

SYMMETRIES OF ACCOLA-MACLACHLAN AND KULKARNI SURFACES

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ABSTRACT. For all $g \geq 2$ there is a Riemann surface of genus g whose automorphism group has order $8g+8$, establishing a lower bound for the possible orders of automorphism groups of Riemann surfaces. Accola and Maclachlan established the existence of such surfaces; we shall call them Accola-Maclachlan surfaces. Later Kulkarni proved that for sufficiently large g the Accola-Maclachlan surface was unique for $g = 0, 1, 2 \pmod{4}$ and produced exactly one additional surface (the Kulkarni surface) for $g = 3 \pmod{4}$. In this paper we determine the symmetries of these special surfaces, computing the number of ovals and the separability of the symmetries. The results are then applied to classify the real forms of these complex algebraic curves. Explicit equations of these real forms of Accola-Maclachlan surfaces are given in all but one case.

1. INTRODUCTION

In the 1960's Accola [1] and Maclachlan [12] proved independently that for every $g \geq 2$ there is a Riemann surface X_g (*Accola-Maclachlan surface*) of genus g whose automorphism group has order $8g + 8$, by explicitly giving the equation of the surface, $w^2 = z^{2g+2} - 1$, and calculating its automorphism group. This result is interesting in that it is the largest order of an automorphism group that can be uniformly constructed for every g . This extended a result of Wiman in the last century in which he explicitly constructed a surface (*Wiman surface*) of genus g , $w^2 = z^{2g+1} - 1$ with an automorphism of maximal order $4g + 2$. Much later, Kulkarni [8] considered the question of uniqueness of the surfaces, i.e., whether they were the only surfaces of genus g with an automorphism group of order $8g + 8$. Kulkarni demonstrated uniqueness for $g = 0, 1, 2 \pmod{4}$ and g sufficiently large. For $g = 3 \pmod{4}$ and sufficiently large g he also proved, that in addition to the Accola-Maclachlan surface, there is exactly one other surface of genus g whose automorphism group has order $8g + 8$, though he did not give explicit equations for the surfaces. We shall call this surface the *Kulkarni surface* denoted by Y_g . He also showed that the conformal automorphism group of X_g has the following

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presentation $\langle a, b \mid a^{2(g+1)}, b^4, (ab)^2, ab^2a^{-1}b^2 \rangle$ and the one of Y_g has the following presentation $\langle a, b \mid a^{2(g+1)}, b^4, (ab)^2, b^2ab^2a^g \rangle$.

In this paper we prove that both the Accola-Maclachlan and the Kulkarni surfaces are symmetric and we determine their symmetries. The results are given in the Table in Theorem 3.2. Our purpose is to not only examine the symmetries themselves, but also to determine the real forms of these two special families of Riemann surfaces. Before proceeding, let us recall a few facts about symmetries and real forms. Let X be a surface of genus g . The group of conformal automorphisms of X is denoted by $\text{Aut}^+(X)$, and the group of all automorphisms of X , both conformal and anti-conformal, is denoted by $\text{Aut}^\pm(X)$. A *symmetry* σ of X is an anti-conformal, involutory automorphism, i.e., an element of $\text{Aut}^\pm(X) \setminus \text{Aut}^+(X)$ of order 2. The fixed point set $X_\sigma \subset X$ of a symmetry σ is a finite number of disjoint curves diffeomorphic to circles, which are called *ovals* of the symmetry. We denote by $k(X, \sigma)$ the number of ovals of σ . Define the symbol $\varepsilon(X, \sigma)$ to be 0 if $X \setminus X_\sigma$ is not connected - (we say that σ is *separating*) - and 1 otherwise - (we say that σ is *non-separating*). To abbreviate we define the *species* of (X, σ) as

$$(1) \quad \text{sp}(X, \sigma) = \begin{cases} +k(X, \sigma) & \text{if } \varepsilon(X, \sigma) = 0, \\ -k(X, \sigma) & \text{if } \varepsilon(X, \sigma) = 1. \end{cases}$$

Two pairs (X, σ) and (Y, τ) are said to be topologically equivalent if there exists a homeomorphism $h : X \rightarrow Y$ such that $\tau h = h\sigma$. Weichold [16] proved that (X, σ) and (Y, τ) are topologically equivalent if and only if X and Y have the same genus and $\text{sp}(X, \sigma) = \text{sp}(Y, \tau)$. Later, Harnack [5] proved that given a triple of integers (g, k, ε) there exists a pair (X, σ) such that g is the genus of X , $k = k(X, \sigma)$ and $\varepsilon = \varepsilon(X, \sigma)$ if and only if

$$(2) \quad 1 \leq k + \varepsilon \leq g + 1 ; k \equiv g + 1 \pmod{2 - \varepsilon}.$$

Clearly, two conjugate symmetries σ and τ on X with respect to $\text{Aut}^\pm(X)$ have the same species since the pairs (X, σ) and (X, τ) are topologically equivalent. The *symmetry type* of a surface is the list of species of all conjugacy classes of symmetries occurring in $\text{Aut}^\pm(X)$.

There is a 1-1 correspondence between symmetries and real forms of algebraic curves. In particular, if an algebraic curve can be defined by polynomial equations with real coefficients, then complex conjugation defines a symmetry of the corresponding Riemann surface. The ovals of a symmetry are the connected components of the corresponding real form. A real form is separating if it disconnects its complexification, and non-separating otherwise.

We shall determine the symmetry types of the Accola-Maclachlan surfaces and the Kulkarni surfaces, computing the number of ovals and separability of the symmetries. The results are then applied to determine the real forms of these complex algebraic curves. Explicit equations of the real forms of the curves are given in all cases but one.

2. NUMBER OF OVALS

From the Riemann uniformization theorem, every compact Riemann surface X of genus $g \geq 2$ can be represented as the quotient H/Γ of the upper half complex plane H under the action of a fuchsian surface group Γ . Thus there exists a Fuchsian group Δ containing Γ as a normal subgroup such that $G = \text{Aut}^+(X) = \Delta/\Gamma$. Although the method we are going to outline works in a more general setting, as in

[4], for $X = X_g$ or Y_g it is sufficient to analyze what happens when Δ is generated by three elliptic elements of even distinct orders k, l and m (actually in our case $k = 2(g + 1), l = 4, m = 2$).

The unique NEC (non-euclidean crystallographic) group Λ containing Δ as a subgroup of index 2 has the presentation

$$\Lambda = \langle c_0, c_1, c_2 \mid c_0^2, c_1^2, c_2^2, (c_0c_1)^k, (c_1c_2)^l, (c_0c_2)^m \rangle$$

and we shall refer to c_0, c_1 and c_2 as the canonical reflections of Λ .

From the presentations it is trivial to see that the assignment $a \mapsto a^{-1}, b \mapsto b^{-1}$ induces an automorphism

$$(3) \quad \varphi : G \rightarrow G$$

and so applying the criterion of Singerman [15] X is symmetric. This is equivalent to saying that Γ is a normal subgroup of Λ , and precisely, the symmetries on X are the involutions $x\Gamma \in \Lambda/\Gamma = \tilde{G}, x \in \Lambda \setminus \Delta$. We shall see later that the images $\theta(c_i) = \sigma_i, i = 0, 1, 2$, of the c -reflections under the canonical projection $\theta : \Lambda \rightarrow \Lambda/\Gamma$ are not conjugate in \tilde{G} . Since all reflections in Λ are conjugate to one of the c -reflections and each symmetry σ in X with a non-empty set of fixed points X_σ is the image $\sigma = \theta(c)$ of some reflection $c \in \Lambda$ (see [4]), classifying all symmetries σ with X_σ non-empty reduces to computing the species $\text{sp}(X, \sigma_i), i = 0, 1, 2$. In this section we calculate $k(X, \sigma_i) = k_i$. If we represent the subgroup $\langle \sigma_i \rangle$ as Λ_i/Γ for some group $\Lambda_i \subset \Lambda$ it follows that k_i is the number of boundary components of the Klein surface $X/\langle \sigma_i \rangle = H/\Lambda_i$. In other words, k_i is the number of conjugacy classes of reflections in Λ_i . If C_i is the preimage under θ of the centralizer $C(\tilde{G}, \sigma_i)$ of σ_i in \tilde{G} , the set of all reflections of Λ_i is

$$R_i = \{c_i^w = wc_iw^{-1} : w \in C_i\}.$$

Now given $u, v \in C_i$, the reflections c_i^u and c_i^v are conjugate in Λ_i if and only if $u^{-1}\gamma v \in C(\Lambda, c_i)$ for some $\gamma \in \Lambda_i$. On the other hand, $u^{-1}\gamma v = (u^{-1}v)(v^{-1}\gamma v)$. So as C_i normalizes Λ_i , we see that $u^{-1}\gamma v \in C(\Lambda, c_i)$ if and only if $u^{-1}v \in C(\Lambda, c_i)\Lambda_i = C(\Lambda, c_i)\Gamma$. Hence we get

$$(4) \quad k_i = [C_i : C(\Lambda, c_i)\Gamma] = [C_i/\Gamma : C(\Lambda, c_i)\Gamma/\Gamma] = [C(\tilde{G}, \sigma_i) : \theta(C(\Lambda, c_i))].$$

In particular, for $i = 0$ it is easy to prove that

$$(5) \quad C(\Lambda, c_0) = \langle c_0 \rangle \oplus \left(\langle (c_0c_1)^{k/2} \rangle * \langle (c_0c_2)^{m/2} \rangle \right)$$

and so

$$(6) \quad k_0 = \frac{\#C(\tilde{G}, \sigma_0)}{4\#\langle (\sigma_0\sigma_1)^{k/2}(\sigma_0\sigma_2)^{m/2} \rangle}$$

and similar formulae can be found for k_1 and k_2 (see [4] and [15]).

Let us attack the explicit computations in case $X = X_g$. Here we have

$$(7) \quad \theta : \Lambda \rightarrow \Lambda/\Gamma = \text{Aut}^+(X) \times_\varphi Z_2 = \langle a, b \rangle \times_\varphi \langle t \rangle$$

defined by

$$\theta(c_0) = at, \theta(c_1) = t, \theta(c_2) = tb,$$

where φ is defined in (3).

To apply (6) we must check that $\sigma_0, \sigma_1, \sigma_2$ are not conjugate in \tilde{G} . In fact, each element $x \in \tilde{G}$ can be written in a unique way as $x = a^r b^s t^\mu$ with $0 \leq r \leq$

$2g + 1, 0 \leq s \leq 3, 0 \leq \mu \leq 1$. Hence, $xtx^{-1} = a^{2r}b^{2s}t$ as b^2 commutes with a . So $\sigma_0 = at$ and $\sigma_1 = t$ are not conjugate. Analogously we check the other two cases. Let us calculate the order of $C(\tilde{G}, \sigma_0)$. The element $x \in \tilde{G}$ above commutes with σ_0 if and only if either

$$a = a^r b^s a b^s a^r, \mu = 0 \quad \text{or} \quad a = a^r b^s a^{-1} b^s a^r, \mu = 1.$$

In both cases we have eight solutions. In the first case we have

$$s = 0, 2; r = 0, g + 1 \quad \text{and} \quad s = 1, 3; r = 1, g + 2.$$

while in the the second case we have

$$s = 0, 2; r = 1, g + 2 \quad \text{and} \quad s = 1, 3; r = 0, g + 1.$$

Consequently, $\#C(\tilde{G}, \sigma_0) = 16$. On the other hand, $(\sigma_0 \sigma_1)^{k/2} (\sigma_0 \sigma_2)^{m/2} = a^{g+2} b$ has order 2 for odd g and order 4 for even g . Hence from (6), $k_0 = 2$ for odd g and $k_0 = 1$ for even g . Similarly we prove that $k_1 = 1$ and $k_2 = g + 1$.

We still have to determine how many conjugacy classes of symmetries without fixed points exist. Such symmetries are elements of order 2, $y = a^r b^s t \in \tilde{G}$ which are not conjugate to t, at, tb and it is easy to see that there is exactly one class of such elements for even g represented by ab^2t , and two classes represented by ab^2t and $a^{g+1}bt$ if g is odd.

Concluding, for odd g , X_g has five classes of symmetries: two without fixed points, and three $\sigma_0, \sigma_1, \sigma_2$ with 2, 1 and $g + 1$ ovals, respectively. For even g there are four conjugacy classes of symmetries: one without fixed points, two with one oval and one more with $g + 1$ ovals.

The analysis of the symmetries of Y_g for $g \equiv 3 \pmod 4$ follows the same technical lines. There are three classes of symmetries $\sigma_0, \sigma_1, \sigma_2$ with fixed points having 2, 1 and $(g + 1)/4$ ovals and if $g \equiv 3 \pmod 8$ there is exactly one more class without fixed points.

3. SEPARABILITY CHARACTER OF THE SYMMETRIES

Let σ be a symmetry on $X = X_g$ or Y_g . If X_σ is empty, then σ is non-separating. Thus, in what follows we are going to compute $\varepsilon(X, \sigma_i), i = 0, 1, 2$, with the notation of the preceding sections. It follows from (1) that $\varepsilon = 0$ if $k = g + 1$, and $\varepsilon = 1$ when $k = 1$ and g is odd. Furthermore since $(g + 1)/4$ is odd for $g \equiv 3 \pmod 8$, we conclude that $\varepsilon = 1$ in the case $k = \frac{g+1}{4}, X = Y_g, g \equiv 3 \pmod 8$. Consequently

$$(8) \quad \text{sp}(X, \sigma) = \begin{cases} +(g + 1) & \text{if } (X, \sigma) = (X_g, \sigma_2), \\ -1 & \text{if } \sigma = \sigma_1 \text{ except if } X = X_g \text{ and } g \text{ is even,} \\ -(g + 1)/4 & \text{if } (X, \sigma) = (Y_g, \sigma_2) \text{ and } g \equiv 3 \pmod 8. \end{cases}$$

On the other hand, surfaces admitting a symmetry σ_2 with species $+(g + 1)$ have been studied by Natanzon in [14] and by Bujalance and Costa in [3] and it follows directly from Theorem 3.3 in the latter paper that

$$(9) \quad \text{sp}(X_g, \sigma_0) = +2 \quad \text{if } g \text{ is odd.}$$

To calculate $\varepsilon(X, \sigma)$ in the remaining cases we use two distinct strategies. First we prove:

$$(10) \quad \text{sp}(X, \sigma) = \begin{cases} -1 & \text{if } (X, \sigma) = (X_g, \sigma_1) \text{ and } g \text{ is even,} \\ -(g + 1)/4 & \text{if } (X, \sigma) = (Y_g, \sigma_2) \text{ and } g \equiv 7 \pmod 8. \end{cases}$$

For this purpose we need the following result which combines Theorem 1 in [11] and Lemma 2.8 in [2]:

Proposition 3.1. *Let $G = \text{Aut}^+(X)$, $X = X_g$ or Y_g , with the presentation of the introduction. Let σ, τ be two commuting symmetries on X . Let us define for each $x \in G$ the integer δ_x to be either 1 if $h = \sigma\tau$ is conjugate in G to some power of x and 0 otherwise. Then, if σ is separating, the following inequality holds:*

$$(11) \quad \#C(G, h) \left(\frac{\delta_a}{2g+2} + \frac{\delta_b}{4} + \frac{\delta_{ab}}{2} \right) \leq 2k(X, \sigma).$$

Now if we apply this with $X = X_g$, $\sigma = \sigma_1, \tau = \sigma_2\sigma_1\sigma_2^{-1}$, then $h = b^2$ has order 2, $\delta_a = \delta_{ab} = 0, \delta_b = 1$ and $\#C(G, h) = 8(g+1)$. Hence, if σ_1 is separating we get from (11) $g+1 \leq k(X_g, \sigma_1) \leq 1$, a contradiction. Analogously, if $X = Y_g, \sigma = \sigma_2$, and $\tau = \sigma_1\sigma_2\sigma_1^{-1}$ we get again $h = b^2$ has order 2, $\delta_a = \delta_{ab} = 0, \delta_b = 1$ and now $\#C(G, h) = 4(g+1)$. Thus if σ_2 is separating, $(g+1)/4 = k(Y_g, \sigma_2) \geq (g+1)/2$ which is absurd. This proves (10).

To finish we prove the following

$$(12) \quad \text{sp}(X_g, \sigma_0) = +1 \text{ if } g \text{ is even; } \text{sp}(Y_g, \sigma_0) = -2.$$

In this case we apply a different approach due to Hoare and Singerman [6]. Let X be either X_g or Y_g and with the notation in Section 2 let Γ be a Fuchsian surface group uniformizing X . Let us consider the subgroup $K = \langle \sigma_0, \sigma_1 \rangle$ of $\tilde{G} = \text{Aut}^\pm(X)$, which is isomorphic to the dihedral group D_{2g+2} of $4g+4$ elements. Then K can be written as Γ'/Γ for some NEC-group Γ' whose signature

$$\sigma(\Gamma') = (0; +; [-]; \{(2g+2, 2g+2, g+1)\})$$

can be easily computed by the Riemann-Hurwitz formula (see e.g. [13] pg. 438). Let ρ_0, ρ_1, ρ_2 be a set of canonical reflections generating Γ' such that the natural projection $\Phi : \Gamma' \rightarrow K$ maps $\rho_j \mapsto \sigma_j, j = 0, 1$. It is easy to see that

$$\Phi(\rho_2) = \begin{cases} \sigma_0(\sigma_1\sigma_0)^{2g} & \text{if } X = X_g, \\ \sigma_0(\sigma_1\sigma_0)^{g-1} & \text{if } X = Y_g. \end{cases}$$

Following the quoted result in [6], the symmetry σ_0 is separating if and only if the Schreier graph \mathcal{S} of the set of cosets $K/\langle \sigma_0 \rangle$ corresponding to the system of generators $\{\Phi(\rho_j) : j = 0, 1, 2\}$ is bipartite. Assume now $X = X_g$. The $2g+2$ classes in $K/\langle \sigma_0 \rangle$ are denoted by $K/\langle \sigma_0 \rangle = \{[i] : 0 \leq i \leq 2g+1\}$, where $[i] = \{(\sigma_0\sigma_1)^i, (\sigma_0\sigma_1)^i\sigma_0\}$. To produce the graph \mathcal{S} it is necessary to know how each $\Phi(\rho_j), j = 0, 1, 2$, links two vertices of $K/\langle \sigma_0 \rangle$. But it is easy to check that the action of $\Phi(\rho_j)$ on the vertices is given by

$$\Phi(\rho_j)([i]) = [2g+2-i-j], 0 \leq i \leq 2g+1, 0 \leq j \leq 2.$$

Then the Schreier graph \mathcal{S} (with the loops deleted) admits a bipartition

$$V_1 = \{[i] : 0 \leq i \leq g\}; V_2 = \{[i] : g+1 \leq i \leq 2g+1\}$$

of the set of vertices of \mathcal{S} , and so, σ_0 is separating in this case. In a similar way it is proved that σ_0 is not separating if $X = Y_g$. We summarize our results in the following Theorem.

Theorem 3.2. *The symmetry types of the Accola-Maclachlan surface and the Kulkarni surface are as in the following table:*

X	g	σ_0	σ_1	σ_2
X_g	odd	+2	-1	$+(g + 1)$
X_g	even	+1	-1	$+(g + 1)$
Y_g	$\equiv 7 \pmod 8$	-2	-1	$-(g + 1)/4$
Y_g	$\equiv 3 \pmod 8$	-2	-1	$-(g + 1)/4$

4. REAL PLANE CURVES REPRESENTING THE REAL FORMS

Let $N = g + 1$, $P(z) = z^{2N} - 1$ and consider the complex plane curve

$$C = \{(z, w) \in \mathbf{C}^2 : w^2 = P(z)\}.$$

It follows from Theorem 3.2 that there exist three polynomials $Q_i(z) \in \mathbf{R}[z]$, $i = 0, 1, 2$, such that each curve $C_i = \{(z, w) \in \mathbf{C}^2 : w^2 = Q_i(z)\}$ is \mathbf{C} -birationally isomorphic to C , and their real parts $C_i(\mathbf{R}) = \{(z, w) \in \mathbf{R}^2 : w^2 = Q_i(z)\}$ have the species of the symmetries σ_i , $i = 0, 1, 2$. Let X_i denote a projective, smooth, real model of $C_i(\mathbf{R})$, i.e, a smooth, projective real algebraic curve, \mathbf{R} -birationally isomorphic to $C_i(\mathbf{R})$. Let us denote by X_g a (common) nonsingular complexification of X_i . We must prove that:

- 4.1. X_1 and $X_g - X_1$ are connected.
- 4.2. X_0 is connected if N is odd and it has two connected components if N is even. In both cases $X_g - X_0$ is not connected.
- 4.3. X_2 has N connected components, thus $X_g - X_0$ is not connected.

In order to construct the polynomials Q_i , we examine the automorphism group of X_g . Recall that a presentation of the automorphism group G of X_g is

$$G = \langle a, b \mid a^{2N}, b^4, (ab)^2, ab^2a^{-1}b^2 \rangle.$$

We can explicitly determine the generators a and b , in fact, since C and X_g are \mathbf{C} -birationally isomorphic, G is isomorphic to the group of automorphisms of the function field $k_g = (\mathbf{C}[z, w]/w^2 - P(z))_{(0)}$. It is easy to see that if $\xi = e^{\frac{2\pi i}{2N}}$, the maps $a, b : k_g \rightarrow k_g$ defined by

$$a(z) = \xi z, a(w) = -w; b(z) = \frac{1}{z}, b(w) = \frac{iw}{z^N}; a, b \mid \mathbf{C} = \text{identity},$$

are isomorphisms satisfying the required relations.

Moreover, complex conjugation on C corresponds to the \mathbf{R} -isomorphism σ of k_g defined by $\sigma(z) = z$, $\sigma(w) = w$, $\sigma(i) = -i$. We can explicitly compute the action of a, b and σ on z, w and i and it follows that $\sigma a \sigma = a^{-1}$; $\sigma b \sigma = b^{-1}$; $a \sigma, b \sigma$ are symmetries of X_g . Observe the similarity with the equalities $t a t^{-1} = a^{-1}$; $t b t^{-1} = b^{-1}$; $a t = \theta(c_0)$, $t b = \theta(c_2)$ from Section 2. For technical reasons we have chosen $b \sigma$ instead of $\sigma b = \sigma(b \sigma) \sigma^{-1}$, but both are conjugates. We will see that σ , $a \sigma$ and $b \sigma$ are representatives of the conjugacy classes of symmetries with fixed points in X_g . This will allow us to calculate the polynomials Q_i . In fact we prove that, with the notations of Section 3, $\sigma \equiv \sigma_1$, $a \sigma \equiv \sigma_0$ and $b \sigma \equiv \sigma_2$.

The fixed subfield of σ is $\mathbf{R}(z, w)$, so we choose $Q_1(z) = P(z)$. Let us prove 4.1. The only point at infinity $(0 : 1 : 0)$ of $C = C_1$ is singular. From $w^2 = z^{2N} - 1$ it follows $(\frac{w}{z^N})^2 = 1 - \frac{1}{z^{2N}}$ and so $z = \infty$ if and only if $\frac{w}{z^N} = \pm 1$. Hence, X_g is the affine curve C_1 with two points adjoined. We can represent X_g as a glueing

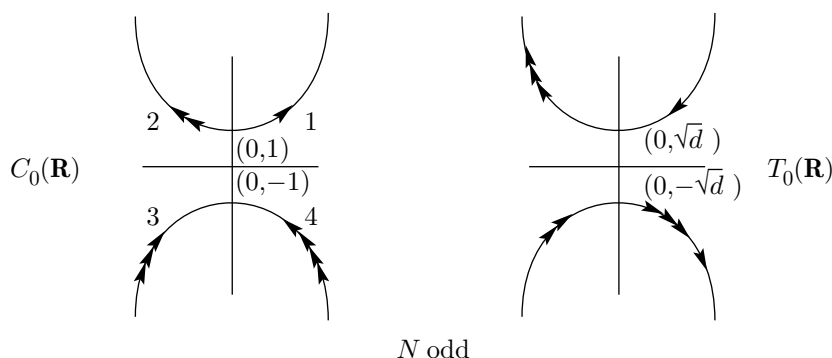


FIGURE 1

by defining $t = \frac{1}{z}$, and $s = \frac{w}{z^N}$. Note that $s^2 = 1 - t^{2N}$ if $(z, w) \in C_1$. Then, $\varphi : C_1 \rightarrow T_1 = \{(t, s) \in \mathbf{C}^2 : s^2 = 1 - t^{2N}\}$ defined by $\varphi(z, w) = (\frac{1}{z}, \frac{w}{z^N})$ is a birational isomorphism preserving the real parts. Note that $t = 0, s = \pm 1$ at the points at infinity, and so X_g can be viewed as the glueing of C_1 and T_1 via φ , along the points $z \neq 0, t \neq 0$. Since $\varphi(C_1(\mathbf{R})) = T_1(\mathbf{R})$ and $C_1(\mathbf{R})$ does not intersect the line $(z = 0)$, it follows that X_1 is connected because $T_1(\mathbf{R})$ is. Let us check that $X_g - X_1$ is also connected. The map $T_1 \rightarrow \mathbf{C} : (t, s) \rightarrow t$ is a ramified double covering which ramifies at $t = \xi^k, k = 0, 1, \dots, 2N - 1$, with two sheets $s_\epsilon : \mathbf{C} \rightarrow T_1 : t \rightarrow (t, \epsilon\sqrt{1 - t^{2N}}), \epsilon = \pm 1$. The real points of T_1 project onto the closed interval $[-1, 1] \subset \mathbf{R}$. Hence, using circles around the branching points, one can connect, avoiding points with $t \in [-1, 1]$, points in the upper sheet with points in the bottom sheet, and so $X_g - X_1$ is connected.

We now construct Q_0 . To do this, we determine two algebraically independent elements in k_g , fixed under the action of $a\sigma$. However, it is easy to see that $z_1 = (1 + \xi)z$ and $w_1 = iw$ are fixed by $a\sigma$. Since $k_g = \mathbf{C}(z, w) = \mathbf{C}(z_1, w_1)$, we can express the equation $w^2 = z^{2N} - 1$ of C in terms of z_1 and w_1 . In fact $w = -iw_1$, and $z = \frac{z_1}{1 + \xi}$. Thus, $-w_1^2 = \frac{z_1^{2N}}{(1 + \xi)^{2N}} - 1$. By defining $d = \frac{-1}{(1 + \xi)^{2N}}$, we obtain $w_1^2 = 1 + dz_1^{2N}$. Note that $|\xi| = 1$ and $Arg(\xi) = \frac{\pi}{N}$, thus $Arg(1 + \xi) = \frac{\pi}{2N}$ and $Arg(1 + \xi)^{2N} = \pi$. Thus d is a positive real number. We define $Q_0(z) = 1 + dz^{2N}$. It is obvious that C_0 is \mathbf{C} -isomorphic to C .

We now prove 4.2. First we count the number of connected components of X_0 . We represent X_g as before as the glueing of C_0 with $T_0 = \{(t, s) \in \mathbf{C}^2 : s^2 = d + t^{2N}\}$ via the isomorphism $\varphi : C_0 \rightarrow T_0 : (z, w) \rightarrow (\frac{1}{z}, \frac{w}{z^N})$, along the points $z \neq 0, t \neq 0$. This glueing behaves quite differently according to the parity of N . If N is even, the intersection of $C_0(\mathbf{R})$ with the j -quadrant of $\mathbf{R}^2, j = 1, 2, 3, 4$, labelled counterclockwise, is identified with the intersection of $T_0(\mathbf{R})$ with the same quadrant. If N is odd, the intersections with quadrants 2 and 3 are interchanged.

Considering Figure 1, one realizes that X_0 has two connected components if N is even, since the points at infinity $(0, 1)$ and $(0, -1)$ of $C_0(\mathbf{R})$ cannot be joined by a path, while X_0 is connected if N is odd, because you can jump from the points in $T_0(\mathbf{R})$ lying in the second quadrant to the points in $C_0(\mathbf{R})$ lying on the third quadrant.

To check that $X_g - X_0$ is disconnected, note that it is included in C_0 , since X_0 contains the two points at infinity $(0, \pm\sqrt{d})$ of T_0 . Now, consider two points $P_0 = (z_0, w_0)$ and $P'_0 = (z'_0, w'_0)$ in $X_g - X_0$ such that $I_m(z_0) > 0, I_m(z'_0) < 0$. These two points cannot be joined by a path in $X_g - X_0$. Otherwise, let $\gamma : [0, 1] \rightarrow X_g - X_0 : s \rightarrow (\gamma_1(s), \gamma_2(s))$ be such a path. Since $\gamma_1(0)$ and $\gamma_1(1)$ are complex numbers whose imaginary parts have opposite signs, there exists $s \in (0, 1)$ such that $\gamma_1(s) \in \mathbf{R}$. Then, $1 + \gamma_1(s)^{2N}$ is a positive real number, whose square is $\gamma_2(s)$, and so $\gamma(s) \in X_0$, a contradiction. This finishes 4.2.

We now construct the polynomial Q_2 . As before, we try to find two elements in k_g , fixed by $b\sigma$ and algebraically independent over \mathbf{C} . Note that X_2 has N connected components, so $C_2(\mathbf{R})$, should have $N + 1$ components, and for this to occur, Q_2 must have $2N$ distinct real roots. So we transform the polynomial $P(z) = z^{2N} - 1$ with roots $\xi^k, k = 0, 1, \dots, 2N - 1$, into $Q_2(z)$ with $2N$ real roots, such that C and C_2 are birationally isomorphic over \mathbf{C} . Consider the roots $\xi, 1, -1$ of P ; we will transform these to the roots $0, 1, -1$ of Q_2 . The only Möbius transformation $\Phi : \mathbf{P}_1(\mathbf{C}) \rightarrow \mathbf{P}_1(\mathbf{C})$ which maps $\xi \rightarrow 0, 1 \rightarrow 1$ and $-1 \rightarrow -1$ is $\Phi(z) = \frac{z-\xi}{1-\xi z} = z_1$, and this element is fixed by $b\sigma$, as can be easily checked!

It is evident that each $a_k = \Phi(\xi^k) \in \mathbf{R}, k = 0, 1, \dots, 2N - 2$, and $a_{2N-1} = \Phi(\xi^{2N-1}) = \infty$, and these numbers a_k are pairwise distinct because Φ is injective. The inverse map of Φ is $\Psi(z) = \frac{z+\xi}{1+\xi z}$ and the rational function $R(z) = P(\Psi(z))$ vanishes at each a_k . But,

$$R(z) = \left(\frac{z + \xi}{1 + \xi z}\right)^{2N} - 1 = \frac{(z + \xi)^{2N} - (1 + \xi z)^{2N}}{(1 + \xi z)^{2N}}$$

and $S(z) = (z + \xi)^{2N} - (1 + \xi z)^{2N} \in \mathbf{C}[z]$ is a polynomial of degree $2N - 1$ which vanishes at a_0, \dots, a_{2N-2} , and whose leading coefficient is $2N(\xi - \xi^{-1}) = 4Ni \sin \frac{\pi}{N} = \rho^2$, where $\rho = \eta e^{\frac{\pi i}{4}}$ and η is a real positive number. Thus, $S(z) = \rho^2 Q_2(z)$, where

$$Q_2(z) = (z - a_0) \dots (z - a_{2N-2})$$

is the polynomial we are looking for.

Note that the map $\alpha : C \rightarrow C_2 : (z, w) \rightarrow (\Phi(z), \rho^{-1}w(1 + \xi\Phi(z))^N)$ is a rational mapping for, if $(z, w) \in C$, then upon setting $\Phi(z) = u$, i.e., $\Psi(u) = z$, we obtain

$$\begin{aligned} w^2 = z^{2N} - 1 &= \Psi(u)^{2N} - 1 = P(\Psi(u)) = R(u) = \frac{S(u)}{(1 + \xi u)^{2N}} \\ &= \frac{\rho^2(u - a_0) \dots (u - a_{2N-2})}{(1 + \xi u)^{2N}}, \end{aligned}$$

i.e.,

$$(u - a_0) \dots (u - a_{2N-2}) = [\rho^{-1}w(1 + \xi u)^N]^2.$$

In fact α is an isomorphism with inverse $\alpha^{-1} : C_2 \rightarrow C : (u, v) \rightarrow (\Psi(u), \frac{\rho v}{(1 + \xi u)^N})$.

Thus C_2 is \mathbf{C} -isomorphic to C and X_2 has N components. We are in a position to prove also that $b\sigma$ represents the symmetry σ_2 of the previous section. In fact, $w_1 = \rho^{-1}w(1 + \xi z_1)^N \in k_g$ is fixed by $b\sigma$ because $b\sigma(w_1) = (\bar{\rho})^{-1} \frac{iw}{z^N} (1 + \bar{\xi} z_1)^N$, and since $\bar{\xi} = \xi^{-1}$ and $\bar{\rho} = -i\rho$, we get

$$b\sigma(w_1) = \frac{i}{\rho} \frac{iw}{z^N} \left(1 + \frac{z_1}{\xi}\right)^N = \frac{w}{\rho} \frac{(\xi + z_1)^N}{z^N} = w_1.$$

Hence the map $\alpha : C \rightarrow C_2$ is $\alpha(z, w) = (z_1, w_1)$ and $\mathbf{R}(z_1, w_1)$ is the fixed field of $b\sigma$, and so it represents σ_2 .

For the sake of completeness we also consider the symmetries $ab^2\sigma$ and $a^N b\sigma$ (the last only for even N) which are fixed point free. The first fixes both w and $u = (1 + \xi)z$. From the equation $w^2 = z^{2N} - 1$ one gets $w^2 = \frac{u^{2N}}{(1+\xi)^{2N}} - 1$. We already noted that $\gamma = \frac{1}{(1+\xi)^{2N}}$ is a real negative number, and so the curve $D_1 = \{(u, w) \in \mathbf{C}^2 : w^2 = \gamma u^{2N} - 1\}$ is \mathbf{C} -isomorphic to C via the map $(z, w) \rightarrow (u, w)$, but the set $D_1(\mathbf{R})$ of real points of D_1 is empty.

Finally, if N is even, $a^N b\sigma$ is a symmetry which is not a conjugate of $ab^2\sigma$, and it maps z to $\frac{-1}{z}$ and w to $\frac{iw}{z}$. Let $M = \frac{N}{2}$. Then $a^N b\sigma$ fixes $y = z - \frac{1}{z}$, $x = i(z + \frac{1}{z})$ and $t = \frac{\lambda w}{z^M}$, where $\lambda = 1 + (-1)^M i$. Then $z = \frac{x+iy}{2i}$, $\frac{1}{z} = \frac{x-iy}{2i}$ and $w = \frac{tz^M}{\lambda}$. After substituting in $z^{2N} - 1 = w^2$, we get $0 = F(x, y, t) = t^2 + \frac{1}{2^{N-2}} \sum_{j=0}^{M-1} (-1)^j \binom{N}{2j+1} x^{N-1-2j} y^{2j+1}$. Also, $G(x, y) = x^2 + y^2 + 4 = 0$. Thus C is \mathbf{C} -isomorphic to $D_2 = \{(x, y, t) \in \mathbf{C}^3 : F(x, y, t) = G(x, y) = 0\}$ and $D_2(\mathbf{R})$ is empty.

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