aforementioned cases \((m = 2, 3, 4, \) and \( 6 \)) of the general result (1.10).

The following properties of the Riemann \( \zeta \)-function will be required in our investigation:

\[ \zeta(0) = -\frac{1}{2}, \quad \zeta(-2n) = 0 \quad (n \in \mathbb{N}); \quad \zeta'(0) = -\frac{1}{2} \log(2\pi), \]

and (in general)

\[ \zeta'(-2n) = \lim_{\epsilon \to 0} \frac{\zeta(-2n + \epsilon)}{\epsilon} = \frac{(-1)^n}{2(2\pi)^{2n}} (2n)! \zeta(2n + 1) \quad (n \in \mathbb{N}), \]

(1.15)

where use is made of the familiar functional equation:

\[ 2^s \Gamma(1 - s) \zeta(1 - s) \sin \left( \frac{1}{2} \pi s \right) = \pi^{1-s} \zeta(s). \]

(1.16)

Furthermore, by l'Hôpital’s rule, we have

\[ \lim_{s \to 2n} \left\{ \frac{\sin \left( \frac{1}{2} \pi s \right)}{s + 2n} \right\} = \frac{(-1)^n \pi}{2} \quad (n \in \mathbb{N}) \]

(1.17)

and

\[ \lim_{s \to 2n} \left\{ \frac{\zeta(s + 2k)}{s + 2n} \right\} = \frac{(-1)^{n-k}}{2(2\pi)^{2(n-k)}} (2n - 2k)! \zeta(2n - 2k + 1) \]

(1.18)

\((k = 1, \ldots, n - 1; \quad n \in \mathbb{N} \setminus \{1\})\).
We begin with the case $m = 2$ of the general result (1.10). Upon separating the first $n + 1$ terms of the series occurring on the left-hand side, if we transpose the terms for $k = 0$ and $k = n$ to the right-hand side, we obtain

$$
\sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k)}{2^{2k}} + \sum_{k=n+1}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s + 2k)}{2^{2k}} = (2^s - 2) \zeta(s) - 2^{s-1} \frac{(s)_{2n}}{(2n)!} \frac{\zeta(s + 2n)}{2^{2n}},
$$

which readily yields the identity:

$$
\sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} 2^{2(n-k)} \zeta(s + 2k) + \sum_{k=1}^{\infty} \frac{(s)_{2n+2k}}{(2n + 2k)!} \frac{\zeta(s + 2n + 2k)}{2^{2k}} = 2^{s+2n}(2^s - 2) \zeta(s) - 2^{s+2n-1} \frac{(s)_{2n}}{(2n)!} \frac{\zeta(s + 2n)}{2^{2n}} \quad (n \in \mathbb{N}),
$$

it being understood, as before, that an empty sum is to be interpreted as nil.

Now we apply the functional equation (1.16) in the first term on the right-hand side of (2.2) and divide both sides by $s + 2n$. We thus find that

$$
\sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} 2^{2(n-k)} \zeta(s + 2k) + \sum_{k=1}^{\infty} \frac{(s)_{2n+2k}}{(2n + 2k)!} \frac{\zeta(s + 2n + 2k)}{2^{2k}} = 2^{s+2n}(2^s - 2) \pi^{s-1} \Gamma(1-s) \zeta(1-s) \left\{ \frac{\sin \left( \frac{1}{2} \pi s \right)}{s + 2n} \right\}
$$

$$
\left\{ \frac{2^{s+2n-1} + \frac{(s)_{2n}}{(2n)!} \zeta(s + 2n)}{s + 2n} \right\} \quad (s \neq -2n; \ n \in \mathbb{N}).
$$

Since

$$
(-n)_k = (-1)^k \frac{n!}{(n - k)!} \quad (k = 0, 1, \ldots, n; \ n \in \mathbb{N}),
$$

so that, obviously,

$$
(-n)_n = (-1)^n n! \quad (n \in \mathbb{N}),
$$

in view of the definition (1.4), it is easily seen by logarithmic differentiation that

$$
\frac{d}{ds} \left\{ \frac{(s)_{2n}}{2n} \right\} \bigg|_{s = -2n} = -(2n)! \ H_{2n} \quad (n \in \mathbb{N}),
$$

so that

$$
\frac{d}{ds} \left\{ \frac{(s)_{2n}}{2n} \right\} \bigg|_{s = -2n} = -(2n)! \ H_{2n} \quad (n \in \mathbb{N}),
$$
where \( H_n \) denotes the familiar harmonic numbers defined by

\[
(2.7) \quad H_n := \sum_{j=1}^{n} \frac{1}{j} \quad (n \in \mathbb{N}).
\]

We observe also that the limit formula (1.18) is needed in the first sum on the left-hand side of (2.3) only when this sum is nonzero (that is, only when \( n \in \mathbb{N} \setminus \{1\} \)). Furthermore, by l'Hôpital's rule, we have

\[
\lim_{s \to -2n} \left\{ \frac{2^{s+2n-1} + (s)_{2n}}{s+2n} \right\} \zeta(s+2n)
= \left[ 2^{s+2n-1} \log 2 + \frac{d}{ds} \{(s)_{2n}\} \cdot \frac{\zeta(s+2n)}{(2n)!} + \frac{(s)_{2n}}{(2n)!} \zeta'(s+2n) \right]_{s=-2n}
= \frac{1}{2} (H_{2n} - \log \pi) \quad (n \in \mathbb{N}).
\]

Finally, letting \( s \to -2n \) in (2.3), and making use of the limit relationships (1.17), (1.18), and (2.8), we obtain our first series representation for \( \zeta(2n+1) \):

\[
(2.9) \quad \zeta(2n+1) = (-1)^{n-1} \frac{(2\pi)^{2n}}{2^{2n+1} - 1} \left[ H_{2n} - \log \left( \frac{3}{2} \pi \right) \right] + \sum_{k=1}^{n-1} \frac{(-1)^{k}}{(2n-2k)!} \frac{\zeta(2k+1)}{(2n-2k)!} \pi^{2k} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{2^{2k}} \quad (n \in \mathbb{N}).
\]

In precisely the same manner, we can apply the identities (1.11), (1.12), and (1.13) in order to prove the following additional series representations for \( \zeta(2n+1) \):

\[
(2.10) \quad \zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n+1} + 2^{2n+1} - 1} \left[ H_{2n} - \log \left( \frac{5}{2} \pi \right) \right] + \sum_{k=1}^{n-1} \frac{(-1)^{k}}{(2n-2k)!} \frac{\zeta(2k+1)}{(2n-2k)!} \frac{\pi^{2k}}{3^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{4^{2k}} \quad (n \in \mathbb{N});
\]

\[
(2.11) \quad \zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{2^{4n+1} + 2^{2n+1} - 1} \left[ H_{2n} - \log \left( \frac{1}{2} \pi \right) \right] + \sum_{k=1}^{n-1} \frac{(-1)^{k}}{(2n-2k)!} \frac{\zeta(2k+1)}{(2n-2k)!} \pi^{2k} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{4^{2k}} \quad (n \in \mathbb{N});
\]

\[
(2.12) \quad \zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n}(2^{2n+1} + 1) + 2^{2n} - 1} \left[ H_{2n} - \log \left( \frac{1}{3} \pi \right) \right] + \sum_{k=1}^{n-1} \frac{(-1)^{k}}{(2n-2k)!} \frac{\zeta(2k+1)}{(2n-2k)!} \frac{\pi^{2k}}{3^{2k}} + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!} \frac{\zeta(2k)}{6^{2k}} \quad (n \in \mathbb{N}).
\]
3. Remarks and observations

Our series representation (2.9) is markedly different from each of the series representations for \( \zeta(2n+1) \), which were given earlier by Zhang and Williams [29, p. 1591, Equation (3.16)] and (more recently) by Cvijović and Klinowski [8, p. 1265, Theorem A]. Since \( \zeta(2k) \to 1 \) as \( k \to \infty \), the general term in our series representation (2.9) has the order estimate:

\[
O \left( 2^{-2k} \cdot k^{-2n-1} \right) \quad (k \to \infty; \ n \in \mathbb{N}),
\]

whereas the general term in each of these earlier series representations has the order estimate:

\[
O \left( 2^{-2k} \cdot k^{-2n} \right) \quad (k \to \infty; \ n \in \mathbb{N}).
\]

By suitably combining (2.9) and (2.11), it is fairly straightforward to obtain the series representation:

\[
\zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)} \left[ \log 2 \right]_{(2n)!} \left( k \to \infty \right) (n \in \mathbb{N})
\]

\[
+ \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \right]
\]

\[
- 2 \sum_{k=1}^{\infty} \frac{(2k-1)! (2^{2k}-1)}{(2n+2k)!} \frac{\zeta(2k)}{2^{4k}} \right] \quad (n \in \mathbb{N}).
\]

(3.1)

Now, in terms of the Bernoulli numbers \( B_n \) and the Euler polynomials \( E_n(x) \) defined by the generating functions:

\[
\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \quad (|z| < 2\pi)
\]

(3.2)

and

\[
\frac{2e^{xz}}{e^z + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} \quad (|z| < \pi),
\]

(3.3)

respectively, it is known that (cf., e.g., Magnus et al. [20, p. 29])

\[
E_n(0) = (-1)^n E_n(1) = \frac{2(1 - 2n+1)}{n+1} B_{n+1} \quad (n \in \mathbb{N})
\]

(3.4)

and [20, p. 19]

\[
\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \quad (n \in \mathbb{N}),
\]

(3.5)

which, together, imply that

\[
E_{2n-1}(0) = \frac{4(-1)^n}{(2\pi)^{2n}} (2n-1)! (2^{2n}-1) \zeta(2n) \quad (n \in \mathbb{N}).
\]

(3.6)
Making use of this last relationship (3.6), the series representation (3.1) can immediately be put in the form:

\[
\zeta(2n + 1) = (-1)^{n-1} \frac{2 (2\pi)^{2n}}{(2^{2n} - 1)(2^{2n+1} + 1)} \left[ \log \frac{2}{(2n)!} \right] \\
+ \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k} - 1)}{(2n - 2k)!} \frac{\zeta(2k + 1)}{\pi^{2k}} \\
+ \frac{1}{2} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(2n + 2k)!} \left( \frac{\pi}{2} \right)^{2k} E_{2k-1}(0) \right] \quad (n \in \mathbb{N}),
\]

which is a slightly modified (and corrected) version of a result proven in a significantly different way by Tsumura [27, p. 383, Theorem B].

Another interesting combination of our series representations (2.9) and (2.11) leads us to the following variant of Tsumura’s result (3.1) or (3.7):

\[
\zeta(2n + 1) = (-1)^{n-1} \frac{\pi^{2n}}{2^{2n+1} - 1} \left[ H_{2n} - \log \left( \frac{1}{\sqrt[2]{\pi}} \right) \right] \\
+ \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k+1} - 1)}{(2n - 2k)!} \frac{\zeta(2k + 1)}{\pi^{2k}} \\
- 4 \sum_{k=1}^{\infty} \frac{(2k - 1)! (2^{2k-1} - 1)}{(2n + 2k)!} \frac{\zeta(2k)}{2^{4k}} \right] \quad (n \in \mathbb{N}),
\]

which is essentially the same as the determinantal expression for \( \zeta(2n + 1) \) derived recently by Ewell [12, p. 1010, Corollary 3] by employing an entirely different technique from ours.

Other similar combinations of our series representations (2.9) to (2.12) would yield the following (presumably new) companions of Ewell’s result (3.8):

\[
\zeta(2n + 1) = (-1)^{n-1} \frac{2 (2\pi)^{2n}}{(2^{2n} + 1)(3^{2n} + 1)} \left[ H_{2n} - \log \left( \frac{1}{\sqrt[2]{3\pi}} \right) \right] \\
+ \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k+1} - 1)}{(2n - 2k)!} \frac{\zeta(2k + 1)}{(\frac{2}{3})^{2k}} \\
- 6 \sum_{k=1}^{\infty} \frac{(2k - 1)! (2^{2k} - 1)}{(2n + 2k)!} \frac{\zeta(2k)}{2^{6k}} \right] \quad (n \in \mathbb{N});
\]

\[
\zeta(2n + 1) = (-1)^{n-1} \frac{2 (2\pi)^{2n}}{(2^{2n} + 1)(3^{2n+1} + 1)} \left[ 2H_{2n} - \log \left( \frac{\pi^2}{24} \right) \right] \\
+ \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1} - 1)}{(2n - 2k)!} \frac{\zeta(2k + 1)}{\pi^{2k}} \\
- 6 \sum_{k=1}^{\infty} \frac{(2k - 1)! (3^{2k} - 1)}{(2n + 2k)!} \frac{\zeta(2k)}{6^{2k}} \right] \quad (n \in \mathbb{N});
\]
\[
\zeta(2n + 1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n+2} - 2^{2n+3} + 1} \left[ H_{2n} - \log \left( \frac{2\pi}{27} \right) \right] (2n)!
\]
\[
+ \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1} - 2^{2k+1})}{(2n - 2k)!} \frac{\zeta(2k + 1)}{(2\pi)^{2k}}
\]
\[
- 12 \sum_{k=1}^{\infty} \frac{(2k - 1)! (3^{2k-1} - 2^{2k-1}) \zeta(2k)}{(2n + 2k)!} 6^{2k} 
\]  \hspace{1cm} (n \in \mathbb{N});
\]
\[
\zeta(2n + 1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{2^{4n+3} + 2^{n+2} - 3^{2n+2} - 1} \left[ H_{2n} - \log \left( \frac{2\pi}{128} \right) \right] (2n)!
\]
\[
+ \sum_{k=1}^{n-1} \frac{(-1)^k (4^{2k+1} - 3^{2k+1})}{(2n - 2k)!} \frac{\zeta(2k + 1)}{(2\pi)^{2k}}
\]
\[
- 24 \sum_{k=1}^{\infty} \frac{(2k - 1)! (4^{2k-1} - 3^{2k-1}) \zeta(2k)}{(2n + 2k)!} 12^{2k} 
\]  \hspace{1cm} (n \in \mathbb{N}),
\]
and
\[
\zeta(2n + 1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{3^{2n+1}(2^{2n+1} - 2^{n+2} + 2^{2n} - 1)} \left[ H_{2n} - \log \left( \frac{4\pi}{27} \right) \right] (2n)!
\]
\[
+ \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1} - 2^{2k+1})}{(2n - 2k)!} \frac{\zeta(2k + 1)}{\pi^{2k}}
\]
\[
- 12 \sum_{k=1}^{\infty} \frac{(2k - 1)! (3^{2k-1} - 2^{2k-1}) \zeta(2k)}{(2n + 2k)!} 12^{2k} 
\]  \hspace{1cm} (n \in \mathbb{N}).
\]

Next we turn to the identity (1.7). By setting \( t = 1/m \) and differentiating both sides with respect to \( s \), we find from (1.7) that
\[
\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)! m^{2k}} \left[ \zeta'(s + 2k + 1, a) + \zeta(s + 2k + 1, a) \sum_{j=0}^{2k} \frac{1}{s+j} \right]
\]
\[
= \frac{m}{2} \frac{d}{ds} \left\{ \zeta \left( s, a - \frac{1}{m} \right) - \zeta \left( s, a + \frac{1}{m} \right) \right\} \hspace{1cm} (m \in \mathbb{N} \setminus \{1\}),
\]
where we have made use of the derivative formula (2.5). In particular, when \( m = 2 \), (3.14) immediately yields
\[
\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)! 2^{2k}} \left[ \zeta'(s + 2k + 1, a) + \zeta(s + 2k + 1, a) \sum_{j=0}^{2k} \frac{1}{s+j} \right]
\]
\[
= - \left( a - \frac{1}{2} \right)^{-s} \log \left( a - \frac{1}{2} \right). 
\]

By letting \( s \to -2n - 1 \ (n \in \mathbb{N}) \) in the further special of this last identity (3.15) when \( a = 1 \), Wilton \([28, p. 92]\) obtained the following series representation for
\[ \zeta(2n+1) \text{ (see also Hansen [16, p. 357, Entry (54.6.9)]):} \]
\[ \zeta(2n+1) = (-1)^{n-1} \pi^{2n} \left[ H_{2n+1} - \log \pi \frac{1}{(2n+1)!} \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k+1)!} \frac{\zeta(2k+1)}{\pi^{2k}} \right] \]
\[ + 2 \sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k+1)!} \frac{\zeta(2k)}{2^{2k}} \]  
(n \in \mathbb{N}),  
(3.16)

which may be compared with our first series representation (2.9). As a matter of fact, since
\[ \frac{(2k)!}{(2n+2k+1)!} = \frac{(2k-1)!}{(2n+2k-1)!} - 2^{-n} \frac{(2k-1)!}{(2n+2k)!} \]  
(n, k \in \mathbb{N}),  
(3.17)

it is not difficult to deduce from (2.9) and (3.16) (with \( n \) replaced by \( n-1 \)) that
\[ \zeta(2n+1) = (-1)^n \frac{(2\pi)^{2n}}{n(2n+1)!} \left[ \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k)!} \frac{\zeta(2k)}{\pi^{2k}} \right] \]
\[ + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k)!} \frac{\zeta(2k)}{2^{2k}} \]  
(n \in \mathbb{N}),  
(3.18)

which is precisely the aforementioned main result of Cvijović and Klinowski [8, p. 1265, Theorem A] (see also Zhang and Williams [29, p. 1591, Equation (3.16)] where an obviously more complicated version of (3.18) was proven by applying the same identity (1.14) above).

Observing also that
\[ \frac{(2k)!}{(2n+2k+1)!} = \frac{(2k-1)!}{(2n+2k-1)!} - 2^{-n} \frac{(2k-1)!}{(2n+2k)!} \]  
(n, k \in \mathbb{N}),  
(3.19)

we obtain yet another series representation for \( \zeta(2n+1) \) by applying (2.9) and (3.16):
\[ \zeta(2n+1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n-1)!} \left[ \sum_{k=1}^{n-1} \frac{(-1)^{k-1} k}{(2n-2k)!} \frac{\zeta(2k)}{\pi^{2k}} \right] \]
\[ + \sum_{k=0}^{\infty} \frac{(2k)!}{(2n+2k)!} \frac{\zeta(2k)}{2^{2k}} \]  
(n \in \mathbb{N}),  
(3.20)

which provides a significantly simpler (and much more rapidly convergent) version of the other main result of Cvijović and Klinowski [8, p. 1265, Theorem B]:
\[ \zeta(2n+1) = (-1)^n \frac{2(2\pi)^{2n}}{(2n)!} \sum_{k=0}^{\infty} \Omega_{n,k} \frac{\zeta(2k)}{2^{2k}} \]  
(n \in \mathbb{N}),  
(3.21)

where the coefficients \( \Omega_{n,k} \) are given explicitly by
\[ \Omega_{n,k} := \sum_{j=0}^{2n} \binom{2n}{j} \frac{B_{2n-j}}{j+2k+1)(j+1)!2^j} \]  
(n \in \mathbb{N}; \ k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}),  
(3.22)

in terms of the Bernoulli numbers defined by (3.2). Since [20, pp. 27 and 28]
\[ B_1 = -\frac{1}{2} \text{ and } B_{2n+1} = 0 \]  
(n \in \mathbb{N}),
the definition (3.22) can be rewritten at once in the form:

\[
\Omega_{n,k} = \sum_{j=0}^{n} \frac{(2n)\binom{2n-2j}{2j}}{(2j+2k+1)(2j+1)2^{2j}} - \frac{1}{(2n+2k)2^{2n}} \quad (n \in \mathbb{N}; \ k \in \mathbb{N}_0),
\]

(3.23)

or, equivalently,

\[
\Omega_{n,k} = (-1)^{n-1} \frac{2(2n)!}{(2\pi)^{2n}} \sum_{j=0}^{n} \frac{(-\pi^2)^j \zeta(2n-2j)}{(2j+1)! (2j+2k+1)} - \frac{1}{(2n+2k)2^{2n}} \quad (n \in \mathbb{N}; \ k \in \mathbb{N}_0),
\]

(3.24)

by virtue of the relationship (3.5). Combining the partial fractions occurring in (3.23) or (3.24), it is easily seen that

\[
\Omega_{n,k} = \prod_{\ell=0}^{n} \left\{ \frac{(2k+2\ell+1)^{-1}}{(2n+2k)2^{2n}} \right\} \left[ \sum_{j=0}^{n} \frac{(2n)\binom{2n+2k}{2j}}{(2j+1)2^{2n-2j}} B_{2n-2j} \right.
\]

\[
\cdot \prod_{\ell=0}^{n} \left( 2k+2\ell+1 \right) - \prod_{\ell=0}^{n} \left( 2k+2\ell+1 \right) \quad (n \in \mathbb{N}; \ k \in \mathbb{N}_0).
\]

(3.25)

In view of the identity:

\[
\sum_{j=0}^{n} \frac{(2n)\binom{2n-2j}{2j}}{(2j+1)2^{2j+1}} = 1 = \sum_{j=0}^{n} \frac{(2n)\binom{2n}{2j}}{2^{2n-2j}} B_{2n-2j} \cdot \prod_{\ell=0}^{n} \left( 2k+2\ell+1 \right) - \prod_{\ell=0}^{n} \left( 2k+2\ell+1 \right),
\]

(3.26)

which is due essentially to Euler (cf., e.g., Riordan [23, p. 123, Problem 12]), the expression inside brackets in (3.25) is a polynomial in \( k \) of degree \( n \) (not \( n+1 \)), and therefore

\[
\Omega_{n,k} = O(k^{-2}) \quad (k \to \infty; \ n \in \mathbb{N}).
\]

(3.27)

It follows from (3.27) that the general term in (3.21) has the order estimate:

\[
O(2^{-2k} \cdot k^{-2}) \quad (k \to \infty),
\]

(3.28)

whereas the general term in our series representation (3.20) has precisely the same order estimate:

\[
O(2^{-2k} \cdot k^{-2n-1}) \quad (k \to \infty; \ n \in \mathbb{N}),
\]

(3.29)

as that in (2.9). Thus, even in the special case when \( n = 1 \), the series representing \( \zeta(3) \) converges faster in (3.20) than in (3.21).

Various known series representations for \( \zeta(2n+1) (n \in \mathbb{N}) \) of other types include those given (for example) by Ramanujan [21] (see also Berndt [3]), Glaisher [13] (see also Hansen [16, p. 359]), Koshliakov [18], Leshchiner [19], Grosswald ([14] and [15]), Terras [25], Cohen [7], Butzer et al. ([5] and [6]), Dąbrowski [9], and others (see, e.g., Berndt [4, pp. 275 and 276]).

We conclude this paper by remarking that a particular case of our series representation (2.12) when \( n = 1 \) was proven, by an entirely different method, by Zhang.
and Williams [30, p. 707, Theorem 9]. Furthermore, the following particular case of (3.18) when \( n = 1 \):

\[
\zeta(3) = -\frac{4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)} 2^{2k},
\]

which is contained in a 1772 paper entitled Exercitationes Analyticae by Euler (see, e.g., Ayoub [2, pp. 1084–1085]), was rediscovered by Ramaswami [22] and (more recently) by Ewell [10]. In fact, Euler’s formula (3.30) was reproduced by Srivastava [24, p. 7, Equation (2.23)] from the work of Ramaswami [22]. In the current mathematical literature, however, Euler’s formula (3.30) is being attributed to Ewell [10].

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**References**

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