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ON THE ESSENTIAL SELF-ADJOINTNESS OF THE GENERAL SECOND ORDER ELLIPTIC OPERATORS

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ABSTRACT. In this paper, we give sufficient conditions for the essential self-adjointness of second order elliptic operators. It turns out that these conditions coincide with those for the Schrödinger operator on a manifold whose metric essentially depends on the principal coefficients of a given operator.

1. Basic notation and facts

Let us consider a strongly elliptic second order operator

$$Lu(x) = -\nabla (A(x)\nabla u(x)) + q(x)u(x), \tag{1}$$

where $x \in \mathbb{R}^n$, $n \geq 1$, $A(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a positive definite symmetric matrix, that is the inequality $(A(x)\xi, \xi) > 0$ holds for all $\xi \in \mathbb{R}^n$, $\xi \neq 0$, and $x \in \mathbb{R}^n$. We assume that the potential $q \in L^\infty_{loc}(\mathbb{R}^n)$ is a real-valued function, and that the elements $a^{ij}(x)$ of the matrix $A(x)$ belong to $C^\infty(\mathbb{R}^n)$.

It is easy to see that the operator L_0 defined by the expression (1) on $C_0^\infty(\mathbb{R}^n)$ is symmetric in $L^2(\mathbb{R}^n)$. The essential self-adjointness of the operator (1) depends on the behaviour of the principal coefficients at infinity. S. A. Laptev [14] constructed an example of the operator (1) in $L^2(\mathbb{R}^3)$ which is bounded from below but not essentially self-adjoint. On the other hand, it is a well-known fact (see, for example, P. R. Chernoff [5]) that if the Schrödinger operator (1) with $A = \mathbf{I}$ is bounded from below, then it is essentially self-adjoint. Here $\mathbf{I}$ denotes the identity matrix.

It turns out that some restrictions should be imposed on the growth of the principal coefficients $a^{ij}(x)$. A typical example can be found in the paper of T. Ikebe and T. Kato [12] who assumed that for the function

$$\lambda(r) = \sup_{|x|=r} \{(A(x)\xi, \xi) \,|\, |\xi| = 1\} \tag{2}$$

the following integral diverges

$$\int_0^\infty \lambda^{-\frac{1}{2}}(r) = \infty. \tag{3}$$

They proved that the bounded from below operator (1) satisfying (3) is essentially self-adjoint. This can be obtained as a simple corollary of a more general theorem from [12].

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Yu. B. Orochko [16] used the method of hyperbolic equations to prove the essential self-adjointness of L_0. He considered the Cauchy problem for the wave equation

$$\frac{\partial^2 u(t,x)}{\partial t^2} + Lu(t,x) = 0 \tag{4}$$

with the initial conditions

$$u(0,x) = \varphi(x) \in C_0^\infty(\mathbb{R}^n), \frac{\partial u}{\partial t}(0,x) = 0. \tag{5}$$

It turns out that the problem (4)-(5) with the operator L satisfying (3) has *the finite speed propagation property,* that is, for each $t > 0$ the solution $u(t,x)$ has compact support in $\mathbb{R}^n$, and, therefore, due to Theorem 6.2 [2], the operator L_0 is essentially self-adjoint when it is bounded from below.

A. A. Chumak [7] showed that the question of the finite propagation speed for the problem (4)-(5) is closely related to the question of the completeness of the Riemannian manifold $M = (\mathbb{R}^n, A^{-1}(x))$. If the manifold M is complete, then the finite propagation speed property for the problem (4)-(5) holds, and the operator (1) is essentially self-adjoint if it is bounded from below. In fact, F.S. Rofe-Beketov [20] proved that the completeness of M is necessary and sufficient for the finite propagation speed for the Cauchy problem corresponding to the semibounded below operator of the type (1).

Chumak's paper develops a general geometrical view on the principal coefficients of operator (1). We will also mention the well-known paper of M. Riesz [18] who studied the geodesic curves of the manifold M in connection with the characteristic cone for the problem (4)-(5). It turns out that the characteristic cone for the Cauchy problem (4)-(5) with the vertex at an arbitrary point $(t_0, x_0) \in \mathbb{R} \times \mathbb{R}^n$ locally has the form $\Lambda_{(t_0,x_0)} = \{(t_0 + s, \gamma_{x_0}(s)) \mid \gamma_{x_0} \in \Gamma_{x_0}\}$, where Γ_{x_0} is a set of naturally parametrized geodesics on M such that $\gamma_{x_0}(0) = x_0$. Thus Chumak concluded that the solution of (4)-(5) initially supported on a compact set $K_0 \subset \mathbb{R}^n$ will be supported on a set $K_\tau = \bigcup_{x \in K_0} B(x,\tau)$ after a time $\tau > 0$ (possibly very small but fixed). Here $B(x,\tau)$ denotes an open ball in M with the center at x and the radius τ. Due to the Hopf-Rinoff theorem [1] the closure $\overline{K_\tau}$ of the set K_τ is a compact subset in M. Therefore the finite propagation speed property of (4)-(5) follows easily from the completeness of the manifold M.

The goal of this paper is to prove the essential self-adjointness of the operator L_0 with the potential possibly not bounded from below.

Theorem 1. *Let the operator* (1) *satisfy the following conditions:*

$$\text{the manifold } M = (\mathbb{R}^n, A^{-1}) \text{ is complete;} \tag{6}$$

there exists a function $Q > 1$ such that:

$$\forall x \in M \quad q(x) \geq -Q(x); \tag{7}$$

the function $Q^{-\frac{1}{2}}$ is a Lipschitz function on M:

$$|Q^{-\frac{1}{2}}(x) - Q^{-\frac{1}{2}}(y)| \leq K \cdot dist_M(x,y), \tag{8}$$

where $K > 0$, and $x, y \in M$; for an arbitrary piecewise smooth curve ℓ going out to infinity

$$\int_\ell Q^{-\frac{1}{2}}(x) d\ell = \infty. \tag{9}$$

Then the operator L_0 is essentially self-adjoint.

Remarks. The distance between two points $x, y \in M$ in the condition (8) is denoted by $dist_M(x, y)$. The curvilinear integral (9) is taken with respect to the distance on M. The function Q may be equal to $+\infty$ on a set of positive measure. We define $Q^{-1}(x) = 0$ when $Q(x) = +\infty$.

The sufficient conditions on the essential self-adjointness of the operator L_0 from Theorem 1 simply coincide with those for the Schrödinger operator $H_0 = H|_{C_0^\infty}(M)$ on the complete Riemannian manifold M (see Theorem 1 [15]):

$$H = -\Delta + q(x). \tag{10}$$

Here the operator Δ is the Laplace-Beltrami operator on M

$$\Delta u = \frac{1}{(\det(A^{-1}(x)))^{\frac{1}{2}}} \nabla \left((\det(A^{-1}(x)))^{\frac{1}{2}} A(x) \nabla u \right). \tag{11}$$

We shall show that Theorem 1 generalizes some well-known results giving sufficient conditions on the essential self-adjointness of the operator (1).

Recently M. Braverman [4] extended the methods of this paper to the case of operators on differential forms and proved the essential self-adjointness of Schrödinger-type operators on forms of arbitrary degree with Sears-type conditions on the matrix potential.

2. Proof of Theorem 1

We will use arguments similar to the ones used mainly in the proofs of Theorem 1 in [15] and Theorem 1 in [19].

The following lemma gives us an estimate on the behaviour of an arbitrary function from the domain $D(L_0^*)$ of the adjoint operator L_0^*. It is known from the definition of the operator L_0^* that the domain $D(L_0^*)$ consists of those $u \in L^2(\mathbb{R}^n)$ for which $Lu \in L^2(\mathbb{R}^n)$ (derivatives are taken in the distributional sense). Using the regularity of the solutions of elliptic operators (see, for example, Addendum 2 of [3]) we can conclude that u belongs to the Sobolev space $W^2_{2,loc}(\mathbb{R}^n)$ consisting of those $u \in L^2_{loc}(\mathbb{R}^n)$ such that for all $i, j = \overline{1, n}$ $\frac{\partial u}{\partial x_i}, \frac{\partial^2 u}{\partial x_i \partial x_j} \in L^2_{loc}(\mathbb{R}^n)$.

Lemma 2. *Let the coefficients of the operator* (1) *satisfy the conditions* (6),(7) *and* (8). *Then for an arbitrary $u \in D(L_0^*)$ the following integral converges*

$$\int |\mathbf{grad} u|^2 Q^{-1}(x) dx < \infty. \tag{12}$$

Remarks. The integral (12) is taken over $\mathbb{R}^n$. The vector $\mathbf{grad} u$ is the gradient of the function u on the manifold M, and it can be expressed in local coordinates (that coincide with the Cartesian coordinates in $\mathbb{R}^n$) by

$$\mathbf{grad} u = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial}{\partial x_j}, \tag{13}$$

so

$$|\mathbf{grad} u|^2 = \sum_{i,j=1}^{n} a^{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} = (A \nabla u(x), \nabla u(x)). \tag{14}$$

Proof. Locally the integral (12) exists due to the remark before Lemma 2.

Let $m \in M$ be an arbitrary point. Define the function $d(x) = dist_M(m, x)$. Due to the triangle inequality, the function $d(x)$ is Lipschitz on M, but not necessarily smooth. We will approximate the functions $Q^{-\frac{1}{2}}$ and d using the technique of the paper by M. Gaffney [9] in order to get smooth functions with bounded differentials.

Let $j \in C^\infty(\mathbb{R})$ be a function vanishing outside the interval $[-1, 1]$ such that

$$\int_{-1}^{1} j(t)dt = 1.$$

We further define the function $j_a \in C^\infty(\mathbb{R}^n)$ by the formula $j_a(x) = \frac{1}{a^n} j(\frac{x_1}{a}) \cdots j(\frac{x_n}{a})$ so that

$$\int_{\mathbb{R}^n} j_a(x)dx = 1.$$

Define for an arbitrary continuous function h the mollifier operator

$$(J_a h)(x) = \int_{\mathbb{R}^n} j_a(x-y)h(y)dy.$$

It is known that $J_a h \in C^\infty(\mathbb{R}^n)$ and for an arbitrary compact set $K_0 \subset \mathbb{R}^n$ $J_a h(x) \to h(x)$ uniformly on K_0 when $a \to +0$.

We choose a covering of the closed ball $\overline{B}(m, R+2) \subset M,\ R > 1$ consisting of the coordinate neighbourhoods U_i of the M-diameter $\le K^{-1}R^{-1}$, where K is from (8), and let us denote a partition of unity φ_i corresponding to this covering. Moreover, we eliminate those φ_i such that $\operatorname{supp}\varphi_i \cap \overline{B}(m, R+1) = \emptyset$, and we choose a small $a_0 > 0$ such that for all $a \le a_0$ the integrand functions of the mollifier operators

$$J_a^i h(x) = \int_{U_i} j_a(x-y)\varphi_i(y)h(y)dy \tag{15}$$

vanish at the boundaries of U_i. Therefore we can correctly define the operators $J_a^i : C(M) \to C_0^\infty(U_i)$ for all i and for all $a \le a_0$.

Define for $x \in \overline{B}(m, R+1)$ the functions

$$d_{a,R}(x) = \sum_i (J_a^i d)(x),$$

$$Q_{a,R}^{-\frac{1}{2}}(x) = \sum_i (J_a^i \tilde{Q}_R^{-\frac{1}{2}})(x),$$

where

$$\tilde{Q}_R^{-\frac{1}{2}}(\xi) = \begin{cases} Q^{-\frac{1}{2}}(\xi) - R^{-1}, & \text{if } Q^{-\frac{1}{2}}(\xi) - R^{-1} \ge 0; \\ 0, & \text{otherwise.} \end{cases}$$

Using the inequality (8) and the fact that the M-diameter of U_i is $\le K^{-1}R^{-1}$ we have the inequality

$$Q^{-\frac{1}{2}}(x) - 2R^{-1} \le \tilde{Q}_R^{-\frac{1}{2}}(\xi) \le Q^{-\frac{1}{2}}(x)$$

for all $x, \xi \in U_i$. Using the definition of $Q_{a,R}^{-\frac{1}{2}}$ we can derive

$$(Q^{-\frac{1}{2}}(x) - 2R^{-1}) \sum_i J_a^i \mathbf{1}(x) \le Q_{a,R}^{-\frac{1}{2}}(x) \le Q^{-\frac{1}{2}}(x) \sum_i J_a^i \mathbf{1}(x). \tag{16}$$

Due to Lemma 4 from [9] we can show that uniformly for all $x \in \overline{B}(m, R+1)$

$$\limsup_{a \to +0} |\mathbf{grad} d_{a,R}(x)| \leq 1 \tag{17}$$

and

$$\limsup_{a \to +0} |\mathbf{grad} Q_{a,R}^{-\frac{1}{2}}(x)| \leq K, \tag{18}$$

where K is taken from the condition (8).

Due to the operator mollifier property, we can choose a small $a \leq a_1$ such that $|d_{a,R}(x) - d(x)| \leq \frac{1}{2}$ for all $x \in \overline{B}(m, R+1)$. Therefore the set $\Omega_{a,R} = \{x \in M | \ d_{a,R}(x) < R\}$ has the property $B(m, R-1) \subset \Omega_{a,R} \subset B(m, R+1)$ for all $a \leq a_1$ with a fixed $R > 0$. From now on, we will assume that $a < \min\{a_0, a_1\}$.

Define on $[0, \infty)$ a cut-off function $0 \leq \Phi \leq 1$ which equals one on $[0, \frac{1}{2}]$ and vanishes outside of $[0, 1]$. We further define a function

$$\phi_{a,R}(x) = \begin{cases} \Phi\left(\frac{d_{a,R}(x)}{R}\right) Q_{a,R}^{-\frac{1}{2}}(x), & \text{when } x \in \Omega_{a,R}; \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

It is clear from (16) and (18) that

$$\phi_{a,R}(x) \leq Q_{a,R}^{-\frac{1}{2}}(x) \leq Q^{-\frac{1}{2}}(x) \sum_i J_a^i \mathbf{1}(x) \tag{20}$$

and

$$\limsup_{a \to +0} |\mathbf{grad} \phi_{a,R}(x)| \leq K_1, \tag{21}$$

where K_1 does not depend on R.

We consider the integral

$$I_{a,R}^2 = \int_{\mathbb{R}^n} \phi_{a,R}^2(x) |\mathbf{grad} u(x)|^2 dx. \tag{22}$$

Assuming that the function u is real-valued, we have

$$\begin{aligned} |\phi_{a,R} \mathbf{grad} u|^2 &= \phi_{a,R}^2 (A\nabla u, \nabla u) \\ &= \nabla(\phi_{a,R}^2 u A \nabla u) - 2\phi_{a,R} u (\nabla \phi_{a,R}, A\nabla u) - \phi_{a,R}^2 u \nabla\left(A \nabla u\right). \end{aligned}$$

Using (7), (19), (21), the Schwarz inequality, and the fact that

$$\begin{aligned} |(\nabla \phi_{a,R}, A\nabla u)| &\leq (A\nabla\phi_{a,R}, \nabla\phi_{a,R})^{\frac{1}{2}} (A\nabla u, \nabla u)^{\frac{1}{2}} \\ &= |\mathbf{grad}\phi_{a,R}||\mathbf{grad} u| \end{aligned}$$

we get the estimate

$$\begin{aligned} \limsup_{a \to +0} I_{a,R}^2 &\leq 2K_1 \|u\| \limsup_{a \to +0} I_{a,R} + \limsup_{a \to +0} \int_{\mathbb{R}^n} \phi_{a,R}^2 (uLu - q(x)u^2) dx \\ &\leq 2K_1 \|u\| \limsup_{a \to +0} I_{a,R} + \|u\| \cdot \|Lu\| + \|u\|^2. \end{aligned}$$

Therefore we obtain

$$\limsup_{R \to \infty} \limsup_{a \to +0} I_{a,R} < \infty.$$

This last inequality, together with (16) and (19), gives us the estimate (12). □

As in [15], we make the following

Definition. We define the generalized distance between two points $x, y \in M$ as

$$\rho(x,y) = \inf_{\ell} \int_{\ell} Q^{-\frac{1}{2}} d\ell,$$

where infimum is taken over all piecewise smooth curves connecting x and y.

We define the function $P(x) = \rho(m,x)$ as the generalized distance between a fixed point m and an arbitrary point x. It is clear that the function P is Lipschitz and, due to (9), $P(x) \to \infty$ uniformly when $dist_M(m,x) \to \infty$.

The following lemma gives an estimate on the Lipschitz constant for P

Lemma 3. *For arbitrary $x, y \in M$, we have the inequality*

$$|P(x) - P(y)| \le Q^{-\frac{1}{2}}(x) dist_M(x,y) + \frac{K}{2}(dist_M(x,y))^2. \tag{23}$$

Proof. See Lemma 2 [15]. □

Proof of Theorem 1. We have to show that the operator L_0^* is symmetric.

We define the set $\Omega_t = \{x \in M \mid P(x) \le t\}$. It is easy to see from the definition of P, that for an arbitrary $t > 0$, Ω_t is a compact set on M. We choose a covering U_i of Ω_{t+2} with M-diameter $\le \frac{1}{2}$. We take a partition of unity ψ_i corresponding to this covering, and we do not take into consideration those ψ_i for which $\mathrm{supp}\psi_i \cap \Omega_{t+1} = \emptyset$.

Define on Ω_{t+1} the function $\tilde{P}_{a,t} = \sum_i J_a^i(P)$. We take such a small $a \le a_0$ that for an arbitrary i the integrand in the expression for $P_{a,t}$ vanishes at the boundary of U_i and, furthermore, that the inequality $|\tilde{P}_{a,t} - P| \le \frac{1}{2}$ is satisfied.

We denote by $P_{a,t}$ the function

$$P_{a,t} = \begin{cases} \tilde{P}_{a,t}, & \text{when } \tilde{P}_{a,t} \le t; \\ t, & \text{otherwise.} \end{cases} \tag{24}$$

The function $P_{a,t}$ is piecewise smooth, and due to (23) and Lemma 4 [9] it satisfies almost everywhere the inequality

$$\limsup_{a \to +0} |\mathbf{grad} P_{a,t}(x)| \le Q^{-\frac{1}{2}}(x). \tag{25}$$

We estimate the integral $I_{a,t}$ for arbitrary functions $u, v \in D(L_0^*)$

$$\begin{aligned} I_{a,t} &= \int_{\mathbb{R}^n} \left(1 - \frac{P_{a,t}}{t}\right)(u \nabla A \nabla v - v \nabla A \nabla u) dx \\ &= \frac{1}{t} \int_{\mathbb{R}^n} (uA\nabla v - vA\nabla u, \nabla P_{a,t}) dx. \end{aligned}$$

We obtain from (25), Lemma 2 and the Lebesque dominated convergence theorem

$$\begin{aligned} \limsup_{a \to +0} I_{a,t} &\le \limsup_{a \to +0} \frac{1}{t} \int_{\mathbb{R}^n} (|u| \cdot |\mathbf{grad} v| + |v| \cdot |\mathbf{grad} u|) \cdot |\mathbf{grad} P_{a,t}| dx \\ &\le \frac{1}{t} \int_{\mathbb{R}^n} (|u| \cdot |\mathbf{grad} v| + |v| \cdot |\mathbf{grad} u|) Q^{-\frac{1}{2}} dx \\ &\le \frac{1}{t} \, \|u\| \left(\int_{\mathbb{R}^n} |\mathbf{grad} v|^2 Q^{-1} dx \right)^{\frac{1}{2}} + \frac{1}{t} \, \|v\| \left(\int_{\mathbb{R}^n} |\mathbf{grad} u|^2 Q^{-1} dx \right)^{\frac{1}{2}} \le \frac{c}{t}, \end{aligned}$$

where c is a positive constant not depending on t. Since the integral $I_{a,m}$ converges absolutely, then we have

$$\lim_{t\to\infty} \limsup_{a\to+0} I_{a,t} = 0,$$

and this proves the symmetry of operator L_0^* by Fatou's theorem. □

Remark. From the proof of Theorem 1, we can see that the condition (8) was imposed to prove the estimate (12). Actually we may consider the condition (8) outside some compact subset in $\mathbb{R}^n$ because of the local smoothness of elements $D(L_0^*)$ (see the remark before Lemma 2). The condition (8) is essential for the essential self-adjointness of the operator (1). We refer the reader to Example 2 of Appendix to X.1 of [17] where an example was constructed of a one-dimensional Schrödinger operator that is not self-adjoint with the potential satisfying (9) at infinity. A potential q satisfying (9) is called *classically complete at infinity* on M (see a detailed survey about this in [17], Appendix to X.1).

3. Some corollaries of Theorem 1

In this section, we discuss some results related to the essential self-adjointness of the operator (1). First of all we give another proof of Theorem 2 [8].

Theorem 4. *Let the operator* (1) *satisfy the following.*

There exists a nonnegative function $\chi(t) \in C^1([0,\infty))$ such that

$$\int_0^\infty \chi\lambda^{-\frac{1}{2}} = \infty, \tag{26}$$

$$\chi^2(t) \le a\left[\int_0^t \chi\lambda^{-\frac{1}{2}}\right]^2 + b, \tag{27}$$

$$\begin{aligned}\chi^2(|x|)q(x) \ge -c\left[\int_0^{|x|} \chi\lambda^{-\frac{1}{2}}\right]^2 - d \\ + \gamma\sum_{i,j=1}^n a_{jk}(x)\frac{x_jx_k}{|x|^2}\left[\frac{d}{dt}\chi(|x|)\right]^2,\end{aligned} \tag{28}$$

where $1 < \gamma < \infty$, and a, b, c and d are positive constants, and the function λ is defined in (2). *Then the operator* (1) *is essentially self-adjoint.*

Remark. Theorem 2 [8] gives conditions sufficient for the essential self-adjointness of the general Schrödinger operators with magnetic potentials. We present here a variant of Theorem 2 for the operator (1) without a magnetic potential and principal coefficients which are more smooth. It was shown in [8] that magnetic potentials, when assumed smooth enough, do not influence the essential self-adjointness of such operators.

Proof. We will show that conditions (26) and (27) imply completeness of the manifold M. Divide both sides of (27) by $\left[\int_0^t \chi\lambda^{-\frac{1}{2}}\right]^2$ and take square roots. We obtain

$$\frac{\chi(t)}{\int_0^t \chi\lambda^{-\frac{1}{2}}} \le a'$$

for some sufficiently large t and for some positive constant a'. The last inequality, when multiplied through by $\lambda^{-\frac{1}{2}}(t)$ and integrated over the interval $[t_0, T]$, leads to the inequality

$$\ln \int_0^T \chi\lambda^{-\frac{1}{2}} - \ln \int_0^{t_0} \chi\lambda^{-\frac{1}{2}} \leq a' \int_{t_0}^T \lambda^{-\frac{1}{2}}.$$

Hence the condition (3) is satisfied, and M is complete.

Using the condition (26) we can find a sufficiently large constant $R_0 > 0$ such that $b < a\left[\int_0^t \chi\lambda^{-\frac{1}{2}}\right]^2$ and $d < c\left[\int_0^t \chi\lambda^{-\frac{1}{2}}\right]^2$, where a, b, c and d are taken from Theorem 4. Therefore we can replace the inequalities (27) and (28) outside the closed ball of the radius R_0 by the inequalities

$$\chi^2(|x|) \leq 2a\left[\int_0^{|x|} \chi\lambda^{-\frac{1}{2}}\right]^2 \tag{29}$$

and

$$\chi^2(|x|)q(x) \geq -2c\left[\int_0^{|x|} \chi\lambda^{-\frac{1}{2}}\right]^2 + \gamma \sum_{i,j=1}^{n} a_{jk}(x)\frac{x_j x_k}{|x|^2}\left[\frac{d}{dt}\chi(|x|)\right]^2, \tag{30}$$

respectively.

Denote the right-hand side of (30) by $h(x)$, and we define a minorant Q for $|x| > R_0$ using (30) and the remark after the proof of Theorem 1 as follows

$$Q(x) = \begin{cases} 1, & \text{if } h(x) \geq \chi^2(|x|); \\ 2c\left[\frac{\int_0^{|x|} \chi\lambda^{-\frac{1}{2}}}{\chi(|x|)}\right]^2, & \text{if } h(x) < \chi^2(|x|). \end{cases}$$

Then we have

$$Q^{-\frac{1}{2}} = \begin{cases} 1, & \text{if } h(x) \geq \chi^2(|x|); \\ (2c)^{-\frac{1}{2}}\left[\frac{\chi(|x|)}{\int_0^{|x|} \chi\lambda^{-\frac{1}{2}}}\right], & \text{if } h(x) < \chi^2(|x|). \end{cases} \tag{31}$$

We will prove (8) for $Q^{-\frac{1}{2}}$ on the subset $H_0 = \{x \,|\, h(x) < \chi^2(|x|), |x| > R_0\}$. Using (29), (30) and the definition of H_0, we have the following inequality

$$\begin{aligned} |\,\mathbf{grad}\chi\,|^2 &= \sum_{i,j=1}^{n} a_{jk}(x)\frac{x_j x_k}{|x|^2}\left[\frac{d}{dt}\chi(|x|)\right]^2 \\ &\leq \gamma^{-1}\chi^2 + 2c\gamma^{-1}\left[\int_0^{|x|} \chi\lambda^{-\frac{1}{2}}\right]^2 \leq 2(a+c)\gamma^{-1}\left[\int_0^{|x|} \chi\lambda^{-\frac{1}{2}}\right]^2 \\ &= A^2\left[\int_0^{|x|} \chi\lambda^{-\frac{1}{2}}\right]^2, \end{aligned}$$

where A is defined from the relation $A^2 = 2(a+c)\gamma^{-1}$. Therefore the gradient of χ on H_0 is estimated by

$$|\,\mathbf{grad}\chi\,| \le A \int_0^{|x|} \chi\lambda^{-\frac{1}{2}}.$$

Using the last relation, the definition of λ and (29) we have

$$|\,\mathbf{grad}Q^{-\frac{1}{2}}\,| = \left|\mathbf{grad}\left(\frac{\chi(|x|)}{\int_0^{|x|}\chi\lambda^{-\frac{1}{2}}}\right)\right| = \left|\frac{\mathbf{grad}\chi\,\int_0^{|x|}\chi\lambda^{-\frac{1}{2}} - \chi^2\,\lambda^{-\frac{1}{2}}\,\mathbf{grad}|x|}{\left[\int_0^{|x|}\chi\lambda^{-\frac{1}{2}}\right]^2}\right|$$

$$\le \frac{|\,\mathbf{grad}\chi\,|}{\int_0^{|x|}\chi\lambda^{-\frac{1}{2}}} + \frac{\chi^2\left[\lambda^{-1}\sum_{j,k}a_{jk}(x)\,\frac{x_jx_k}{|x|^2}\right]^{\frac{1}{2}}}{\left[\int_0^{|x|}\chi\lambda^{-\frac{1}{2}}\right]^2} \le c,$$

so (8) is fulfilled.

To prove (9), we choose an arbitrary curve Γ going out to infinity on M. We will use the parameter r, the Euclidean distance from the origin, on the curve Γ. Using (31) and (29), we have for sufficiently small $\delta > 0$ and for $|x| > R_0$

$$\int_\Gamma Q^{-\frac{1}{2}}dl \ge \int_\Gamma \min\left(1,\,(2c)^{-\frac{1}{2}}\,\frac{\chi(|x|)}{\int_0^{|x|}\chi\,\lambda^{-\frac{1}{2}}}\right)dl$$

$$\ge \delta\int_\Gamma \frac{\chi(|x|)}{\int_0^{|x|}\chi\,\lambda^{-\frac{1}{2}}}\,dl \ge \delta\int_\Gamma \frac{\chi(|x|)\,\lambda^{-\frac{1}{2}}(|x|)}{\int_0^{|x|}\chi\,\lambda^{-\frac{1}{2}}}\,dr$$

$$= \delta\lim_{R\to\infty}\ln\left(\int_0^R\chi\,\lambda^{-\frac{1}{2}}\right) - \delta\,\ln\left(\int_0^{R_0}\chi\,\lambda^{-\frac{1}{2}}\right) = \infty.$$

□

The next theorem is a slight modification of a theorem due to T. Ikebe and T. Kato [12] (see also [6] and [16] for similar results).

Theorem 5. *Let the function λ from* (2) *satisfy* (3)*. The potential q of the operator* (1) *is estimated from below by*

$$q(x) \ge -(a + b\,\theta(r)\,)^2, \tag{32}$$

where $a, b \in \mathbb{R}^1$ are fixed, $\theta(r) = \int_0^r\lambda^{-\frac{1}{2}}$. Then the operator (1) *is essentially self-adjoint.*

Proof. Using (3), we can define the minorant $Q(r) = b^2\theta^2(r)$ for large r. We have

$$|\mathbf{grad}\theta^{-1}|^2 = \theta^{-2}\lambda^{-1}\sum_{ij}a_{ij}\frac{x_ix_j}{|x|^2} \le const,$$

so the condition (8) is fulfilled. We have an estimate

$$\int_\ell Q^{-\frac{1}{2}}d\ell = b^{-1}\int_\ell\left(\int_0^r\lambda^{-\frac{1}{2}}\right)^{-1}d\ell \ge b^{-1}\int_0^\infty\left(\int_0^r\lambda^{-\frac{1}{2}}\right)^{-1}\lambda^{-\frac{1}{2}}(r)dr = \infty,$$

so the condition (9) is satisfied. □

Remark. It was shown in [6], [12] and [16] that the functional inequality (32) can be replaced by a weaker operator inequality $L \ge -(a + b\,\theta(r)\,)^2$.

Theorems 1 and 4 in [19] give sufficient conditions on the essential self-adjointness of the operator (1) in cases when the matrix A is either the identity matrix or it is a diagonal matrix $A = p(x)\mathbf{I}$ with a positive smooth function p. The next theorem gives a generalization of these results for an arbitrary positive matrix A.

Theorem 6. *Assume that the condition* (6) *holds and that the potential q of the operator* (1) *satisfies* (7) *with a new minorant Q for which the condition* (8) *holds. We further assume that there exist a smooth function $0 < P \to \infty$ for $|x| \to \infty$ and a sequence of domains Ω_k, $\overline{\Omega_k} \subset \Omega_{k+1}$ with smooth boundaries $S_k = \partial\Omega_k$,*

$$P(x)|_{S_k} = N_k \to \infty;\ P(x) \le N_k,\ x \in \Omega_k, \tag{33}$$

such that

$$|\mathbf{grad} P(x)| \le C_k Q^{-\frac{1}{2}}(x),\ x \in \Omega_k,\ C_k = \overline{o}(N_k) \tag{34}$$

for $k \to \infty$. (The possibility that $|\mathbf{grad} P(x)| = 0$ on a set of positive measure is not excluded.)

Then the operator (1) *is essentially self-adjoint.*

Proof. It is enough to prove that the minorant Q satisfies (9). Let us choose an arbitrary piecewise smooth curve Γ going out to infinity, and we assume that $\Gamma_k = \Gamma \cap \Omega_k$ is nonempty. Due to Sard's theorem, for almost every $t > 0$ the preimage $P^{-1}(t)$ is a regular submanifold in M, and the Riemannian metric on M in its neighbourhood can be represented by

$$d\ell^2 = \frac{1}{|\mathbf{grad} P(x)|^2} dt^2 + d\sigma^2,$$

where $d\sigma^2$ is an induced metric on the submanifold $P^{-1}(t)$. From this, in particular, we obtain

$$|\mathbf{grad} P| d\ell \ge dt. \tag{35}$$

From the condition (34), we can derive the inequality

$$|\mathbf{grad} P(x)|^{-1} Q^{-\frac{1}{2}}(x) \ge \frac{1}{C_k},\ x \in \Omega_k \tag{36}$$

for a.e. $t \in [0, N_k]$. Using (35) and (36) we have

$$\begin{aligned}\int_{\Gamma_k} Q^{-\frac{1}{2}}(x) d\ell &= \int_{\Gamma_k} |\mathbf{grad} P|^{-1} Q^{-\frac{1}{2}} |\mathbf{grad} P| d\ell \\ &\ge \int_{t_0}^{N_k} \frac{1}{C_k} dt = \frac{N_k - t_0}{C_k} \to \infty.\end{aligned}$$

□

The next theorem gives sufficient conditions on the essential self-adjointness of the operator (1) in case when an estimate from below on the potential q is given on a sequence of disjoint layers going out to infinity. This is a variant of the Hartman-Ismagilov criterion given in the one-dimensional case for the operator (1) whose principal coefficient A equals one (see for details [13] and [10]). We will not provide a proof for this theorem, noting that it can be proved using the scheme of the proof of Theorem 2 in [19].

Theorem 7. *Let* $\{\Omega_k\}_{k=0}^{\infty}$, $\overline{\Omega_k} \subset \Omega_{k+1}$ *be a sequence of simply-connected bounded regions such that* $\bigcup \Omega_k = \mathbb{R}^n$. *Let*

$$q(x) \geq -C\gamma_k,\ x \in T_k = \Omega_{2k+1} \backslash \Omega_{2k}, \tag{37}$$

where $\gamma_k \geq 1$ *and* $C > 0$ *is independent of* k. *If*

$$\sum_{k=0}^{\infty} \min\{h_k^2,\ h_k \gamma_k^{-\frac{1}{2}}\} = \infty, \tag{38}$$

where $h_k = dist_M(\Omega_{2k},\ \mathbb{R}^n \backslash \Omega_{2k+1})$ *is the minimum* M*-thickness of* T_k, *then the operator* (1) *is essentially self–adjoint.*

Consider the following example given by B. Hellwig [11].

Example. Let

$$Bu = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(e^{-|x|^2}\, \frac{\partial u}{\partial x_i}\right) + q(x)u$$

with

$$A(x) = e^{-|x|^2}\mathbf{I},$$
$$q(x) \geq -\mathrm{Const.}\, e^{|x|^2}|x|^2 \ln^2(|x|).$$

The inverse matrix equals $A^{-1} = e^{|x|^2}\mathbf{I}$, so the manifold $M = \big(\mathbb{R}^n,\ A^{-1}(x)\big)$ is obviously complete. The norm of the gradient of the function

$$Q^{-\frac{1}{2}}(x) = \frac{e^{-\frac{1}{2}\,|x|^2}}{|x|\ln(|x|)}$$

is uniformly bounded with respect to the metric given by A^{-1}, hence the condition (8) is satisfied. We estimate the integral of $Q^{-\frac{1}{2}}$ over an arbitrary piecewise smooth curve Γ going out to infinity

$$\int_{\Gamma} Q^{-\frac{1}{2}}\, dl \geq \int_{r_0}^{\infty} \frac{e^{-\frac{1}{2}\,r^2}}{r\ln r}\, e^{\frac{1}{2}\,r^2}\, dr = \int_{r_0}^{\infty} \frac{dr}{r\ln r} = \infty.$$

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