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SOME RAPIDLY CONVERGING SERIES FOR $\zeta(2n+1)$

H. M. SRIVASTAVA

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ABSTRACT. For a natural number n, the author derives several families of series representations for the Riemann Zeta function $\zeta(2n+1)$. Each of these series representing $\zeta(2n+1)$ converges remarkably rapidly with its general term having the order estimate:

$$O(k^{-2n-1} \cdot m^{-2k}) \qquad (k \to \infty; \quad m = 2, 3, 4, 6).$$

Relevant connections of the results presented here with many other known series representations for $\zeta(2n+1)$ are also pointed out.

1. INTRODUCTION AND PRELIMINARIES

The Riemann Zeta function $\zeta(s)$ and the (Hurwitz's) generalized Zeta function $\zeta(s,a)$, defined usually by (see, *e.g.*, Titchmarsh [26])

$$\zeta(s) := \begin{cases} \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1-2^{-s}} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^s} & (\Re(s) > 1), \\ \displaystyle\frac{1}{1-2^{1-s}} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} & (\Re(s) > 0; \quad s \neq 1) \end{cases} \tag{1.1}$$

and

$$\zeta(s,a) := \sum_{n=0}^{\infty} \frac{1}{(n+a)^s} \qquad (\Re(s) > 1; \quad a \neq 0, -1, -2, \cdots), \tag{1.2}$$

so that

$$\zeta(s,1) = \zeta(s) \quad \text{and} \quad \zeta(s,2) = \zeta(s) - 1, \tag{1.3}$$

are known to be meromorphic (that is, analytic everywhere in the complex s-plane except for a simple pole at $s = 1$ with residue 1). Making use of the binomial theorem *and* the Pochhammer symbol $(\lambda)_n$ defined by

$$(\lambda)_n := \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0), \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (n \in \mathbb{N} := \{1, 2, 3, \cdots\}), \end{cases} \tag{1.4}$$

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it is easily seen from the definition (1.2) that

$$\sum_{k=0}^{\infty} \frac{(s)_k}{k!}\, \zeta(s+k,a)\, t^k = \zeta(s,\, a-t) \qquad (|t| < |a|)\,, \tag{1.5}$$

which immediately yields the familiar identities (*cf.*, *e.g.*, Hansen [16, p. 359] where other references are also cited):

$$\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!}\, \zeta(s+2k,a)\, t^{2k} = \frac{1}{2}\, [\zeta(s,a-t) + \zeta(s,a+t)] \qquad (|t| < |a|) \tag{1.6}$$

and

$$\sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!}\, \zeta(s+2k+1,a)\, t^{2k+1} = \frac{1}{2}\, [\zeta(s,a-t) - \zeta(s,a+t)] \qquad (|t| < |a|) \tag{1.7}$$

or, equivalently,

$$\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k+1)!}\, \zeta(s+2k,a)\, t^{2k+1} = \frac{1}{2(s-1)}\, [\zeta(s-1,\, a-t) - \zeta(s-1,\, a+t)] \qquad (|t| < |a|)\,. \tag{1.8}$$

Since

$$\zeta(s) = \frac{1}{m^s - 1} \sum_{j=1}^{m-1} \zeta\left(s,\, \frac{j}{m}\right) \qquad (m \in \mathbb{N} \setminus \{1\}), \tag{1.9}$$

which follows readily from the definitions (1.1) and (1.2), the special case of the identity (1.6) when $a = 1$ and $t = 1/m$ can be rewritten in the form:

$$\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!}\, \frac{\zeta(s+2k)}{m^{2k}} = \begin{cases} (2^s - 1)\,\zeta(s) - 2^{s-1} & (m = 2), \\ \dfrac{1}{2}\left[(m^s - 1)\zeta(s) - m^s - \displaystyle\sum_{j=2}^{m-2} \zeta\left(s,\, \frac{j}{m}\right)\right] & (m \in \mathbb{N} \setminus \{1, 2\}), \end{cases} \tag{1.10}$$

where (and *throughout this paper*) an empty sum is to be interpreted as nil. In addition to the case $m = 2$, the formula (1.10) simplifies also in the cases when $m = 3$, 4, and 6, and we thus obtain the identities:

$$\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!}\, \frac{\zeta(s+2k)}{3^{2k}} = \frac{1}{2}\, [(3^s - 1)\,\zeta(s) - 3^s]\,, \tag{1.11}$$

$$\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!}\, \frac{\zeta(s+2k)}{4^{2k}} = \frac{1}{2}\, [(4^s - 2^s)\,\zeta(s) - 4^s]\,, \tag{1.12}$$

and

$$\sum_{k=0}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{6^{2k}} = \frac{1}{2}\left[(6^s - 3^s - 2^s + 1)\,\zeta(s) - 6^s\right], \tag{1.13}$$

respectively.

Identities of this kind seem to have first appeared in the work of Ramaswami [22], who actually proved the cases $m = 2$, 3, and 6 of the general result in (1.10). Each of these three identities of Ramaswami [22] can also be found in the work of Hansen [16, p. 357], who referred to Apostol [1] as his source for the identities (1.11) and (1.13) only. As a matter of fact, Apostol [1] reproduced the identities (1.11) and (1.13) from Ramaswami's work [22] and then proved an interesting *arithmetical* generalization of these identities (see also Klusch [17, p. 520]).

In its slightly variant form:

$$\sum_{k=1}^{\infty} \frac{(s+1)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{2^{2k}} = (2^s - 2)\,\zeta(s), \tag{1.14}$$

which can indeed be proven *directly* from the known special cases of (1.6) and (1.7) when $a = 1$ and $t = \frac{1}{2}$, the case $m = 2$ of the general result (1.10) was applied by Zhang and Williams [29] (and, more recently, by Cvijović and Klinowski [8]) with a view to finding two seemingly different series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$). The main object of this paper is to obtain much more rapidly converging series representations for $\zeta(2n+1)$ ($n \in \mathbb{N}$) chiefly by appealing appropriately to each of the aforementioned cases ($m = 2$, 3, 4, and 6) of the general result (1.10).

The following properties of the Riemann ζ-function will be required in our investigation:

$$\zeta(0) = -\frac{1}{2}; \quad \zeta(-2n) = 0 \quad (n \in \mathbb{N}); \quad \zeta'(0) = -\frac{1}{2}\log(2\pi),$$

and (in general)

$$\zeta'(-2n) = \lim_{\epsilon \to 0} \frac{\zeta(-2n+\epsilon)}{\epsilon} = \frac{(-1)^n}{2(2\pi)^{2n}}\,(2n)!\,\zeta(2n+1) \qquad (n \in \mathbb{N}), \tag{1.15}$$

where use is made of the familiar functional equation:

$$2^s\,\Gamma(1-s)\,\zeta(1-s)\sin\left(\frac{1}{2}\pi s\right) = \pi^{1-s}\,\zeta(s). \tag{1.16}$$

Furthermore, by l'Hôpital's rule, we have

$$\lim_{s \to -2n} \left\{\frac{\sin\left(\frac{1}{2}\pi s\right)}{s+2n}\right\} = (-1)^n \frac{\pi}{2} \qquad (n \in \mathbb{N}) \tag{1.17}$$

and

$$\lim_{s \to -2n} \left\{\frac{\zeta(s+2k)}{s+2n}\right\} = \frac{(-1)^{n-k}}{2(2\pi)^{2(n-k)}}\,(2n-2k)!\,\zeta(2n-2k+1) \quad (k = 1, \cdots, n-1; \quad n \in \mathbb{N} \setminus \{1\}). \tag{1.18}$$

2. A set of series representations

We begin with the case $m = 2$ of the general result (1.10). Upon separating the first $n+1$ terms of the series occurring on the left-hand side, if we transpose the terms for $k = 0$ and $k = n$ to the right-hand side, we obtain

$$\sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{2^{2k}} + \sum_{k=n+1}^{\infty} \frac{(s)_{2k}}{(2k)!} \frac{\zeta(s+2k)}{2^{2k}}$$
$$= (2^s - 2)\zeta(s) - 2^{s-1} - \frac{(s)_{2n}}{(2n)!} \frac{\zeta(s+2n)}{2^{2n}}, \tag{2.1}$$

which readily yields the identity:

$$\sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} 2^{2(n-k)} \zeta(s+2k) + \sum_{k=1}^{\infty} \frac{(s)_{2n+2k}}{(2n+2k)!} \frac{\zeta(s+2n+2k)}{2^{2k}}$$
$$= 2^{2n}(2^s - 2)\,\zeta(s) - 2^{s+2n-1} - \frac{(s)_{2n}}{(2n)!}\,\zeta(s+2n) \qquad (n \in \mathbb{N}), \tag{2.2}$$

it being understood, as before, that an empty sum is to be interpreted as nil.

Now we apply the functional equation (1.16) in the first term on the right-hand side of (2.2) and divide both sides by $s + 2n$. We thus find that

$$\sum_{k=1}^{n-1} \frac{(s)_{2k}}{(2k)!} 2^{2(n-k)} \left\{ \frac{\zeta(s+2k)}{s+2n} \right\} + \sum_{k=1}^{\infty} \frac{(s)_{2n}(s+2n+1)_{2k-1}}{(2n+2k)!} \frac{\zeta(s+2n+2k)}{2^{2k}}$$
$$= 2^{s+2n}(2^s-2)\,\pi^{s-1}\,\Gamma(1-s)\,\zeta(1-s) \left\{ \frac{\sin\left(\frac{1}{2}\pi s\right)}{s+2n} \right\}$$
$$- \left\{ \frac{2^{s+2n-1} + \frac{(s)_{2n}}{(2n)!}\,\zeta(s+2n)}{s+2n} \right\} \qquad (s \neq -2n;\;\; n \in \mathbb{N}). \tag{2.3}$$

Since

$$(-n)_k = (-1)^k \frac{n!}{(n-k)!} \qquad (k = 0, 1, \cdots, n; \quad n \in \mathbb{N}),$$

so that, obviously,

$$(-n)_n = (-1)^n\; n! \qquad (n \in \mathbb{N}), \tag{2.4}$$

in view of the definition (1.4), it is easily seen by logarithmic differentiation that

$$\frac{d}{ds}\;\{(s)_n\} = (s)_n \sum_{j=0}^{n-1} \frac{1}{s+j} \qquad (n \in \mathbb{N}), \tag{2.5}$$

so that

$$\frac{d}{ds}\;\{(s)_{2n}\}\bigg|_{s=-2n} = -(2n)!\;\, H_{2n} \qquad (n \in \mathbb{N}), \tag{2.6}$$

where H_n denotes the familiar harmonic numbers defined by

$$H_n := \sum_{j=1}^{n} \frac{1}{j} \qquad (n \in \mathbb{N}). \tag{2.7}$$

We observe also that the limit formula (1.18) is needed in the first sum on the left-hand side of (2.3) only when this sum is nonzero (that is, only when $n \in \mathbb{N} \setminus \{1\}$). Furthermore, by l'Hôpital's rule, we have

$$\begin{aligned}
&\lim_{s\to -2n} \left\{ \frac{2^{s+2n-1} + \frac{(s)_{2n}}{(2n)!}\,\zeta(s+2n)}{s+2n} \right\} \\
&\quad = \left[2^{s+2n-1}\log 2 + \frac{d}{ds}\{(s)_{2n}\} \cdot \frac{\zeta(s+2n)}{(2n)!} + \frac{(s)_{2n}}{(2n)!}\,\zeta'(s+2n) \right]\Bigg|_{s=-2n} \\
&\quad = \frac{1}{2}\left(H_{2n} - \log \pi\right) \qquad (n \in \mathbb{N}).
\end{aligned} \tag{2.8}$$

Finally, letting $s \to -2n$ in (2.3), and making use of the limit relationships (1.17), (1.18), and (2.8), we obtain our first series representation for $\zeta(2n+1)$:

$$\begin{aligned}
\zeta(2n+1) = (-1)^{n-1}\,\frac{(2\pi)^{2n}}{2^{2n+1}-1} &\left[\frac{H_{2n} - \log \pi}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\pi^{2k}} \right. \\
&\left. + 2\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!}\,\frac{\zeta(2k)}{2^{2k}} \right] \qquad (n \in \mathbb{N}).
\end{aligned} \tag{2.9}$$

In precisely the same manner, we can apply the identities (1.11), (1.12), and (1.13) in order to prove the following *additional* series representations for $\zeta(2n+1)$:

$$\begin{aligned}
\zeta(2n+1) = (-1)^{n-1}\,\frac{2(2\pi)^{2n}}{3^{2n+1}-1} &\left[\frac{H_{2n} - \log\left(\frac{2}{3}\pi\right)}{(2n)!} + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\left(\frac{2}{3}\pi\right)^{2k}} \right. \\
&\left. + 2\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!}\,\frac{\zeta(2k)}{3^{2k}} \right] \qquad (n \in \mathbb{N});
\end{aligned} \tag{2.10}$$

$$\begin{aligned}
\zeta(2n+1) = (-1)^{n-1}\,\frac{2(2\pi)^{2n}}{2^{4n+1}+2^{2n}-1} &\left[\frac{H_{2n} - \log\left(\frac{1}{2}\pi\right)}{(2n)!} \right. \\
&\left. + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\left(\frac{1}{2}\pi\right)^{2k}} + 2\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!}\,\frac{\zeta(2k)}{4^{2k}} \right] \qquad (n \in \mathbb{N});
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
\zeta(2n+1) = (-1)^{n-1}\,\frac{2(2\pi)^{2n}}{3^{2n}(2^{2n}+1)+2^{2n}-1} &\left[\frac{H_{2n} - \log\left(\frac{1}{3}\pi\right)}{(2n)!} \right. \\
&\left. + \sum_{k=1}^{n-1} \frac{(-1)^k}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\left(\frac{1}{3}\pi\right)^{2k}} + 2\sum_{k=1}^{\infty} \frac{(2k-1)!}{(2n+2k)!}\,\frac{\zeta(2k)}{6^{2k}} \right] \qquad (n \in \mathbb{N}).
\end{aligned} \tag{2.12}$$

3. Remarks and observations

Our series representation (2.9) is markedly different from each of the series representations for $\zeta(2n+1)$, which were given earlier by Zhang and Williams [29, p. 1591, Equation (3.16)] and (more recently) by Cvijović and Klinowski [8, p. 1265, Theorem A]. Since $\zeta(2k) \to 1$ as $k \to \infty$, the general term in our series representation (2.9) has the order estimate:

$$O\left(2^{-2k} \cdot k^{-2n-1}\right) \qquad (k \to \infty; \quad n \in \mathbb{N}),$$

whereas the general term in each of these earlier series representations has the order estimate:

$$O\left(2^{-2k} \cdot k^{-2n}\right) \qquad (k \to \infty; \quad n \in \mathbb{N}).$$

By suitably combining (2.9) and (2.11), it is fairly straightforward to obtain the series representation:

$$\begin{aligned}\zeta(2n+1) = (-1)^{n-1} \frac{2(2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)} \Bigg[& \frac{\log 2}{(2n)!} \\ & + \sum_{k=1}^{n-1} \frac{(-1)^k (2^{2k}-1)}{(2n-2k)!} \frac{\zeta(2k+1)}{\pi^{2k}} \\ & -2 \sum_{k=1}^{\infty} \frac{(2k-1)!\,(2^{2k}-1)}{(2n+2k)!} \frac{\zeta(2k)}{2^{4k}} \Bigg] \qquad (n \in \mathbb{N}).\end{aligned} \tag{3.1}$$

Now, in terms of the Bernoulli numbers B_n and the Euler polynomials $E_n(x)$ defined by the generating functions:

$$\frac{z}{e^z-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!} \qquad (|z| < 2\pi) \tag{3.2}$$

and

$$\frac{2e^{xz}}{e^z+1} = \sum_{n=0}^{\infty} E_n(x) \frac{z^n}{n!} \qquad (|z| < \pi), \tag{3.3}$$

respectively, it is known that (*cf.*, *e.g.*, Magnus *et al.* [20, p. 29])

$$E_n(0) = (-1)^n E_n(1) = \frac{2(1-2^{n+1})}{n+1} B_{n+1} \qquad (n \in \mathbb{N}) \tag{3.4}$$

and [20, p. 19]

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n} \qquad (n \in \mathbb{N}), \tag{3.5}$$

which, together, imply that

$$E_{2n-1}(0) = \frac{4(-1)^n}{(2\pi)^{2n}} (2n-1)!\,(2^{2n}-1)\,\zeta(2n) \qquad (n \in \mathbb{N}). \tag{3.6}$$

Making use of this last relationship (3.6), the series representation (3.1) can immediately be put in the form:

$$\zeta(2n+1) = (-1)^{n-1}\frac{2(2\pi)^{2n}}{(2^{2n}-1)(2^{2n+1}-1)}\left[\frac{\log 2}{(2n)!} + \sum_{k=1}^{n-1}\frac{(-1)^k(2^{2k}-1)}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\pi^{2k}} + \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2n+2k)!}\left(\frac{\pi}{2}\right)^{2k} E_{2k-1}(0)\right] \qquad (n\in\mathbb{N}), \tag{3.7}$$

which is a slightly modified (and corrected) version of a result proven in a significantly different way by Tsumura [27, p. 383, Theorem B].

Another interesting combination of our series representations (2.9) and (2.11) leads us to the following variant of Tsumura's result (3.1) or (3.7):

$$\zeta(2n+1) = (-1)^{n-1}\frac{\pi^{2n}}{2^{2n+1}-1}\left[\frac{H_{2n}-\log\left(\frac{1}{4}\pi\right)}{(2n)!} + \sum_{k=1}^{n-1}\frac{(-1)^k(2^{2k+1}-1)}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\pi^{2k}} - 4\sum_{k=1}^{\infty}\frac{(2k-1)!\,(2^{2k-1}-1)}{(2n+2k)!}\,\frac{\zeta(2k)}{2^{4k}}\right] \qquad (n\in\mathbb{N}), \tag{3.8}$$

which is essentially the same as the *determinantal* expression for $\zeta(2n+1)$ derived recently by Ewell [12, p. 1010, Corollary 3] by employing an entirely different technique from ours.

Other similar combinations of our series representations (2.9) to (2.12) would yield the following (presumably new) companions of Ewell's result (3.8):

$$\zeta(2n+1) = (-1)^{n-1}\frac{2(2\pi)^{2n}}{(2^{2n+1}-1)(3^{2n}+1)}\left[\frac{H_{2n}-\log\left(\frac{1}{6}\pi\right)}{(2n)!} + \sum_{k=1}^{n-1}\frac{(-1)^k(2^{2k+1}-1)}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\left(\frac{2}{3}\pi\right)^{2k}} - 4\sum_{k=1}^{\infty}\frac{(2k-1)!(2^{2k-1}-1)}{(2n+2k)!}\,\frac{\zeta(2k)}{6^{2k}}\right] \qquad (n\in\mathbb{N}); \tag{3.9}$$

$$\zeta(2n+1) = (-1)^{n-1}\frac{2(2\pi)^{2n}}{(2^{2n}+1)(3^{2n+1}-1)}\left[\frac{2H_{2n}-\log\left(\frac{\pi^2}{27}\right)}{(2n)!} + \sum_{k=1}^{n-1}\frac{(-1)^k(3^{2k+1}-1)}{(2n-2k)!}\,\frac{\zeta(2k+1)}{\pi^{2k}} - 6\sum_{k=1}^{\infty}\frac{(2k-1)!\,(3^{2k-1}-1)}{(2n+2k)!}\,\frac{\zeta(2k)}{6^{2k}}\right] \qquad (n\in\mathbb{N}); \tag{3.10}$$

$$
\begin{aligned}
\zeta(2n+1) = (-1)^{n-1}\, \frac{2(2\pi)^{2n}}{3^{2n+2} - 2^{2n+3} + 1} & \left[\frac{H_{2n} - \log\left(\frac{8\pi}{27}\right)}{(2n)!} \right. \\
& + \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1} - 2^{2k+1})}{(2n-2k)!}\, \frac{\zeta(2k+1)}{(2\pi)^{2k}} \\
& \left. - 12 \sum_{k=1}^{\infty} \frac{(2k-1)!\,(3^{2k-1} - 2^{2k-1})}{(2n+2k)!}\, \frac{\zeta(2k)}{6^{2k}} \right] \qquad (n \in \mathbb{N}); \qquad (3.11)
\end{aligned}
$$

$$
\begin{aligned}
\zeta(2n+1) = (-1)^{n-1}\, \frac{2(2\pi)^{2n}}{2^{4n+3} + 2^{2n+2} - 3^{2n+2} - 1} & \left[\frac{H_{2n} - \log\left(\frac{27\pi}{128}\right)}{(2n)!} \right. \\
& + \sum_{k=1}^{n-1} \frac{(-1)^k (4^{2k+1} - 3^{2k+1})}{(2n-2k)!}\, \frac{\zeta(2k+1)}{(2\pi)^{2k}} \\
& \left. - 24 \sum_{k=1}^{\infty} \frac{(2k-1)!\,(4^{2k-1} - 3^{2k-1})}{(2n+2k)!}\, \frac{\zeta(2k)}{12^{2k}} \right] \qquad (n \in \mathbb{N}), \qquad (3.12)
\end{aligned}
$$

and

$$
\begin{aligned}
\zeta(2n+1) = (-1)^{n-1}\, \frac{2(2\pi)^{2n}}{3^{2n+1}(2^{2n}+1) - 2^{4n+2} + 2^{2n} - 1} & \left[\frac{H_{2n} - \log\left(\frac{4\pi}{27}\right)}{(2n)!} \right. \\
& + \sum_{k=1}^{n-1} \frac{(-1)^k (3^{2k+1} - 2^{2k+1})}{(2n-2k)!}\, \frac{\zeta(2k+1)}{\pi^{2k}} \\
& \left. - 12 \sum_{k=1}^{\infty} \frac{(2k-1)!\,(3^{2k-1} - 2^{2k-1})}{(2n+2k)!}\, \frac{\zeta(2k)}{12^{2k}} \right] \qquad (n \in \mathbb{N}). \qquad (3.13)
\end{aligned}
$$

Next we turn to the identity (1.7). By setting $t = 1/m$ and differentiating both sides with respect to s, we find from (1.7) that

$$
\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!\, m^{2k}} \left[\zeta'(s+2k+1, a) + \zeta(s+2k+1, a) \sum_{j=0}^{2k} \frac{1}{s+j} \right] \\
& \qquad = \frac{m}{2}\, \frac{d}{ds} \left\{ \zeta\left(s,\, a - \frac{1}{m}\right) - \zeta\left(s,\, a + \frac{1}{m}\right) \right\} \qquad (m \in \mathbb{N} \setminus \{1\}), \qquad (3.14)
\end{aligned}
$$

where we have made use of the derivative formula (2.5). In particular, when $m = 2$, (3.14) immediately yields

$$
\begin{aligned}
& \sum_{k=0}^{\infty} \frac{(s)_{2k+1}}{(2k+1)!\; 2^{2k}} \left[\zeta'(s+2k+1,\, a) + \zeta(s+2k+1,\, a) \sum_{j=0}^{2k} \frac{1}{s+j} \right] \\
& \qquad = -\left(a - \frac{1}{2}\right)^{-s} \log\left(a - \frac{1}{2}\right). \qquad (3.15)
\end{aligned}
$$

By letting $s \to -2n-1$ $(n \in \mathbb{N})$ in the *further* special of this last identity (3.15) when $a = 1$, Wilton [28, p. 92] obtained the following series representation for

$\zeta(2n+1)$ (see also Hansen [16, p. 357, Entry (54.6.9)]):

$$\zeta(2n+1) = (-1)^{n-1}\,\pi^{2n}\left[\frac{H_{2n+1}-\log\pi}{(2n+1)!} + \sum_{k=1}^{n-1}\frac{(-1)^k}{(2n-2k+1)!}\,\frac{\zeta(2k+1)}{\pi^{2k}}\right.$$

$$\left. + 2\sum_{k=1}^{\infty}\frac{(2k-1)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{2^{2k}}\right] \qquad (n\in\mathbb{N}), \tag{3.16}$$

which may be compared with our first series representation (2.9). As a matter of fact, since

$$\frac{(2k)!}{(2n+2k)!} = \frac{(2k-1)!}{(2n+2k-1)!} - 2n\,\frac{(2k-1)!}{(2n+2k)!} \qquad (n,k\in\mathbb{N}), \tag{3.17}$$

it is not difficult to deduce from (2.9) and (3.16) (with n replaced by $n-1$) that

$$\zeta(2n+1) = (-1)^n\,\frac{(2\pi)^{2n}}{n(2^{2n+1}-1)}\left[\sum_{k=1}^{n-1}\frac{(-1)^{k-1}\,k}{(2n-2k)!}\,\frac{\zeta(2k)}{\pi^{2k}}\right.$$

$$\left. + \sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k)!}\,\frac{\zeta(2k)}{2^{2k}}\right] \qquad (n\in\mathbb{N}), \tag{3.18}$$

which is precisely the aforementioned *main* result of Cvijović and Klinowski [8, p. 1265, Theorem A] (see also Zhang and Williams [29, p. 1591, Equation (3.16)] where an obviously more complicated version of (3.18) was proven by applying the same identity (1.14) above).

Observing also that

$$\frac{(2k)!}{(2n+2k+1)!} = \frac{(2k-1)!}{(2n+2k)!} - (2n+1)\,\frac{(2k-1)!}{(2n+2k+1)!} \qquad (n,k\in\mathbb{N}), \tag{3.19}$$

we obtain yet another series representation for $\zeta(2n+1)$ by applying (2.9) and (3.16):

$$\zeta(2n+1) = (-1)^n\,\frac{2(2\pi)^{2n}}{(2n-1)\,2^{2n}+1}\left[\sum_{k=1}^{n-1}\frac{(-1)^{k-1}\,k}{(2n-2k+1)!}\,\frac{\zeta(2k+1)}{\pi^{2k}}\right.$$

$$\left. + \sum_{k=0}^{\infty}\frac{(2k)!}{(2n+2k+1)!}\,\frac{\zeta(2k)}{2^{2k}}\right] \qquad (n\in\mathbb{N}), \tag{3.20}$$

which provides a significantly simpler (and much more rapidly convergent) version of the other *main* result of Cvijović and Klinowski [8, p. 1265, Theorem B]:

$$\zeta(2n+1) = (-1)^n\,\frac{2(2\pi)^{2n}}{(2n)!}\sum_{k=0}^{\infty}\Omega_{n,k}\,\frac{\zeta(2k)}{2^{2k}} \qquad (n\in\mathbb{N}), \tag{3.21}$$

where the coefficients $\Omega_{n,k}$ are given explicitly by

$$\Omega_{n,k} := \sum_{j=0}^{2n}\binom{2n}{j}\frac{B_{2n-j}}{(j+2k+1)(j+1)\,2^j} \qquad \left(n\in\mathbb{N};\ \ k\in\mathbb{N}_0 := \mathbb{N}\cup\{0\}\right), \tag{3.22}$$

in terms of the Bernoulli numbers defined by (3.2). Since [20, pp. 27 and 28]

$$B_1 = -\frac{1}{2} \quad \text{and} \quad B_{2n+1} = 0 \qquad (n\in\mathbb{N}),$$

the definition (3.22) can be rewritten at once in the form:

$$\Omega_{n,k} = \sum_{j=0}^{n} \binom{2n}{2j} \frac{B_{2n-2j}}{(2j+2k+1)(2j+1)\,2^{2j}} - \frac{1}{(2n+2k)\,2^{2n}} \qquad (n \in \mathbb{N};\ \ k \in \mathbb{N}_0)\,, \tag{3.23}$$

or, equivalently,

$$\begin{aligned}\Omega_{n,k} = (-1)^{n-1}\,\frac{2(2n)!}{(2\pi)^{2n}} \sum_{j=0}^{n} \frac{(-\pi^2)^j\,\zeta(2n-2j)}{(2j+1)!(2j+2k+1)} \\ - \frac{1}{(2n+2k)\,2^{2n}} \qquad (n \in \mathbb{N};\ \ k \in \mathbb{N}_0),\end{aligned} \tag{3.24}$$

by virtue of the relationship (3.5). Combining the partial fractions occurring in (3.23) or (3.24), it is easily seen that

$$\begin{aligned}\Omega_{n,k} = \frac{\prod_{\ell=0}^{n}\left\{(2k+2\ell+1)^{-1}\right\}}{(2n+2k)\,2^{2n}} \Bigg[\sum_{j=0}^{n} \binom{2n}{2j} \frac{2n+2k}{2j+1}\, 2^{2n-2j}\, B_{2n-2j} \\ \cdot \prod_{\substack{\ell=0 \\ (\ell \neq j)}}^{n} (2k+2\ell+1) - \prod_{\ell=0}^{n}(2k+2\ell+1)\Bigg] \qquad (n \in \mathbb{N};\ \ k \in \mathbb{N}_0).\end{aligned} \tag{3.25}$$

In view of the identity:

$$\sum_{j=0}^{n} \binom{2n}{2j} \frac{2^{2n-2j}\,B_{2n-2j}}{2j+1} = 1 = \sum_{j=0}^{n} \binom{2n}{2j} \frac{2^{2j}\,B_{2j}}{2n-2j+1}, \tag{3.26}$$

which is due essentially to Euler (*cf.*, *e.g.*, Riordan [23, p. 123, Problem 12]), the expression inside brackets in (3.25) is a polynomial in k of degree n (not $n+1$), and therefore

$$\Omega_{n,k} = O(k^{-2}) \qquad (k \to \infty;\ \ n \in \mathbb{N}). \tag{3.27}$$

It follows from (3.27) that the general term in (3.21) has the order estimate:

$$O(2^{-2k} \cdot k^{-2}) \qquad (k \to \infty), \tag{3.28}$$

whereas the general term in our series representation (3.20) has precisely the same order estimate:

$$O(2^{-2k} \cdot k^{-2n-1}) \qquad (k \to \infty;\ \ n \in \mathbb{N}), \tag{3.29}$$

as that in (2.9). Thus, even in the special case when $n = 1$, the series representing $\zeta(3)$ converges faster in (3.20) than in (3.21).

Various known series representations for $\zeta(2n+1)$ $(n \in \mathbb{N})$ of other types include those given (for example) by Ramanujan [21] (see also Berndt [3]), Glaisher [13] (see also Hansen [16, p. 359]), Koshliakov [18], Leshchiner [19], Grosswald ([14] and [15]), Terras [25], Cohen [7], Butzer *et al.* ([5] and [6]), Dąbrowski [9], and others (see, *e.g.*, Berndt [4, pp. 275 and 276]).

We conclude this paper by remarking that a particular case of our series representation (2.12) when $n = 1$ was proven, by an entirely different method, by Zhang and Williams [30, p. 707, Theorem 9]. Furthermore, the following particular case of (3.18) when $n = 1$:

$$\zeta(3) = -\frac{4\pi^2}{7} \sum_{k=0}^{\infty} \frac{\zeta(2k)}{(2k+1)(2k+2)\,2^{2k}}, \tag{3.30}$$

which is contained in a 1772 paper entitled *Exercitationes Analyticae* by Euler (see, *e.g.*, Ayoub [2, pp. 1084–1085]), was rediscovered by Ramaswami [22] and (more recently) by Ewell [10]. In fact, Euler's formula (3.30) was reproduced by Srivastava [24, p. 7, Equation (2.23)] from the work of Ramaswami [22]. In the *current* mathematical literature, however, Euler's formula (3.30) is being attributed to Ewell [10].

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Department of Mathematics and Statistics, University of Victoria, Victoria, British Columbia, Canada V8W 3P4
E-mail address: HMSRI@UVVM.UVIC.CA