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# A VARIANT OF THE DIAMOND PRINCIPLE FOR COMBINATORIAL IDEALS

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Abstract. We use a variant of the diamond principle to show many ideals on $\kappa$ are not $2^\kappa$-saturated if $\kappa$ is large. For instance, the $\Pi^1_1$-indescribable ideal is not $2^\kappa$-saturated if $\kappa$ is almost ineffable.

Kunen proved that the diamond principle for $\kappa$, $\diamondsuit(\kappa)$ holds if $\kappa$ is subtle. A consequence of $\diamondsuit(\kappa)$ is that the nonstationary ideal on $\kappa$ is not $2^\kappa$-saturated.

Meanwhile Baumgartner, Taylor and Wagon [2] proved that the ethereal ideal on $\kappa$ is not $\kappa^+$-saturated if $\kappa$ is almost ineffable.

These two facts have a point in common. If $\kappa$ has a strong property, then an ideal corresponding to a weaker property is less saturated.

For a regular uncountable cardinal $\kappa$, $\diamondsuit(\kappa)$ can be regarded as a property of the nonstationary ideal. We consider the following principle for an ideal $I$ on $\kappa$:

**The Diamond Principle for $I$, $\diamondsuit(I)$.** There is a sequence $\langle S_\alpha \subset \alpha \mid \alpha < \kappa\rangle$ such that for every $X \subset \kappa$,

$$\{\alpha < \kappa \mid X \cap \alpha = S_\alpha\} \notin I.$$

We modify Kunen's construction of a diamond sequence assuming $\kappa$ has a sufficiently strong property so that $\diamondsuit(I)$ holds. It is clear that no ideal $J \subseteq I$ is $2^\kappa$-saturated if $\diamondsuit(I)$ holds. Specifically we prove the following.

**Theorem.** *(1) If $\kappa$ is almost ineffable, then any ideal extended by the $\Pi^1_1$-indescribable ideal on $\kappa$ is not $2^\kappa$-saturated.*

*(2) If $\kappa$ is completely ineffable, then any ideal extended by the ineffable ideal on $\kappa$ is not $2^\kappa$-saturated.*

Before proving the theorem we state the definition of these ideals. Throughout the rest of this paper, $\kappa$ is a regular uncountable cardinal and $I$ is a $\kappa$-complete ideal on $\kappa$. The filter dual to an ideal $I$ is denoted by $I^*$, and $I^+$ is the set $\{X \subset \kappa \mid X \notin I\}$.

**Definition.** Let $X \subset \kappa$.

(i) $X$ is *$\Pi^1_1$-indescribable* if for any $R \subset V_\kappa$ and $\Pi^1_1$ sentence $\varphi$ such that $\langle V_\kappa, \in, R\rangle \models \varphi$, there is $\alpha \in X$ such that $\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi$.

(ii) $X$ is *almost ineffable* if for any sequence $\langle S_\alpha \subset \alpha \mid \alpha < \kappa\rangle$ there is $S \subset \kappa$ such that $\{\alpha \in X \mid S_\alpha = S \cap \alpha\}$ is unbounded in $\kappa$.

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(iii) $X$ is *ineffable* if for any sequence $\langle S_\alpha \subset \alpha \mid \alpha < \kappa\rangle$ there is $S \subset \kappa$ such that $\{\alpha \in X \mid S_\alpha = S \cap \alpha\}$ is stationary in $\kappa$.

(iv) *The completely ineffable ideal* on $\kappa$ is the minimal normal ideal $I$ such that for any $X \in I^+$ and any sequence $\langle S_\alpha \subset \alpha \mid \alpha < \kappa\rangle$ there is $S \subset \kappa$ such that $\{\alpha \in X \mid S_\alpha = S \cap \alpha\} \in I^+$. $X \in I^+$ is called *completely ineffable*.

(v) $X$ is *subtle* if for any sequence $\langle S_\alpha \subset \alpha \mid \alpha < \kappa\rangle$ and $C$ closed unbounded in $\kappa$, there exist $\alpha < \beta$ both in $C \cap X$ such that $S_\alpha = S_\beta \cap \alpha$.

For each property $A$ stated above, we consider the set

$$\{X \subset \kappa \mid X \text{ does not have property } A\},$$

which is a normal ideal on $\kappa$. For instance the $\Pi^1_1$-indescribable ideal is the set

$$\{X \subset \kappa \mid X \text{ is not } \Pi^1_1\text{-indescribable}\}.$$

These ideals were studied in Baumgartner [1] and Johnson [4].

*Proof of the Theorem.* (1) Suppose that $\kappa$ is almost ineffable. Let $NAIn_\kappa$ denote the almost ineffable ideal on $\kappa$ and $P_\alpha$ the $\Pi^1_1$-indescribable ideal on $\alpha$ for $\alpha \leq \kappa$. We use the fact that $P_\kappa \subset NAIn_\kappa$ and for every $X \in P^*_\kappa$,

$$\{\alpha \in X \mid X \cap \alpha \in P^*_\alpha\} \in NAIn^*_\kappa.$$

We recursively define $(S_\alpha, C_\alpha)$ for $\alpha < \kappa$ such that $S_\alpha \subset \alpha$ and $C_\alpha \in P^*_\alpha$ as follows.

Suppose that $\alpha < \kappa$ and $(S_\beta, C_\beta)$ has been defined for $\beta < \alpha$. Set $(S_\alpha, C_\alpha) = (\emptyset, \alpha)$ except in the case that

($\heartsuit$): There exist $S \subset \alpha$ and $C \in P^*_\alpha$ such that $S \cap \beta \neq S_\beta$ for any $\beta \in C$.

In this case, let $(S_\alpha, C_\alpha)$ be one such pair $(S, C)$.

We show that $\langle S_\alpha \mid \alpha < \kappa\rangle$ is a diamond sequence for $P_\kappa$. Suppose to the contrary that there are $X \subset \kappa$ and $C \in P^*_\kappa$ such that $X \cap \alpha \neq S_\alpha$ for $\alpha \in C$. Let $D = \{\alpha \in C \mid C \cap \alpha \in P^*_\alpha\}$. For $\alpha \in D$, $(S \cap \alpha, C \cap \alpha)$ satisfies the condition of ($\heartsuit$). Hence $C_\alpha \in P^*_\alpha$ and $S_\alpha \cap \beta \neq S_\beta$ for $\beta \in C_\alpha$. Since $D \in NAIn^*_\kappa$, $D$ is subtle. By Theorem 4.1 in Baumgartner [1],

$$\{\alpha \in D \mid \{\beta \in D \cap \alpha \mid S_\beta \neq S_\alpha \cap \beta\} \in P_\alpha\} \text{ is not subtle.}$$

Thus we have

$$E = \{\alpha \in D \mid \{\beta \in D \cap \alpha \mid S_\beta = S_\alpha \cap \beta\} \in P^+_\alpha\} \in NAIn^*_\kappa.$$

For any $\alpha \in E$, $C_\alpha \in P^*_\alpha$. Hence we can find $\beta \in C_\alpha$ such that $S_\beta = S_\alpha \cap \beta$ contradicting the definition of $(S_\alpha, C_\alpha)$.

(2) Suppose that $\kappa$ is completely ineffable. Let $NCIn_\kappa$ denote the completely ineffable ideal on $\kappa$ and $NIn_\alpha$ the ineffable ideal on $\alpha$ for $\alpha \leq \kappa$. We need only replace $P^*_\alpha$ by $NIn^*_\alpha$ in the definition of $(S_\alpha, C_\alpha)$ to get a diamond sequence for $NIn^*_\kappa$.

Consider the notion of forcing $Q = (NCIn^+_\kappa, \subseteq)$ and let $G$ be a $V$ generic filter on $Q$ and $M = Ult_G(V)$ the generic ultrapower. Since $NCIn_\kappa$ is normal $(\kappa, \kappa)$ distributive, $V^V_{\kappa+1} = V^M_{\kappa+1}$. (See [3], [4].) Hence, $NIn^V_\kappa = NIn^M_\kappa$ and, for any $X \in NIn^*_\kappa$,

$$\{\alpha \in X \mid X \cap \alpha \in NIn^*_\alpha\} \in NCIn^*_\kappa.$$

If $\langle S_\alpha \mid \alpha < \kappa\rangle$ is not a diamond sequence for $NIn_\kappa$, there is $Y \in NIn_\kappa^* \subset NCIn_\kappa^*$ such that, for any $\alpha \in Y$,

$$C_\alpha \in NIn_\alpha^* \text{ and } S_\beta \neq S_\alpha \cap \beta \text{ for } \beta \in C_\alpha.$$

By complete ineffability, there exist $T, U \subset \kappa$ such that

$$H = \{\beta \in Y \mid S_\alpha = T \cap \alpha \text{ and } C_\alpha = U \cap \alpha\} \in NCIn_\kappa^+.$$

Since $H \Vdash U \in NIn_\kappa^*$, $U \cap H \in NIn_\kappa^+$. For any $\beta < \alpha$ both in $U \cap H$, $\beta \in U \cap \alpha = C_\alpha$ and $S_\beta = T \cap \beta = (T \cap \alpha) \cap \beta = S_\alpha \cap \beta$, which contradicts the fact that $\alpha \in Y$. $\square$

There are several facts which can be proved by the same argument. For instance:

*If $\kappa$ is ineffable, then the $\Pi^1_2$-indescribable ideal on $\kappa$ is not $2^\kappa$-saturated.*

*If $\kappa$ is 2-subtle, then the ineffable ideal on $\kappa$ is not $2^\kappa$-saturated.*

*If $\kappa$ is measurable, then the completely ineffable ideal on $\kappa$ is not $2^\kappa$-saturated.*

Such an argument can be carried out for ideals on $P_\kappa\lambda$ as well.

Johnson proved in [4] that the completely ineffable ideal is not precipitous if $\kappa$ is completely ineffable. Thus it seems natural to ask:

**Question.** (1) *Can it be proved that these combinatorial ideals mentioned above are not precipitous?*

(2) *Is it possible to prove the ideal corresponding to property A is not $2^\kappa$-saturated just assuming $\kappa$ has property A? For instance, in order to prove the ineffable ideal on $\kappa$ is not $2^\kappa$-saturated, does it suffice to assume $\kappa$ is ineffable?*

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