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MULTINOMIAL COEFFICIENTS MODULO A PRIME

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Abstract. We say that the multinomial coefficient (m.c.) $(j_1,\dots,j_l) = n!/(j_1!\cdots j_l!)$ has order l and power $n = j_1+\cdots+j_l$. Let $G(n,l,p)$ be the number of m.c. that are not divisible by p and have order l with powers which are not larger than n. If $\theta = \log_p(l,p-1)$ and

$$q_{l,p}^{(r)} = \min_{p^r\le n<p^{r+1}} G(n,l,p)/(n+1)^\theta,$$

then for any integer $r = 1,2,\dots$

$$0 < q_{l,p}^{(r)} - \liminf_{n\to\infty} G(n,l,p)/n^\theta \le \frac{1}{\theta p^r}\left(1+\frac{1}{p^r}\right)^{\theta-1}.$$

1. Introduction

We consider the nth row of multinomial coefficients of order l:

$$(j_1,j_2,\dots,j_l) = \frac{n!}{j_1!j_2!\cdots j_l!},$$

where $j_i \ge 0$, $i = 1,\dots,l$, and $j_1+j_2+\cdots+j_l = n$. Let $g(n,l,p^N)$ be the number of multinomial coefficients in the nth row not divisible by p^N, where p is a prime and N is a fixed integer. We also define

$$G(n,l,p^N) = \sum_{k=1}^{n} g(k,l,p^N).$$

The properties and behaviour of functions $g(n,l,p^N)$ and $G(n,l,p^N)$ were studied in [1]–[14] where an interested person could find the history of the topic and references on earlier papers.

$G(n,l,p)$ is the subject of this paper, hence we will report some results concerning this function. The function $G(n,l,p)$ has been studied by K. B. Stolarsky [10], [11] and H. Harborth [3] for $l = p = 2$; by A. H. Stein [9] for $l = 2$ and arbitrary p; and by Volodin [12], [13] for arbitrary l and p.

Theorem (Volodin, 1989). *If $n+1 = a_0 + a_1p + \cdots + a_mp^m$, then*

$$G(n,l,p) = \sum_{k=0}^{m}(l,p-1)^k\frac{a_k}{l}\prod_{i=k}^{m}(l-1,a_i). \tag{1}$$

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Theorem (Volodin, 1994). *If* $\theta = \log_p(l, p-1)$*, then*

$$\limsup_{n\to\infty} G(n,l,p)/n^\theta \equiv 1. \tag{2}$$

For $l = 2$ the last theorem was proved by H. Harborth [3] for $p = 2$ and by A. H. Stein [9] for arbitrary p.

There are fewer results for

$$\beta_{l,p} = \liminf_{n\to\infty} G(n,l,p)/n^\theta \equiv 1.$$

Theorem (Harborth, 1977). $\beta_{2,2} = 0.812556\ldots$ *and* $\beta_{2,2}$ *is determined to six decimal places.*

Theorem (Wilson, 1996a). $\beta_{2,p}$ *are calculated to six decimal places for* $p = 3, 5, 7, \ldots, 37$ *and*

$$\lim_{p\to\infty} \beta_{2,p} = 0.5.$$

Theorem (Wilson, 1996b). $\lim_{p\to\infty} \beta_{l,p} = \frac{1}{l!}$.

In this paper we give a general approach to the calculation of $\beta_{l,p}$ for arbitrary l and p.

2. Main result

Let $q_{l,p}^{(r)} = \min_{p^r \le n < p^{r+1}-1} G(n,l,p)/(n+1)^\theta$.

Theorem. *For any* $r = 1, 2, \ldots$

$$0 < q_{l,p}^{(r)} - \beta_{l,p} \le \frac{1}{\theta p^r}\left(1 + \frac{1}{p^r}\right)^{\theta-1}.$$

We give some preliminary lemmas.

Lemma 1. *The sequence* $q_{l,p}^{(r)}$*,* $r = 1, 2, \ldots,$ *is nonincreasing and* $\lim_{r\to\infty} q_{l,p}^{(r)} = \beta_{l,p}$.

Proof. At the beginning we make a note that $\min_{p^r \le n < p^{r+1}} G(n,l,p)/(n+1)^\theta$ cannot be reached for $n = p^{r+1} - 1$ because

$$G(p^{r+1}-1, l, p)/p^{r\theta} = 1 \equiv \limsup_{n\to\infty} G(n,l,p)/(n+1)^\theta$$

and, because of that,

$$\min_{p^r \le n < p^{r+1}} G(n,l,p)/(n+1)^\theta = \min_{p^r \le n < p^{r+1}-1} G(n,l,p)/(n+1)^\theta.$$

Now, let n_r be chosen such that $G(n_r,l,p)/(n_r+1)^\theta = q_{l,p}^{(r)}$ and $p^r < n_r + 1 < p^{r+1}$, $n_r + 1 = a_0 + a_1 p + \cdots + a_r p^r$. It is clear that $p^{r+1} < p(n_r+1) < p^{r+2}$ and

$$\begin{aligned} q_{l,p}^{(r+1)} &\le G(n_r p + p - 1, l, p)/((n_r+1)p)^\theta \\ &= (l, p-1) G(n_r, l, p)/(p^\theta (n_r+1)^\theta) \\ &= G(n_r, l, p)/(n_r+1)^\theta = q_{l,p}^{(r)}. \end{aligned}$$

This implies

$$q_{l,p}^{(r)} = \min_{l \le n \le p^{r+1}} G(n,l,p)/(n+1)^\theta \tag{3}$$

and

$$\lim_{r\to\infty} q_{l,p}^{(r)} = \lim_{r\to\infty} \min_{1\le n\le p^{r+1}} G(n,l,p)/(n+1)^\theta = \liminf_{n\to\infty} G(n,l,p)/(n+1)^\theta = \beta_{l,p}.$$

Lemma 1 is proved completely. □

Lemma 2. *For any $r = 1, 2, \dots$ and any positive integer m*

$$0 < q_{l,p}^{(r)} - q_{l,p}^{(m+r)} \le \frac{1}{\theta p^r}\left(1 + \frac{1}{p^r}\right)^{\theta-1}. \tag{4}$$

Proof. Let $n_{m+r} + 1 = a_0 + a_1 p + \cdots + a_m p^m + a_{m+1} p^{m+1} + \cdots + a_{m+r} p^{m+r}$ and n_{m+r} provide a minimum for $q_{l,p}^{(m+r)}$. In other words

$$G(n_{m+r}, l, p)/(n_{m+r}+1)^\theta = q_{l,p}^{(m+r)}.$$

Now we consider a number $\tilde{n}_r$ such that $\tilde{n}_r + 1 = a_m + a_{m+1} p + \cdots + a_{m+r} p^r$. As $\tilde{n}_r + 1 < p^{r+1}$, then $G(\tilde{n}_r, l, p)/(\tilde{n}_r+1)^\theta \ge q_{l,p}^{(r)}$ and $n_{m+r} + 1 = (\tilde{n}_r + 1)p^m + x$, where $x < p^m$. Using this notation and (1) we can write

$$\begin{aligned}
0 < \Delta_r &\equiv q_{l,p}^{(r)} - q_{l,p}^{(m+r)} \\
&= q_{l,p}^{(r)} - G(\tilde{n}_r, l, p)/(\tilde{n}_r+1)^\theta + G(\tilde{n}_r p^m + p^m - 1, l, p)/((\tilde{n}_r+1)p^m)^\theta - q_{l,p}^{(m+r)} \\
&\le G((\tilde{n}_r+1)p^m - 1, l, p)/((\tilde{n}_r+1)p^m)^\theta \\
&\quad - G((\tilde{n}_r+1)p^m + x, l, p)/((\tilde{n}_r+1)p^m + x)^\theta \\
&\le (G((\tilde{n}_r+1)p^m - 1, l, p) - G((\tilde{n}_r+1)p^m + x, l, p))/((\tilde{n}_r+1)p^m)^\theta \\
&\quad + G((\tilde{n}_r+1)p^m + x, l, p)(1/((\tilde{n}_r+1)p^m)^\theta - 1/((\tilde{n}_r+1)p^m + x)^\theta) \\
&\le \frac{G((\tilde{n}_r+1)p^m + x, l, p)}{((\tilde{n}_r+1)p^m + x)^\theta}\left(\frac{((\tilde{n}_r+1)p^m + x)^\theta - ((\tilde{n}_r+1)p^m)^\theta}{((\tilde{n}_r+1)p^m)^\theta}\right) \\
&< \frac{((\tilde{n}_r+1)p^m + x)^{\theta-1} x}{\theta((\tilde{n}_r+1)p^m)^\theta} \le \frac{1}{\theta(\tilde{n}_r+1)}\left(1 + \frac{1}{\tilde{n}_r+1}\right)^{\theta-1} < \frac{1}{\theta p^r}\left(1 + \frac{1}{p^r}\right)^{\theta-1},
\end{aligned}$$

which proves Lemma 2. □

The proof of the Theorem follows from (3) and (4). □

3. Some numerical results

The approximation stated in the theorem allows us to calculate $\beta_{l,p}$ with accuracy that is only restricted by the speed and float precision of the computer used for the calculations. In Tables 1 and 2 we give some approximations for $\beta_{l,p}$ for different l and p. In these tables we follow H. Harborth's [3] presentation where he did not round approximations but gave the first six correct digits for $\beta_{2,2}$.

Example. For $l = 4$ and $p = 5$ we give the number 0.4547060. It means that $\beta_{l,p} = 0.4547060\dots$ even when $\beta_{l,p} = 0.454706096\dots$.

TABLE 1. Values of $\beta_{l,p}$ with seven correct digits for $p = 2, \ldots, 29$ and $l = 2, \ldots, 6$.

p	l				
	2	3	4	5	6
2	0.8125565	0.6769587	0.5811259	0.5122216	0.4594156
3	0.7742813	0.6106028	0.5001563	0.4217362	0.3630086
5	0.7582265	0.5770771	0.4547060	0.3673130	0.3049740
7	0.7491176	0.5579384	0.4279898	0.3395384	0.2716954
11	0.7364950	0.5360269	0.4000547	0.3019502	0.2298251
13	0.7326634	0.5290109	0.3877508	0.2835307	0.2118529
17	0.7275821	0.5167484	0.3622237	0.2572953	0.1851124
19	0.7257546	0.5090111	0.3529557	0.2472798	0.1761556
23	0.7228956	0.4966850	0.3383623	0.2323059	0.1615501
29	0.7168185	0.4830690	0.3220694	0.2155294	0.1433217

TABLE 2. Values of $\beta_{l,p}$ with seven correct digits for $p = 2, 3, 5, 7$ and $l = 7, \ldots, 32$.

l	p			
	2	3	5	7
7	0.4113033	0.3187192	0.2590327	0.2204959
8	0.3745607	0.2847163	0.2243479	0.1825242
9	0.3436618	0.2544162	0.1930142	0.1536125
10	0.3176270	0.2295480	0.1669928	0.1311626
11	0.2955003	0.2086933	0.1458796	0.1134714
12	0.2764950	0.1911916	0.1286715	0.0987069
13	0.2599888	0.1763485	0.1144720	0.0867433
14	0.2455116	0.1636174	0.1026171	0.0769178
15	0.2336864	0.1525884	0.0926147	0.0687172
16	0.2212529	0.1429478	0.0840944	0.0617951
17	0.2109785	0.1344518	0.0767735	0.0559291
18	0.2015317	0.1269030	0.0704336	0.0509033
19	0.1929696	0.1201641	0.0649030	0.0465707
20	0.1851717	0.1140793	0.0600466	0.0428067
21	0.1780362	0.1085546	0.0557545	0.0395941
22	0.1714759	0.1035478	0.0519443	0.0362998
23	0.1654284	0.0989897	0.0485448	0.0331847
24	0.1598345	0.0948237	0.0454825	0.0303715
25	0.1546442	0.0910016	0.0427250	0.0278834
26	0.1498142	0.0874829	0.0402338	0.0256638
27	0.1453075	0.0842309	0.0379746	0.0236850
28	0.1410919	0.0812182	0.0359187	0.0219155
29	0.1371393	0.0782420	0.0340414	0.0203282
30	0.1334252	0.0754630	0.0323222	0.0188998
31	0.1299256	0.0728736	0.0307432	0.0176101
32	0.1266244	0.0704556	0.0292627	0.0164424

References

1. L. Carlitz (1967), *The number of binomial coefficients divisible by a fixed power of a prime*, Rend Circ. Mat. Palermo **16**, 299–320. MR **40:**2554
2. L. Carlitz (1970), *Distribution of binomial coefficients*, Riv. Mat. Univ. Parma **11**, 45–64. MR **47:**45
3. H. Harborth (1977), *Number of odd binomial coefficients*, Proc. Amer. Math. Soc. **62**, 19–22. MR **55:**2725
4. F. T. Howard (1971), *The number of binomial coefficients divisible by a fixed power of* 2, Proc. Amer. Math. Soc. **29**, 236–242. MR **46:**1603
5. F. T. Howard (1973), *The number of binomial coefficients divisible by a fixed power of a prime*, Proc. Amer. Math. Soc. **37**, 358–362. MR **46:**8842
6. F. T. Howard (1974), *The number of multinomial coefficients divisible by a fixed power of a prime*, Pacific J. Math. **50**, 99–108. MR **49:**2513
7. F. T. Howard (1993), *Multinomial and* Q*-binomial coefficients modulo* 4 *and modulo* p, The Fibonacci Quarterly **31**, 53–64. MR **94c:**11019
8. R. J. Martin and G. L. Mullen (1984), *Reducing multinomial coefficients modulo a prime power*, Computers and Mathematics with Applications **10**, 37–41. MR **85a:**05007
9. A. H. Stein (1989), *Binomial coefficients not divisible by a prime*, Number Theory (New York, 1985/1988), Lecture Notes in Mathematics, vol. 1383, Berlin-New York, Springer, 170–177. MR **91c:**11012
10. K. B. Stolarsky (1975), *Digital sums and binomial coefficients*, Notices Amer. Math. Soc. **22**, A-669.
11. K. B. Stolarsky (1977), *Power and exponential sums of digital sums related to binomial coefficient parity*, SIAM J. Appl. Math. **32**, 717–730. MR **55:**12621
12. N. A. Volodin (1989), *Distribution* $\bmod p^N$ *of polynomial coefficients*, (Russian) Mat. Zametki **45** (1989), 24–29. MR **90i:**11022
13. N. A. Volodin (1994), *Number of multinomial coefficients not divisible by a prime*, The Fibonacci Quarterly **32**, 402–406. MR **95j:**11017
14. B. Wilson (1996a), *Asymptotic behavior of Pascal's triangle modulo a prime*, Acta Arith. **83** (1998), 105–116. CMP 98:06
15. B. Wilson (1996b), *LIM INF bounds for multinomial coefficients modulo a prime*, Preprint, SUNY College at Brockport, New York, 16 p.

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