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CLOSURES OF TOTALLY GEODESIC IMMERSIONS INTO LOCALLY SYMMETRIC SPACES OF NONCOMPACT TYPE

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Abstract. It is established that if $\mathcal{M}_1$ and $\mathcal{M}_2$ are connected locally symmetric spaces of noncompact type where $\mathcal{M}_2$ has finite volume, and $\phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a totally geodesic immersion, then the closure of $\phi(\mathcal{M}_1)$ in $\mathcal{M}_2$ is an immersed "algebraic" submanifold. It is also shown that if in addition, the real ranks of $\mathcal{M}_1$ and $\mathcal{M}_2$ are equal, then the the closure of $\phi(\mathcal{M}_1)$ in $\mathcal{M}_2$ is a totally geodesic submanifold of $\mathcal{M}_2$. The proof is a straightforward application of Ratner's Theorem combined with the structure theory of symmetric spaces.

1. Introduction

In this paper, we consider totally geodesic maps and totally geodesic submanifolds. Let $\mathcal{D}$ and $\mathcal{M}$ be Riemannian manifolds. An immersion ϕ from $\mathcal{D}$ to $\mathcal{M}$ is called *totally geodesic* if for every geodesic γ in $\mathcal{D}$, its image $\phi \circ \gamma$ is a geodesic in $\mathcal{M}$. A submanifold $\mathcal{N}$ of a Riemannian manifold $\mathcal{M}$ is called *totally geodesic* if the inclusion map $i : \mathcal{N} \to \mathcal{M}$ is a totally geodesic map.

Let $\mathcal{M}_2$ be a locally symmetric space of noncompact type, and let G_2 be the connected component of the identity in the isometry group of its universal cover $\widetilde{\mathcal{M}}_2$. We let G_2 act on $\mathcal{M}_2$ from the right. Let $p : \widetilde{\mathcal{M}}_2 \to \mathcal{M}_2$ be the covering map.

We will prove

Theorem 1.1. *Let $\mathcal{M}_2$ be a connected locally symmetric space of noncompact type with finite volume. Let $\mathcal{M}_1$ be a connected locally symmetric space of noncompact type. Let ϕ be a totally geodesic immersion of $\mathcal{M}_1$ into $\mathcal{M}_2$. Then the closure in $\mathcal{M}_2$ of the set $\phi(\mathcal{M}_1)$ is an immersed submanifold of $\mathcal{M}_2$ of the form $p(\tilde{x}H)$, where $\tilde{x}$ is a point in $\widetilde{\mathcal{M}}_2$ and $\tilde{x}H$ is the orbit of $\tilde{x}$ under a subgroup H of G_2. If in addition, the rank of $\mathcal{M}_1$ is equal to the rank of $\mathcal{M}_2$, then the closure of $\phi(\mathcal{M}_1)$ is a totally geodesic submanifold of $\mathcal{M}_2$.*

Furthermore, a result due to N. Shah [9], describes the algebraic structure of the subgroup H: if $\mathcal{M}_2$ is either rank one or compact, then H must be a reductive group with compact center.

In [10], A. Zeghib demonstrated that in a compact manifold with variable negative curvature, every immersed geodesic hypersurface of dimension two or more is

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compact, and that there are but a finite number of these objects. In the paper [8], N. Shah proved Theorem 1.1 in the case that $\mathcal{M}_2$ is a compact manifold of dimension $n > 2$ with constant negative curvature and $\mathcal{M}_1$ is equal to $(n-1)$-dimensional hyperbolic space $\mathcal{H}^n$. Shah has informed the author that he also knew proofs for the finite volume and higher codimension cases. After the announcement of M. Ratner's powerful results describing the closures of unipotent orbits in quotients of Lie groups, Shah observed that his theorem followed from Ratner's Theorem and that the compactness hypothesis could be weakened to finite volume [2].

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1.2. **Notation and terminology.** We will use capital letters G, H, etc. to denote Lie groups, and we will denote the corresponding Lie algebras by the corresponding gothic letters $\mathfrak{g}, \mathfrak{h}$, etc. If $\mathfrak{h}$ is a subalgebra of the Lie algebra $\mathfrak{g}$, H will denote the connected subgroup of G corresponding to $\mathfrak{h}$. For an element g of G, we will use c_g to denote conjugation by g. The terms "Cartan subalgebra" and "rank" will always mean an $\mathbf{R}$-split Cartan subalgebra and real rank.

2. Ratner's Theorem

Let G be a second countable real Lie group. Suppose Γ is a lattice in G. The group G acts on the quotient G/Γ. A subset A of G/Γ is called *homogeneous* if there is a closed subgroup H of G and a point x in G/Γ such that A is the orbit Hx of H through x and $H \cap g\Gamma g^{-1}$ is a lattice in H, where $gx \in A$.

For g in G, let $\mathrm{Ad}_g : \mathfrak{g} \to \mathfrak{g}$ denote the adjoint map of g defined by $\mathrm{Ad}_g(X) = dc_g(X)$ for X in $\mathfrak{g}$. Let $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ denote the differential of $\mathrm{Ad} : G \to GL(\mathfrak{g})$. An element u of G is called *unipotent* if $\mathrm{Ad}_u - I$ is nilpotent. A subgroup of G is unipotent if all its elements are unipotent.

Let U be a unipotent subgroup of G. In a series of articles [3], [4], [5], [6], M. Ratner has completely analyzed the closures of orbits Ux of unipotent subgroups and classified the ergodic U-invariant Borel probability measures on G/Γ. See [7] for a survey. We will use the following very powerful theorem of M. Ratner on orbit closures of orbits of unipotent groups.

Theorem 2.1 (M. Ratner [6])**.** *Let G be a connected Lie group and U a connected subgroup of G generated by unipotent elements of G. Then, for any lattice Γ in G and any x in G/Γ, the closure of the orbit Ux in G/Γ is homogeneous.*

An element of X of $\mathfrak{g}$ is called *diagonal* for a nilpotent element Y of $\mathfrak{g}$ if there is a nilpotent element $Y^* \in \mathfrak{g}$ such that $\mathrm{ad}(Y^*)(Y) = X$, $\mathrm{ad}(X)(Y) = -2Y$ and $\mathrm{ad}(X)(Y^*) = 2Y^*$. The elements Y and Y^* generate a subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}(2, \mathbf{R})$ so that X is mapped to $\mathrm{diag}(1, -1)$. The one-parameter subgroup $A = \{\exp(tX) \,|\, t \in \mathbf{R}\}$ of G is called *diagonal* for the one-parameter unipotent subgroup $U = \{\exp(tY) \,|\, t \in \mathbf{R}\}$ of G. Given A diagonal for U, we will denote by $SL_2(U, A)$ the subgroup generated by $A = \{\exp tX\}_{t\in\mathbf{R}}$, $U = \{\exp tY\}_{t\in\mathbf{R}}$ and $U^* = \{\exp tY^*\}_{t\in\mathbf{R}}$.

The next lemma follows from Proposition 2.1 in [5]. It will be used in the part of the proof of Theorem 1.1 that shows that the submanifold is totally geodesic in certain cases.

Proposition 2.2 (M. Ratner)**.** *Let G be a connected Lie group and U a connected subgroup generated by unipotent elements of G. Let $A_1, \dots, A_n$ be diagonal for some one-parameter subgroups $U_1, \dots, U_n$ of U. Let L be the subgroup generated by U and $A_1, \dots, A_n$, and let L' be the subgroup generated by U and $SL_2(U_i, A_i)$. Then $\overline{Lx} = \overline{L'x}$ for all $x \in G/\Gamma$.*

3. Globally symmetric spaces and their totally geodesic submanifolds

Let $\mathcal{M}$ be a globally symmetric space of noncompact type, and let G be the connected component of the identity in its isometry group. Fix a point p in $\mathcal{M}$. Then $\mathcal{M}$ can be identified with the homogeneous space $K\backslash G$, where K is the stabilizer in G of the point p. The group G is a semisimple Lie group of noncompact type, and K is a maximal compact subgroup of G. The point p induces a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ of $\mathfrak{g}$ and a corresponding Cartan involution. We can then identify $T_p\mathcal{M}$ with $\mathfrak{p}$.

Using this set-up, the next theorem from [1] gives an algebraic description of totally geodesic submanifolds.

Theorem 3.1. *Suppose that $\mathcal{N}$ is a totally geodesic submanifold of $\mathcal{M}$ which contains the point p.*

Then $\mathcal{N}$ is a globally symmetric space. Let $\mathfrak{p}^*$ be the subspace of $\mathfrak{p}$ which is identified with $T_p\mathcal{N}$ under the identification of $T_p\mathcal{N}$ and $\mathfrak{p}$. Let $\mathfrak{k}^* = [\mathfrak{p}^*, \mathfrak{p}^*]$ and let $\mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{p}^*$. Let G^* and K^* be the corresponding subgroups of G. The submanifold $\mathcal{N}$ has the structure of G^*/K^* and the corresponding Cartan decomposition relative to the point p is $\mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{p}^*$, with the Cartan involution for $\mathfrak{g}^*$ equal to the restriction of the Cartan involution for $\mathfrak{g}$ to $\mathfrak{g}^*$.

Conversely, if $\mathfrak{g}^*$ is a subalgebra of $\mathfrak{g}$ which is invariant under the Cartan involution for $\mathfrak{g}$, then it gives rise to a totally geodesic submanifold $\mathcal{N}$ of $\mathcal{M}$. Let $\mathfrak{p}^* = \mathfrak{g}^* \cap \mathfrak{p}$. Then

$$\mathcal{N} = \{\exp(X)K \mid X \in \mathfrak{p}^*\}$$

is a totally geodesic submanifold of $\mathcal{M}$.

4. Proof of Theorem 1.1

First, note that we may assume that $\mathcal{M}_1$ is simply connected. If it is not, let ρ_1 be a locally isometric covering map of $\mathcal{M}_1$ by its universal cover $\widetilde{\mathcal{M}}_1$. Then the map $\phi \circ \rho_1$ is a totally geodesic immersion of $\widetilde{\mathcal{M}}_1$ into $\mathcal{M}_2$ so that the closure of $(\phi \circ \rho_1)(\widetilde{\mathcal{M}}_1)$ is equal to the closure of $\phi(\mathcal{M}_1)$.

Now, we find algebraic descriptions of $\mathcal{M}_1$ and $\mathcal{M}_2$ that are compatible through the map ϕ. Let ρ be a locally isometric covering map of $\mathcal{M}_2$ by its universal cover $\widetilde{\mathcal{M}}_2$. The map ϕ lifts to a one-to-one map $\tilde{\phi} : \mathcal{M}_1 \to \widetilde{\mathcal{M}}_2$ such that $\rho \circ \tilde{\phi} = \phi$. Note that $\tilde{\phi}$ is also a totally geodesic isometric immersion, and the submanifold $\tilde{\phi}(\mathcal{M}_1)$ is a totally geodesic submanifold of $\widetilde{\mathcal{M}}_2$. Let $\tilde{x}$ be a point in $\tilde{\phi}(\mathcal{M}_1)$ and let $x = \rho(\tilde{x})$.

Since the map ϕ is totally geodesic, the geodesic symmetries are preserved under ϕ. Since the geodesic symmetries determine the algebraic structure of G_1 and G_2, we may assume, by rescaling the metric on irreducible components of $\mathcal{M}_1$ if necessary, that the map ϕ is a local isometry.

Let G_2 denote the connected component of the identity in the isometry group of $\widetilde{\mathcal{M}}_2$. The globally symmetric space $\widetilde{\mathcal{M}}_2$ has the structure of $K_2\backslash G_2$, where K_2 is the stabilizer of the point $\tilde{x}$ in G_2. The manifold $\mathcal{M}_2$ then has the structure of $K_2\backslash G_2/\Gamma$, where Γ is the group of deck transformations for the covering map ρ. Let p denote the covering map $p : G_2/\Gamma \to K_2\backslash G_2/\Gamma$. Let $\mathfrak{g}_2 = \mathfrak{k}_2 + \mathfrak{p}_2$ be the Cartan decomposition of $\mathfrak{g}_2$ induced by $\tilde{x}$ and let θ be the corresponding Cartan involution.

By Theorem 3.1, the submanifold $\tilde{\phi}(\mathcal{M}_1)$ is globally symmetric. Let G_1 be the subgroup of G_2 that is the connected component of the identity in the isometry group of $\tilde{\phi}(\mathcal{M}_1)$. Then $\tilde{\phi}(\mathcal{M}_1)$ is the orbit $\tilde{x}G_1$.

Let $\hat{x}$ be the identity coset of G_2/Γ, so that $p(\hat{x}) = x$ and $\phi(\mathcal{M}_1) = p(G_1\hat{x})$. If G is a connected semisimple Lie group of noncompact type, then a semisimple subgroup of G without compact factors is generated by unipotent elements of G, so G_1 is generated by unipotent elements. Then by Ratner's Theorem 2.1, the closure of the orbit $G_1\hat{x}$ in G_2/Γ is homogeneous. There is a closed subgroup H of G_2 such that the closure of $G_1\hat{x}$ equals the orbit $H\hat{x}$ and $H \cap \Gamma$ is a lattice in H.

The projection $p(H\hat{x})$ of the homogeneous set $H\hat{x}$ equals $\overline{\phi(\mathcal{M}_1)}$. Since the kernel of p is compact, if $\mathcal{A}$ is a set contained in G/Γ, then $p(\overline{\mathcal{A}}) = \overline{p(\mathcal{A})}$. Thus,

$$\overline{\phi(\mathcal{M}_1)} = \overline{p(G_1\hat{x})} = p(\overline{G_1\hat{x}}) = p(H\hat{x}).$$

Since $\Gamma \cap H$ is a lattice in H and Γ has no torsion, the set $H\hat{x}$ is a submanifold of G_2/Γ with the structure of $H/\Gamma \cap H$. Then the set $p(H\hat{x})$ is a submanifold of $\mathcal{M}_2$ with the structure of $K_2 \cap H\backslash H/\Gamma \cap H$.

Now we would like to show that if the ranks of $\mathcal{M}_1$ and $\mathcal{M}_2$ are equal, then the subset $p(H\hat{x})$ of $\mathcal{M}_2$ is a totally geodesic submanifold.

Let $\mathfrak{a}$ be a Cartan subalgebra for $\mathfrak{g}_1$ contained in $\mathfrak{p}_1$. Since the ranks of $\mathcal{M}_1$ and $\mathcal{M}_2$ are equal, $\mathfrak{a}$ is a Cartan subalgebra for $\mathfrak{g}_2$. Let

$$\mathfrak{g}_2 = \mathfrak{g}_0^2 + \sum_{\beta\in\Lambda_2} \mathfrak{g}_\beta$$

be the corresponding root space decomposition. Because $\mathfrak{g}_1 < \mathfrak{h}$, we know that $\mathfrak{a} < \mathfrak{h}$. The Cartan decomposition for $\mathfrak{g}_2$ restricts to a Cartan decomposition for $\mathfrak{h}$:

$$\mathfrak{h} = \mathfrak{h}_0 \oplus \sum_{\alpha\in\Lambda} \mathfrak{h}_\alpha.$$

Note that for all $\alpha \in \Lambda$, the root space $\mathfrak{h}_\alpha$ is contained in a root space $\mathfrak{g}_\beta$ for some β in Λ_2.

The subalgebra $\mathfrak{k}_2$ of $\mathfrak{g}_2$ is pointwise fixed under θ and all one-dimensional subspaces of $\mathfrak{a}$ are fixed under θ, and hence any subspace of $\mathfrak{k}_2 \oplus \mathfrak{a}$ is preserved under θ. Since $\mathfrak{h}_0$ is contained in $\mathfrak{g}_2^0 = (\mathfrak{k}_2 \cap \mathfrak{g}_2^0) \oplus \mathfrak{a}$, $\mathfrak{h}_0$ is invariant under θ.

Now suppose that $\mathfrak{b}_\alpha$ is a one-dimensional subspace of $\mathfrak{h}_\alpha$ for some $\alpha \in \Lambda$. The subalgebra $\mathfrak{sl}_\alpha$ of $\mathfrak{g}_2$ spanned by the one-dimensional subspaces $\mathfrak{b}_\alpha$, $\theta(\mathfrak{b}_\alpha)$ and $[\mathfrak{b}_\alpha, \theta(\mathfrak{b}_\alpha)]$ is isomorphic to $\mathfrak{sl}(2, \mathbf{R})$, and $[\mathfrak{b}_\alpha, \theta(\mathfrak{b}_\alpha)] < \mathfrak{a}$. Since $\mathfrak{b}_\alpha \oplus [\mathfrak{b}_\alpha, \theta(\mathfrak{b}_\alpha)] < \mathfrak{h}$, by Proposition 2.2, $\mathfrak{sl}_\alpha < \mathfrak{h}$. Letting α vary over Λ, we see that $\sum_{\alpha\in\Lambda} \mathfrak{h}_\alpha$ is invariant under θ. Already knowing that $\mathfrak{h}_0$ is invariant under θ, we conclude that $\mathfrak{h}$ is invariant under θ.

By Theorem 3.1, the orbit $\overline{H\hat{x}}$ in G_2/Γ projects to a totally geodesic submanifold of $K_2\backslash G_2/\Gamma$. Thus, $\overline{\phi(\mathcal{M}_1)} = p(\overline{H\hat{x}})$ is a totally geodesic submanifold of $\mathcal{M}_2$, concluding the proof of Theorem 1.1.

5. Examples

The simplest example of the situation described in Theorem 1.1 is the case when the image of the totally geodesic immersion is closed.

Example 5.1. Let $\mathcal{M}_2$ be a locally symmetric space of noncompact type with finite volume, and let $\mathcal{M}_1$ be a closed totally geodesic submanifold of $\mathcal{M}_2$. Let $\rho : \widetilde{\mathcal{M}_1} \to \mathcal{M}_1$ be the locally isometric covering map of $\mathcal{M}_1$ by its universal cover. Let $i : \mathcal{M}_1 \to \mathcal{M}_2$ be the inclusion map. Then the map $\phi = i \circ \rho$ is totally geodesic. The closure of $\phi(\widetilde{\mathcal{M}_1})$ in $\mathcal{M}_2$ is $\mathcal{M}_1$.

It more often happens that the image of the totally geodesic immersion is not closed, as in the example below.

Example 5.2. Let $\mathcal{M}_1, \mathcal{M}_2$ and ϕ be as in Example 5.1. Let γ be a geodesic in $\mathcal{M}_1$ which is dense in $\mathcal{M}_1$, and let $\tilde{\gamma}$ be the geodesic in $\widetilde{\mathcal{M}_1}$ that covers γ. Let $\mathcal{M}_3$ be any totally geodesic submanifold of $\widetilde{\mathcal{M}_1}$ such that $\tilde{\gamma}$ is contained in $\mathcal{M}_3$. The restriction of ϕ to $\mathcal{M}_3$ is a totally geodesic immersion from $\mathcal{M}_3$ to $\mathcal{M}_2$. The geodesic γ is in its image. Then since

$$\mathcal{M}_1 = \overline{(\gamma)} \subset \overline{\phi|_{\mathcal{M}_3}(\mathcal{M}_3)} \subset \mathcal{M}_1,$$

the closure of $\phi|_{\mathcal{M}_3}(\mathcal{M}_3)$ in $\mathcal{M}_2$ is equal to $\mathcal{M}_1$.

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