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A VARIANT OF THE DIAMOND PRINCIPLE FOR COMBINATORIAL IDEALS

Y. ABE

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Abstract. We use a variant of the diamond principle to show many ideals on κ are not 2^κ-saturated if κ is large. For instance, the Π^1_1-indescribable ideal is not 2^κ-saturated if κ is almost ineffable.

Kunen proved that the diamond principle for κ, $\diamondsuit(\kappa)$ holds if κ is subtle. A consequence of $\diamondsuit(\kappa)$ is that the nonstationary ideal on κ is not 2^κ-saturated.

Meanwhile Baumgartner, Taylor and Wagon [2] proved that the ethereal ideal on κ is not κ^+-saturated if κ is almost ineffable.

These two facts have a point in common. If κ has a strong property, then an ideal corresponding to a weaker property is less saturated.

For a regular uncountable cardinal κ, $\diamondsuit(\kappa)$ can be regarded as a property of the nonstationary ideal. We consider the following principle for an ideal I on κ:

The Diamond Principle for I, $\diamondsuit(I)$. There is a sequence $\langle S_\alpha \subset \alpha \mid \alpha < \kappa\rangle$ such that for every $X \subset \kappa$,

$$\{\alpha < \kappa \mid X \cap \alpha = S_\alpha\} \notin I.$$

We modify Kunen's construction of a diamond sequence assuming κ has a sufficiently strong property so that $\diamondsuit(I)$ holds. It is clear that no ideal $J \subseteq I$ is 2^κ-saturated if $\diamondsuit(I)$ holds. Specifically we prove the following.

Theorem. *(1) If κ is almost ineffable, then any ideal extended by the Π^1_1-indescribable ideal on κ is not 2^κ-saturated.*

(2) If κ is completely ineffable, then any ideal extended by the ineffable ideal on κ is not 2^κ-saturated.

Before proving the theorem we state the definition of these ideals. Throughout the rest of this paper, κ is a regular uncountable cardinal and I is a κ-complete ideal on κ. The filter dual to an ideal I is denoted by I^*, and I^+ is the set $\{X \subset \kappa \mid X \notin I\}$.

Definition. Let $X \subset \kappa$.

(i) X is Π^1_1-*indescribable* if for any $R \subset V_\kappa$ and Π^1_1 sentence φ such that $\langle V_\kappa, \in, R\rangle \models \varphi$, there is $\alpha \in X$ such that $\langle V_\alpha, \in, R \cap V_\alpha\rangle \models \varphi$.

(ii) X is *almost ineffable* if for any sequence $\langle S_\alpha \subset \alpha \mid \alpha < \kappa\rangle$ there is $S \subset \kappa$ such that $\{\alpha \in X \mid S_\alpha = S \cap \alpha\}$ is unbounded in κ.

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(iii) X is *ineffable* if for any sequence $\langle S_\alpha \subset \alpha \mid \alpha < \kappa \rangle$ there is $S \subset \kappa$ such that $\{\alpha \in X \mid S_\alpha = S \cap \alpha\}$ is stationary in κ.

(iv) *The completely ineffable ideal* on κ is the minimal normal ideal I such that for any $X \in I^+$ and any sequence $\langle S_\alpha \subset \alpha \mid \alpha < \kappa \rangle$ there is $S \subset \kappa$ such that $\{\alpha \in X \mid S_\alpha = S \cap \alpha\} \in I^+$. $X \in I^+$ is called *completely ineffable.*

(v) X is *subtle* if for any sequence $\langle S_\alpha \subset \alpha \mid \alpha < \kappa \rangle$ and C closed unbounded in κ, there exist $\alpha < \beta$ both in $C \cap X$ such that $S_\alpha = S_\beta \cap \alpha$.

For each property A stated above, we consider the set

$$\{X \subset \kappa \mid X \text{ does not have property } A\},$$

which is a normal ideal on κ. For instance the Π^1_1-indescribable ideal is the set

$$\{X \subset \kappa \mid X \text{ is not } \Pi^1_1\text{-indescribable}\}.$$

These ideals were studied in Baumgartner [1] and Johnson [4].

Proof of the Theorem. (1) Suppose that κ is almost ineffable. Let $NAIn_\kappa$ denote the almost ineffable ideal on κ and P_α the Π^1_1-indescribable ideal on α for $\alpha \leq \kappa$. We use the fact that $P_\kappa \subset NAIn_\kappa$ and for every $X \in P^*_\kappa$,

$$\{\alpha \in X \mid X \cap \alpha \in P^*_\alpha\} \in NAIn^*_\kappa.$$

We recursively define (S_α, C_α) for $\alpha < \kappa$ such that $S_\alpha \subset \alpha$ and $C_\alpha \in P^*_\alpha$ as follows.

Suppose that $\alpha < \kappa$ and (S_β, C_β) has been defined for $\beta < \alpha$. Set $(S_\alpha, C_\alpha) = (\emptyset, \alpha)$ except in the case that

($\heartsuit$): There exist $S \subset \alpha$ and $C \in P^*_\alpha$ such that $S \cap \beta \neq S_\beta$ for any $\beta \in C$.

In this case, let (S_α, C_α) be one such pair (S, C).

We show that $\langle S_\alpha \mid \alpha < \kappa \rangle$ is a diamond sequence for P_κ. Suppose to the contrary that there are $X \subset \kappa$ and $C \in P^*_\kappa$ such that $X \cap \alpha \neq S_\alpha$ for $\alpha \in C$. Let $D = \{\alpha \in C \mid C \cap \alpha \in P^*_\alpha\}$. For $\alpha \in D$, $(S \cap \alpha, C \cap \alpha)$ satisfies the condition of ($\heartsuit$). Hence $C_\alpha \in P^*_\alpha$ and $S_\alpha \cap \beta \neq S_\beta$ for $\beta \in C_\alpha$. Since $D \in NAIn^*_\kappa$, D is subtle. By Theorem 4.1 in Baumgartner [1],

$$\{\alpha \in D \mid \{\beta \in D \cap \alpha \mid S_\beta \neq S_\alpha \cap \beta\} \in P_\alpha\} \text{ is not subtle.}$$

Thus we have

$$E = \{\alpha \in D \mid \{\beta \in D \cap \alpha \mid S_\beta = S_\alpha \cap \beta\} \in P^+_\alpha\} \in NAIn^*_\kappa.$$

For any $\alpha \in E$, $C_\alpha \in P^*_\alpha$. Hence we can find $\beta \in C_\alpha$ such that $S_\beta = S_\alpha \cap \beta$ contradicting the definition of (S_α, C_α).

(2) Suppose that κ is completely ineffable. Let $NCIn_\kappa$ denote the completely ineffable ideal on κ and NIn_α the ineffable ideal on α for $\alpha \leq \kappa$. We need only replace P^*_α by NIn^*_α in the definition of (S_α, C_α) to get a diamond sequence for NIn^*_κ.

Consider the notion of forcing $Q = (NCIn^+_\kappa, \subseteq)$ and let G be a V generic filter on Q and $M = Ult_G(V)$ the generic ultrapower. Since $NCIn_\kappa$ is normal (κ, κ) distributive, $V^V_{\kappa+1} = V^M_{\kappa+1}$. (See [3], [4].) Hence, $NIn^V_\kappa = NIn^M_\kappa$ and, for any $X \in NIn^*_\kappa$,

$$\{\alpha \in X \mid X \cap \alpha \in NIn^*_\alpha\} \in NCIn^*_\kappa.$$

If $\langle S_\alpha \mid \alpha < \kappa \rangle$ is not a diamond sequence for NIn_κ, there is $Y \in NIn^*_\kappa \subset NCIn^*_\kappa$ such that, for any $\alpha \in Y$,

$$C_\alpha \in NIn^*_\alpha \text{ and } S_\beta \neq S_\alpha \cap \beta \text{ for } \beta \in C_\alpha.$$

By complete ineffability, there exist $T, U \subset \kappa$ such that

$$H = \{\beta \in Y \mid S_\alpha = T \cap \alpha \text{ and } C_\alpha = U \cap \alpha\} \in NCIn^+_\kappa.$$

Since $H \Vdash U \in NIn^*_\kappa$, $U \cap H \in NIn^+_\kappa$. For any $\beta < \alpha$ both in $U \cap H$, $\beta \in U \cap \alpha = C_\alpha$ and $S_\beta = T \cap \beta = (T \cap \alpha) \cap \beta = S_\alpha \cap \beta$, which contradicts the fact that $\alpha \in Y$. □

There are several facts which can be proved by the same argument. For instance:

If κ is ineffable, then the Π^1_2-indescribable ideal on κ is not 2^κ-saturated.

If κ is 2-subtle, then the ineffable ideal on κ is not 2^κ-saturated.

If κ is measurable, then the completely ineffable ideal on κ is not 2^κ-saturated.

Such an argument can be carried out for ideals on $P_\kappa\lambda$ as well.

Johnson proved in [4] that the completely ineffable ideal is not precipitous if κ is completely ineffable. Thus it seems natural to ask:

Question. (1) *Can it be proved that these combinatorial ideals mentioned above are not precipitous?*

(2) *Is it possible to prove the ideal corresponding to property A is not 2^κ-saturated just assuming κ has property A? For instance, in order to prove the ineffable ideal on κ is not 2^κ-saturated, does it suffice to assume κ is ineffable?*

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Department of Mathematics, Kanagawa University, Yokohama 221, Japan
E-mail address: yabe@cc.kanagawa-u.ac.jp