

ON THE NON-PRODUCTIVITY OF NORMALITY IN MOORE SPACES

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ABSTRACT. Under Martin's Axiom and the denial of the Continuum Hypothesis, the authors give examples of normal Moore spaces whose squares are not normal.

1. INTRODUCTION

The question of whether the square of a normal Moore space must be normal was probably first raised by F. B. Jones soon after the publication of the *normal Moore space conjecture* in [Jo1937]. In any event it was a question often presented to his students in the 1950's and 1960's. Cook and Fitzpatrick showed in [CF1968] that if normality is preserved under finite products in Moore spaces, then it is preserved under countable products (a result which also follows from [K1948], since normal Moore spaces are perfectly normal). In [C1968], Cook established that Moore spaces whose squares are normal must be continuously semi-metrizable. Alster and Przymusiński in [AP1976] showed that under **MA**, if X is a co-metrizable T_2 -space (i.e., there exists a weaker separable metric topology δ on X such that if $x \in U$ where U is open in X , then there exists open V in X such that $x \in V$ and the closure of V with respect to δ is contained in U) and the cardinality of $X < \mathfrak{c}$, then X^ω is normal. Using this result and the result from [RZ1976] that each normal Moore space of cardinality $X \leq \mathfrak{c}$ is submetrizable (i.e., admits a one-to-one continuous map onto a metric space), Reed observed in [R1975] that under **MA**, the product question has an affirmative answer for locally compact Moore spaces of cardinality $< \mathfrak{c}$. This result was in turn extended by Alster and Przymusiński (see [P1977]) to show that under **MA**, the product question has an affirmative answer for locally compact separable Moore spaces. Recall that from [FT1966] and [F11974], under **V = L**, all normal locally compact Moore spaces are metrizable.

The above results led Reed to conjecture in [R1975] that (1) normality is productive for locally compact Moore spaces in **ZFC**, and (2) normality is productive for all Moore spaces of cardinality $< \mathfrak{c}$ under **MA**. The first conjecture remains open, but the authors of this paper have independently now shown that the second conjecture is false. The first examples were due to Cook, who in 1976 established

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a complex theory linking the various concepts of continuously semi-metrizability, co-metrizability, and special developments for Moore spaces. He used this theory to provide a necessary condition for the product of a normal Moore space to be normal, and then under $\mathbf{MA} + \neg\mathbf{CH}$ demonstrated the existence of normal Moore spaces without this condition. Although Cook's work was available in preprint [C1976], his examples were never published. In [R1986], Reed studied the class of spaces formed by the intersection topology with respect to subsets of the real line and the order topology on ω_1 , and noted that under $\mathbf{MA} + \neg\mathbf{CH}$, Moore spaces over members of this class (in the sense of [R1974a]) are normal but not co-metrizable. Hence, Reed raised the question as to whether such normal Moore spaces would fail to have normal squares. Indeed, prior to publication of [R1986], Reed showed this was the case and presented the new class of normal Moore spaces whose squares are not normal at the 1986 Prague Topology Symposium. The proof that the squares of these spaces are not normal is by direct demonstration of two disjoint closed sets which cannot be separated; furthermore, Reed observed that the same simple technique can be used to show the squares of Cook's original spaces are not normal. Thus, the two authors would like to take this opportunity to unify their examples and jointly to place them in the literature.

2. A MOORE SPACE MACHINE

In [R1974a] and [R1974b], Reed developed construction techniques which associate Moore spaces to first countable spaces. These techniques and their variations have since been frequently used to provide counterexamples in the theory of Moore spaces. The following construction is the simplest of these techniques; it is merely the first stage in an infinitely iterated process for the other constructions. The description given here is adapted for the applications at hand.

Let \mathbf{R} denote the real numbers, \mathbf{I} the set of integers, \mathbf{N} the non-negative integers, and \mathbf{N}^+ the set of positive integers. Finally, let \mathbf{F} denote $\{1/n : n \in \mathbf{N}^+\}$.

The construction \mathcal{M} . Let X denote a first countable T_2 -space. For each $x \in X$, denote by $\{U_n(x)\}$ a non-increasing sequence of open sets in X which forms a local base at x in X .

Let $\mathcal{M}(X) = (X \times \{0\}) \cup (X \times \mathbf{F})$. Denote by x_0 each element $(x, 0)$ of $X \times \{0\}$ and by x_n each element $(x, 1/n)$ of $X \times \mathbf{F}$.

Now, for each x_m in $\mathcal{M}(X)$ and $n \in \mathbf{N}^+$, define the basic open set $G_n(x_m)$ as follows:

$$\begin{aligned} \text{if } m > 0, \quad G_n(x_m) &= \{x_m\}, \\ \text{if } m = 0, \quad G_n(x_m) &= \{x_m\} \cup \bigcup \{U_j(x) \times \{1/j\} : j \geq n\}. \end{aligned}$$

It follows that $\{\mathcal{G}_n\}$, where for each n , $\mathcal{G}_n = \{G_j(p) : p \in \mathcal{M}(X) \text{ and } j \geq n\}$, is a development for the Moore space $\mathcal{M}(X)$.

□

3. A NORMAL MOORE SPACE WHOSE SQUARE IS NOT NORMAL

For each $n \in \mathbf{N}^+$, let $\mathcal{B}_n = \{(i/2^n, (i+1)/2^n) \subseteq \mathbf{R} : i \in \mathbf{I}\}$. Under $\mathbf{MA} + \neg\mathbf{CH}$, let X denote a Q -set on the real line with the inherited topology such that for each n , \mathcal{B}_n covers X . For each $x \in X$ and $n \in \mathbf{N}^+$, let $U_n(x)$ denote $B_x \cap X$, where B_x is the element of \mathcal{B}_n containing x .

Consider the space $Z = \mathcal{M}(X)$. By standard arguments (see [T1969]), Z is a normal Moore space.

Let Z' denote a new topology on the points of Z , where for each x_m in Z' and $n \in \mathbf{N}^+$, basic open sets $G'_n(x_m)$ are defined as follows:

$$\begin{aligned} \text{if } m > 0, \quad G'_n(x_m) &= \{x_m\}, \\ \text{if } m = 0, \quad G'_n(x_m) &= \{x_m\} \cup \bigcup \{(U_j(x) \setminus \{x\}) \times \{1/j\} : j \geq n\}. \end{aligned}$$

Clearly, Z' is also a normal Moore space, since any two subsets of $X \times \{0\}$ which could be separated in Z can also be separated in Z' . However, the square of Z' is trivially not normal. Consider $H = \{(x_0, x_0) : x \in X\}$ and $K = \{(x_0, x_n) : x \in X \text{ and } n \in \mathbf{N}^+\}$. Clearly H and K are mutually separated in $Z' \times Z'$, i.e., they are disjoint, no point of H is a limit point of K and no point of K is a limit point of H . From [M1962] (Theorem 153), each two mutually separated sets in a normal Moore space can be separated by open sets. Hence suppose $Z' \times Z'$ is normal and let V_H and V_K be disjoint open sets in $Z' \times Z'$ such that $H \subseteq V_H$ and $K \subseteq V_K$. Since X is uncountable, there exists an uncountable subset Y of X and $n \leq m \in \mathbf{N}^+$ such that for all $x \in Y$, $G'_n(x_0) \times G'_n(x_0) \subseteq V_H$ and $G'_m(x_0) \times \{(x_n)\} \subseteq V_K$. Now, there exists $z \in Y$ such that z is a limit point of Y in X . It follows that there exists $y \in Y$ with $y \neq z$ such that $z \in U_m(y)$ and thus $z \in U_n(y)$. Let $z' \in U_m(y) \cap U_m(z)$ such that $z' \neq z$. Then, $(z'_m, z_n) \in G'_n(y_0) \times G'_n(y_0) \subseteq V_H$ and $(z'_m, z_n) \in G'_m(z_0) \times \{(z_n)\} \subseteq V_K$. \square

Note. Of course, a tangent disc space over X with points in the upper plane isolated, and vertical lines removed from tangent disc neighborhoods for points on the x -axis would work just as well. We complicate the space here for simplification of future results.

4. A NORMAL SEPARABLE MOORE SPACE WHOSE SQUARE IS NOT NORMAL

To achieve the desired result, let us now embed the space Z' above in a normal separable Moore space Z'' . We will do so by simply “sewing” disjoint spaces T_n onto each “level” of $X \times \mathbf{F}$ in Z' . Each space T_n is a copy of a separable subspace T of Z' .

Consider the construction of Z' . Let L denote a countable dense subset of the Q -set X , and consider $T = (X \times \{0\}) \cup (L \times \mathbf{F})$ as a subspace of Z' . Clearly, T is a normal separable Moore space.

For each $x_0 \in X \times \{0\}$ and $n \in \mathbf{N}^+$, let $G_n^T(x_0) = G'_n(x_0) \cap T$, and let $D_n(x_0)$ denote an open set in T such that $(U_n(x) \setminus \{x\}) \times \{0\} \subseteq D_n(x_0)$, $D_n(x_0) \subseteq U_n(x) \times (\mathbf{F} \cup \{0\})$, and $x_0 \notin \overline{D_n(x_0)}$.

(*) Note that if $Y \subseteq X \times \{0\}$ and $D = \bigcup \{D_n(x) : x \in Y\}$, then $\overline{D} = D$ in T .

For each $n \in \mathbf{N}^+$, let T_n denote a copy of T disjoint from Z' such that if $n_1 \neq n_2$, then $T_{n_1} \cap T_{n_2} = \emptyset$. Also for each n , let π_n denote the identification mapping from T to T_n , and for each $Y \subseteq T$, let $\pi(Y)$ denote $\bigcup \{\pi_n(Y) : n \in \mathbf{N}^+\}$. Now, let $Z'' = Z' \cup \pi(L \times \mathbf{F})$.

For each $z \in Z''$ and $n \in \mathbf{N}^+$, define the basic open set $G''_n(z)$ for z in Z'' as follows:

$$\begin{aligned} & \text{if } z \in \pi(L \times \mathbf{F}), \quad G''_n(z) = \{z\}, \\ & \text{if } z = x_m \in Z' \text{ and } m > 0, \quad G''_n(z) = \{x_m\} \cup (\pi_m(G_n^T(x_0)) \setminus \{\pi_m(x_0)\}), \\ & \text{if } z = x_m \in Z' \text{ and } m = 0, \quad G''_n(z) = G'_n(x_m) \cup \\ & \qquad \qquad \qquad \bigcup \{\pi_j(D_j(x_0)) \setminus \pi_j(U_j(x) \times \{0\}) : j \geq n\}. \end{aligned}$$

It follows immediately that Z'' is a separable Moore space and that Z' is a closed subspace of Z'' . To see that Z'' is normal, let H and K denote disjoint closed subsets of Z'' . Without loss of generality let $H = H_0 \cup H_F$ and $K = K_0 \cup K_F$, where $H_0, K_0 \subseteq X \times \{0\}$ and $H_F, K_F \subseteq X \times \mathbf{F}$. Clearly, since Z' is normal, there exists an open set V_0 in Z'' containing H_0 such that V_0 is the union of basic open sets for elements of H_0 , $((X \times \{0\}) \setminus H_0) \cap V_0 = \emptyset$, and $((X \times \{0\}) \setminus H_0) \cap \bar{V}_0 = \emptyset$. Hence by application of the property (*) above, it follows that $\bar{V}_0 = V_0$ in Z'' and $\bar{V}_0 \cap K = \emptyset$. Now, by construction and the normality of T , for all $n \in \mathbf{N}^+$, there exists an open set V_n in Z'' such that $H_n = H_F \cap X \times \{1/n\} \subseteq V_n$ and $\bar{V}_n \cap K = \emptyset$. By symmetry, it follows that K is also the union of countably many sets, each of which can be separated by open sets from H . Thus, H and K can be separated by open sets in Z'' . \square

5. A CLASS OF NORMAL MOORE SPACES WHOSE SQUARES ARE NOT NORMAL

If \mathcal{T}_1 and \mathcal{T}_2 are topologies defined on the set X , then \mathcal{T} is the *intersection topology* w.r.t. \mathcal{T}_1 and \mathcal{T}_2 defined on X , where $\{U_1 \cap U_2 : U_1 \in \mathcal{T}_1 \text{ and } U_2 \in \mathcal{T}_2\}$ is a basis for \mathcal{T} . In [R1986], Reed studied the class \mathcal{C} of spaces (X, \mathcal{T}) where $X = \{x_\alpha : \alpha < \omega_1\} \subseteq \mathbf{R}$ and \mathcal{T} is the intersection topology w.r.t. the inherited real line topology \mathcal{T}_R on X and an order topology \mathcal{T}_{ω_1} on X of type ω_1 . He showed that each member of \mathcal{C} is a completely regular, first countable, submetrizable, collectionwise Hausdorff, pseudo-normal, countably metacompact space with a base of countable order and a \mathcal{G}_δ -diagonal, but which is neither θ -refinable, subparacompact, metaLindelöf, nor locally compact. Furthermore, he showed that there exists a non-normal member of \mathcal{C} in **ZFC**, and a collectionwise normal, ω_1 -compact, non-perfect member of \mathcal{C} under **CH**. Finally, Reed noted that under **MA** + \neg **CH**, each member of \mathcal{C} is perfect, whereas Kunen in [K1986] later showed that no member of \mathcal{C} is both perfect and normal, yet there exist models of set theory in which some members of \mathcal{C} are perfect and others are normal.

Notation. If $(X, \mathcal{T}) \in \mathcal{C}$, let \mathcal{B}_R denote a countable basis for \mathcal{T}_R , and for each $x_\alpha \in X$, let $\{B_i(x_\alpha) : i \in \mathbf{N}^+\}$ denote a fixed decreasing local base for x_α in (X, \mathcal{T}_R) where for each i , $B_i(x_\alpha) \in \mathcal{B}_R$. For each $x_\alpha \in X$ such that α is a limit ordinal in ω_1 , let $\{\alpha_i : i \in \mathbf{N}^+\}$ denote a fixed increasing sequence converging to α in ω_1 . If $x_\alpha \in X$ and α is a limit ordinal, $\{U_i(x_\alpha) = B_i(x_\alpha) \cap \{x_\beta : \alpha_i < \beta \leq \alpha\} : i \in \mathbf{N}^+\}$ denotes a fixed local base for x_α in (X, \mathcal{T}) . If $x_\alpha \in X$ and α is a non-limit ordinal, then $\{U_i(x_\alpha) = \{x_\alpha\} : i \in \mathbf{N}^+\}$ denotes a fixed local base for x_α in (X, \mathcal{T}) .

Now, let $X = (X, \mathcal{T}) \in \mathcal{C}$, and consider the Moore space $\mathcal{M}(X)$. As above, under **MA** + \neg **CH**, $\mathcal{M}(X)$ is normal. However, we will show that the square of $\mathcal{M}(X)$ is never normal.

Consider $H = \{(x_\alpha)_0, (x_\alpha)_0) : x_\alpha \in X\}$ and $K = \{(x_\alpha)_0, (x_\beta)_n) : n \in \mathbf{N}^+, x_\alpha \in X \text{ and } x_\beta \in B_n(x_\alpha) \setminus U_n(x_\alpha)\}$. As before, H and K are mutually separated in $\mathcal{M}(X) \times \mathcal{M}(X)$, hence if we suppose that $\mathcal{M}(X) \times \mathcal{M}(X)$ is normal there exist disjoint open sets V_H and V_K in $\mathcal{M}(X) \times \mathcal{M}(X)$ such that $H \subseteq V_H$ and $K \subseteq V_K$. By repeated uses of the Pressing Down Lemma, it follows that there exists a stationary subset S of ω_1 , $B \in \mathcal{B}$, and $n < m \in \mathbf{N}^+$ such that for all x_α where $\alpha \in S$,

- (1) $G_n((x_\alpha)_0) \times G_n((x_\alpha)_0) \subseteq V_H$,
- (2) there exists $x_{\alpha'}$ such that $\alpha < \alpha'$ and $x_{\alpha'} \in B_n(x_\alpha) \setminus U_n(x_\alpha)$,
- (3) $G_m((x_\alpha)_0) \times \{\{(x_{\alpha'})_n\}\} \subseteq V_K$, and
- (4) $B_m(x_\alpha) = B$.

Now, since S is stationary, it follows by construction that there exist $\alpha, \gamma \in S$ such that $((x_\alpha)_m, (x_{\alpha'})_n) \in G_n((x_\gamma)_0) \times G_n((x_\gamma)_0) \subseteq V_H$ and $((x_\alpha)_m, (x_{\alpha'})_n) \in G_m((x_\alpha)_0) \times \{\{(x_{\alpha'})_n\}\} \subseteq V_K$. Thus $\mathcal{M}(X) \times \mathcal{M}(X)$ is not normal.

□

6. BEYOND $\mathbf{MA} + \neg\mathbf{CH}$

Define a Moore space S to be a *sequence space* provided $S = H \cup K$, where H is discrete, each point of K is isolated, and basic open neighborhoods for points x in H consist of x together with the tail of a sequence chosen from K . Normal Moore sequence spaces which are not collectionwise Hausdorff exist under $\mathbf{MA} + \neg\mathbf{CH}$ [T1969], but the existence of such spaces is also consistent with \mathbf{GCH} [DS1979] (see [T1984]). Using the techniques above, Reed has recently shown that the existence of a normal Moore sequence space that is not collectionwise Hausdorff implies the existence of a normal Moore space whose square is not normal. This result will appear as part of a study of Moore sequence spaces in [R∞].

7. QUESTIONS

Question 1. Must the square of a normal, locally compact Moore space be normal?

Question 2. Is it consistent with \mathbf{ZFC} that a normal Moore of cardinality $< \mathfrak{c}$ is co-metrizable if and only if its square is normal?

Question 3. Can the normal Moore spaces in section 5 be embedded in normal, separable Moore spaces?

The normal Moore spaces in section 5 can be embedded in separable Moore spaces by methods similar to those in section 4. Of course, the normal Moore space from [F1982] under \mathbf{CH} cannot be embedded in a normal, separable Moore space, since all such spaces are metrizable under \mathbf{CH} . See [P1977] and [R1990] for further details related to the above questions.

Question 4. Does the existence of a normal Moore space (not necessarily a sequence space) which is not collectionwise Hausdorff imply the existence of a normal Moore space whose square is not normal?

Question 5. Under $\mathbf{V} = \mathbf{L}$, is the square of each normal Moore space normal? For starters, is the square of Fleissner's normal non-metrizable Moore space from \mathbf{CH} in [F1982] normal?

ADDED IN PROOF

The second author has now answered Question 1 in the negative.

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