

## A NEW CHARACTERIZATION OF $\text{Proj}^1 \mathcal{X} = 0$ FOR COUNTABLE SPECTRA OF (LB)-SPACES

JOCHEN WENGENROTH

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ABSTRACT. The derived projective limit functor  $\text{Proj}^1$  is a very useful tool for investigating surjectivity problems in various parts of analysis (e.g. solvability of partial differential equations).

We provide a new characterization for vanishing  $\text{Proj}^1$  on projective spectra of (LB)-spaces which improves a classical result of V. P. Palamodov and V. S. Retakh.

### 1. INTRODUCTION

In 1968, V. P. Palamodov [7], [8] developed the theory of the projective limit functor in the categories of vector spaces and locally convex spaces and introduced the derived projective limit functor  $\text{Proj}^1$  as a tool for investigating surjectivity problems in various parts of analysis (like e.g. solvability of partial differential equations). Due to recent progress [5], [6], [11], [12], [14] this theory could be successfully applied by Braun, Meise and Vogt [3], [4] to characterize surjectivity of partial differential and convolution operators on various classes of functions and distributions, by Vogt [10] and Frerick and the author [6] to the splitting theory for Fréchet spaces, or by Bonet and Domański [1] to investigate real analytic vector valued functions and parameter dependent partial differential equations.

By a projective spectrum  $\mathcal{X} = (X_n, \varrho_m^n)$  we always mean a sequence  $(X_n)_{n \in \mathbb{N}}$  of linear spaces (over the same field of real or complex numbers) and linear spectral maps  $\varrho_m^n : X_m \rightarrow X_n$ ,  $n \leq m$ , satisfying

$$\varrho_m^n \circ \varrho_k^m = \varrho_k^n \text{ and } \varrho_n^n = \text{id}_{X_n} \text{ for } n \leq m \leq k.$$

The projective limit is defined as

$$\text{Proj} \mathcal{X} = \left\{ (x_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} X_n : \varrho_m^n(x_m) = x_n \text{ for all } n \leq m \right\},$$

and  $\varrho^n : \text{Proj} \mathcal{X} \rightarrow X_n$  denotes the canonical projection onto the  $n^{\text{th}}$  component. A morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  between projective spectra  $\mathcal{X} = (X_n, \varrho_m^n)$  and  $\mathcal{Y} = (Y_n, \tau_m^n)$

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is a sequence of linear maps  $f_n : X_n \rightarrow Y_n$  which commute with the spectral maps, i.e.  $\tau_m^n \circ f_m = f_n \circ \varrho_m^n$  for  $n \leq m$ .

Proj may be considered as a functor acting from the category of projective spectra into the category of linear spaces (a morphism  $f : \mathcal{X} \rightarrow \mathcal{Y}$  as above is transformed into the linear map  $\text{Proj}\mathcal{X} \rightarrow \text{Proj}\mathcal{Y}$ ,  $(x_n)_n \mapsto (f_n(x_n))_n$ ), and then the first derived functor  $\text{Proj}^1$  can be constructed within homological algebra. Roughly speaking,  $\text{Proj}^1$  measures the lack of exactness of the functor Proj, in particular,  $\text{Proj}^1\mathcal{X} = 0$  for a spectrum  $\mathcal{X}$  means that for all exact sequences

$$0 \rightarrow \mathcal{X} \rightarrow \mathcal{Y} \rightarrow \mathcal{Z} \rightarrow 0$$

of projective spectra the induced sequence

$$0 \rightarrow \text{Proj}\mathcal{X} \rightarrow \text{Proj}\mathcal{Y} \rightarrow \text{Proj}\mathcal{Z} \rightarrow 0$$

is again exact, i.e. the last map  $\text{Proj}\mathcal{Y} \rightarrow \text{Proj}\mathcal{Z}$  is surjective.

Instead of following the homological approach, one can take a concrete representation of  $\text{Proj}^1$  obtained by Palamodov [7] as the definition (cf. [11], [12]). For a spectrum  $\mathcal{X} = (X_n, \varrho_m^n)$  we define

$$\text{Proj}^1\mathcal{X} = \prod_{n \in \mathbb{N}} X_n / \text{Im } \Psi, \text{ where } \Psi : \prod_{n \in \mathbb{N}} X_n \rightarrow \prod_{n \in \mathbb{N}} X_n, \\ (x_n)_{n \in \mathbb{N}} \mapsto (x_n - \varrho_{n+1}^n x_{n+1})_{n \in \mathbb{N}}.$$

This definition is convenient for calculations, in particular, it is not too hard to compute that  $\text{Proj}^1\mathcal{X} = 0$  if  $\text{Proj}^1\tilde{\mathcal{X}} = 0$  for some ‘‘subspectrum’’  $\tilde{\mathcal{X}} = (X_{n_k}, \varrho_{n_m}^{n_k})$ , where  $(n_k)_k$  is a strictly increasing sequence of natural numbers.

In applications in analysis the spaces  $X_n$  usually carry some natural topology and the spectral maps are continuous. It turned out, that linear topological properties of the spaces are useful to characterize  $\text{Proj}^1 = 0$ . The important case of spectra consisting of spaces endowed with a Fréchet space topology has been considered by Palamodov [8, Corollary 5.1], who proved by using an abstract Mittag-Leffler procedure, that if  $\mathcal{X} = (X_n, \varrho_m^n)$  is a spectrum of Fréchet spaces with continuous spectral maps, then  $\text{Proj}^1\mathcal{X} = 0$  if

$$\forall n \in \mathbb{N} \exists m > n \forall k > m \quad \varrho_m^n(X_m) \subseteq \overline{\varrho_k^n(X_k)},$$

the closure taken in the topology of  $X_n$ . For the (often even more important) dual case of spectra consisting of (LB)-spaces (i.e. countable inductive limits of Banach spaces), Palamodov [8, Theorem 5.4] and Retakh [9] (see also [11], [12]) obtained the following characterization, where we call an absolutely convex subset  $B$  of a vector space a *Banach ball* if its linear span  $[B]$  endowed with the Minkowski functional as a norm is a Banach space.

**Theorem 1.** *Let  $\mathcal{X} = (X_n, \varrho_m^n)_{n \in \mathbb{N}}$  be a projective spectrum of separated (LB)-spaces  $X_n$  and continuous linear maps  $\varrho_m^n$ . The following conditions are equivalent.*

1.  $\text{Proj}^1\mathcal{X} = 0$ .
2. *For every  $n \in \mathbb{N}$  there exist a bounded Banach ball  $B_n \subseteq X_n$  and an  $m > n$  such that*

$$\begin{aligned} (\alpha) \quad & \varrho_{n+1}^n(B_{n+1}) \subseteq B_n \text{ and} \\ (\beta) \quad & \varrho_m^n(X_m) \subseteq \varrho_k^n(X_k) + B_n \text{ for all } k > m. \end{aligned}$$

3. For every  $n \in \mathbb{N}$  there exist a bounded Banach ball  $B_n \subseteq X_n$  and an  $m > n$  such that

$$\begin{aligned} (\alpha) \quad & \varrho_{n+1}^n(B_{n+1}) \subseteq B_n \text{ and} \\ (\tilde{\beta}) \quad & \varrho_m^n(X_m) \subseteq \varrho^n(\text{Proj} \mathcal{X}) + B_n. \end{aligned}$$

There seems to be some mystery around this theorem which is perhaps due to the definition of  $\text{Proj}^1$  in terms of homological algebra. However, the original proof of Retakh (necessity of 2. and 3.) and Palamodov (sufficiency of 2. and 3.) does not use any homological tool. The proof of  $2. \Rightarrow 1.$  can be reduced to the result on Fréchet spaces mentioned above: endow  $X_n$  with the topology induced by the Minkowski functional of  $B_n$  (this is, in general, only a complete metrizable group topology, but this is enough), then  $(\alpha)$  means that the spectral maps remain continuous and  $(\beta)$  is the required density. The proof of  $1. \Rightarrow 3.$  uses a Baire category argument and an open mapping lemma for complete metrizable groups. We recommend the reader to consult the (English translations of the) original articles of Palamodov and Retakh.

As necessary conditions; 2. and 3. are extremely useful, and almost all topological properties of limits of projective spectra with  $\text{Proj}^1 = 0$  are proved via these conditions, e.g. Vogt [11, Theorem 5.7] showed that the projective limit of a spectrum  $\mathcal{X}$  of (LB)-spaces with  $\text{Proj}^1 \mathcal{X} = 0$  is an ultrabornological (hence also barrelled) locally convex space if it is endowed with the relative topology of the product. This result allows us to apply open mapping theorems to show that the product topology coincides with some natural inductive topology on the projective limit (see e.g. [1, Theorems 1 and 31]).

On the other hand, if 2. and 3. are used to check  $\text{Proj}^1 = 0$  to obtain surjectivity results, condition  $(\alpha)$  causes very difficult problems, whereas  $(\beta)$  gives appropriate conditions for the concrete situation.

For projective spectra of Montel (LB)-spaces where the spectral maps have dense range, it could be shown in [14], that the theorem remains true if condition  $(\alpha)$  in 2. and 3. is just dropped. This result is quite satisfactory for applications in distribution theory, where the requirement of Montel steps is almost always satisfied. In other applications, e.g. to the splitting theory for Fréchet spaces, the spaces of the spectrum are never Montel (except in trivial cases). In [6], Frerick and the author gave a sufficient condition for  $\text{Proj}^1 = 0$  like 2. where  $(\alpha)$  is dropped but  $(\beta)$  becomes stronger so that this new condition is no longer a characterization. In the present article we show that at least for the equivalence of 1. and 3. condition  $(\alpha)$  is superfluous.

## 2. THE NEW CHARACTERIZATION

**Theorem 2.** Let  $\mathcal{X} = (X_n, \varrho_m^n)$  be a projective spectrum of separated (LB)-spaces  $X_n$  and continuous linear maps  $\varrho_m^n$ . Then  $\text{Proj}^1 \mathcal{X} = 0$  if and only if for every  $n \in \mathbb{N}$  there exist a bounded Banach ball  $D_n \subseteq X_n$  and an  $m = m_n \geq n$  with

$$\varrho_m^n(X_m) \subseteq \varrho^n(\text{Proj} \mathcal{X}) + D_n.$$

We remark that the assumption  $\mathcal{X}$  consisting of (LB)-spaces is just a convenient way to describe the algebraic properties of the “steps”  $X_n$  which are needed, i.e. each step is covered by an increasing sequence of Banach balls which are mapped by the linking maps into subsets of some multiple of one of the balls

covering the range space. In particular, the theorem applies if the spaces  $X_n$  are Mackey-complete (DF)-spaces (especially, strong duals of not necessarily distinguished Fréchet spaces) and the spectral maps are continuous (just apply the theorem as stated to the associated bornological topologies).

We first collect some (more or less known) facts about Banach balls. The second statement below can be interpreted as an open mapping lemma (“almost open continuous linear operators between Banach spaces are open”).

**Lemma.** *Let  $X$  and  $Y$  be vector spaces,  $f : X \rightarrow Y$  a linear map and  $A \subseteq X$ ,  $B, D \subseteq Y$  be Banach balls.*

1. *If  $f(A) + B$  does not contain subspaces (other than  $\{0\}$ ), then  $f(A) + B$  and  $f^{-1}(B) \cap A$  are again Banach balls.*
2. *If  $B \subseteq D + \frac{1}{2}B$  and  $B + D$  does not contain subspaces, then  $B \subseteq 3D$ .*

*Proof.* This lemma appeared in [6, Lemma 2.4 and proof of 2.5] (with a different proof of 1.), but we give the arguments here for the sake of completeness. The first statement is obtained by considering

$$(\star) \quad 0 \rightarrow [A \cap f^{-1}(B)] \xrightarrow{j} [A] \times [B] \xrightarrow{q} [f(A) + B] \rightarrow 0,$$

where  $j(x) = (x, -f(x))$  and  $q(x, y) = f(x) + y$ , and  $[C]$  is always equipped with the Minkowski functional  $p_C$ .  $(\star)$  is algebraically exact and  $j$  and  $q$  are continuous. Since  $[A] \times [B]$  is Banach and  $[f(A) + B]$  is separated,  $(\star)$  is even topologically exact, which gives the conclusion. (This “structural proof” was kindly provided by S. Dierolf.)

To prove the second part, let  $x \in B \subseteq D + \frac{1}{2}B$ . There are  $y_0 \in D$  and  $x_0 \in B$  with  $x = y_0 + \frac{1}{2}x_0$ . Inductively, we find sequences  $(x_n)_n \in D^{\mathbb{N}_0}$  and  $(y_n)_n \in B^{\mathbb{N}_0}$  such that  $x = \sum_{l=0}^n \frac{1}{2^l} y_l + \frac{1}{2^{n+1}} x_n$ . For every  $n$  we have  $\sum_{l=0}^n \frac{1}{2^l} y_l \in 2D$  and this sequence is Cauchy in  $([D], p_D)$ ; hence, convergent to some  $y \in \overline{2D}^{[D]} \subseteq 3D$ . Finally, we get  $x = \sum_{l=0}^n \frac{1}{2^l} y_l + \frac{1}{2^{n+1}} x_n \rightarrow y$  in the separated space  $([B + D], p_{B+D})$  which implies  $x = y \in 3D$ .  $\square$

*Proof of Theorem 2.* The only if part is immediate by Theorem 1. Since it is enough to show  $\text{Proj}^1 \tilde{\mathcal{X}} = 0$  for the spectrum  $\tilde{\mathcal{X}} = (X_{r_n}, \varrho_{r_k}^{r_n})$ , where  $r_1 = 1$  and  $r_{n+1} = m_{r_n}$ , which is a “subsequence” of the given spectrum, and since this new spectrum again satisfies the hypothesis of the theorem, we may assume  $m_n = n + 1$ .

Let  $X_n = \bigcup_{l \in \mathbb{N}} B_{n,l}$ , where  $(B_{n,l})_l$  is an increasing sequence of bounded Banach balls. By assumption, we have

$$\varrho_2^1(X_2) \subseteq \varrho^1(\text{Proj} \mathcal{X}) + D_1 = \bigcup_{l \in \mathbb{N}} \varrho^1 \left[ (\varrho^3)^{-1}(B_{3,l}) \right] + D_1.$$

$Y = ([\varrho_2^1(D_2)], p_{\varrho_2^1(D_2)})$  is a Banach space with  $Y \subseteq \varrho_2^1(X_2)$ , and Baire’s category theorem implies that there is  $l_3 \in \mathbb{N}$  such that

$$Y \cap \left\{ \varrho^1 \left[ (\varrho^3)^{-1}(B_{3,l_3}) \right] + D_1 \right\}$$

is not meager (of second category) in  $Y$ . Since this set equals

$$\bigcup_{l \in \mathbb{N}} \left( Y \cap \left\{ \varrho^1 \left[ (\varrho^3)^{-1}(B_{3,l_3}) \cap (\varrho^4)^{-1}(B_{4,l}) \right] + D_1 \right\} \right)$$

there is  $l_4 \in \mathbb{N}$  such that

$$Y \cap \left\{ \varrho^1 \left[ (\varrho^3)^{-1}(B_{3,l_3}) \cap (\varrho^4)^{-1}(B_{4,l_4}) \right] + D_1 \right\}$$

is not meager in  $Y$ . Inductively we find a sequence  $(l_m)_{m \geq 3}$  of natural numbers such that for each  $m \geq 3$

$$Y \cap \left\{ \varrho^1 \left[ \bigcap_{j=3}^m (\varrho^j)^{-1}(B_{j,l_j}) \right] + D_1 \right\}$$

is not meager, which therefore also holds for the bigger set

$$Y \cap \left\{ \varrho_m^1 \left[ \bigcap_{j=3}^m (\varrho_m^j)^{-1}(B_{j,l_j}) \right] + D_1 \right\}.$$

Define  $A_m^1 := \bigcap_{j=3}^m (\varrho_m^j)^{-1}(B_{j,l_j})$  which is a Banach ball by the first part of the lemma (and it is clearly bounded in  $X_m$ , recall that  $\varrho_m^m$  is the identity on  $X_m$ ). For  $m \geq 3$  we obviously have  $\varrho_{m+1}^m(A_{m+1}^1) \subseteq A_m^1$ . Since the set

$$Y \cap \left( \varrho_m^1(A_m^1) + D_1 \right)$$

is not meager in  $Y$ , its closure contains an interior point which can be assumed to be 0 (by a simple convexity argument). Hence, for each  $m \geq 3$ , there is  $\varepsilon_m \in (0, 1)$  such that

$$\varepsilon_m \varrho_2^1(D_2) \subseteq \overline{Y \cap \left( \varrho_m^1(A_m^1) + D_1 \right)}^Y \subseteq \left( \varrho_m^1(A_m^1) + D_1 \right) + \frac{\varepsilon_m}{2} \varrho_2^1(D_2).$$

Since  $\varepsilon_m \varrho_2^1(D_2)$  and  $\varrho_m^1(A_m^1) + D_1$  are Banach balls such that their sum is bounded in the separated space  $X_1$ , the second part of the lemma implies

$$\begin{aligned} \frac{\varepsilon_m}{3} \varrho_2^1(D_2) &\subseteq \varrho_m^1(A_m^1) + D_1 \text{ and therefore even,} \\ \varepsilon_m^1 D_2 &\subseteq \varrho_m^2(A_m^1) + (\varrho_2^1)^{-1}(D_1), \end{aligned}$$

with  $\varepsilon_m^1 := \frac{\varepsilon_m}{3}$  for  $m \geq 3$ . If we proceed in the same way with  $D_k$  instead of  $D_1$ , we find bounded Banach balls  $A_m^k \subseteq X_m$  and  $\varepsilon_m^k \in (0, 1)$ ,  $m \geq k + 2$ , such that

- ( $\alpha$ )  $\varepsilon_m^k D_{k+1} \subseteq \varrho_m^{k+1}(A_m^k) + (\varrho_{k+1}^k)^{-1}(D_k)$  and
  - ( $\beta$ )  $\varrho_{m+1}^m(A_{m+1}^k) \subseteq A_m^k$  for  $m \geq k + 2$ .
- Replacing  $A_m^k$  by  $A_m^1 + \dots + A_m^k$  if necessary, we may assume
- ( $\gamma$ )  $A_m^{k-1} \subseteq A_m^k$  for  $m \geq k + 2$ .

Now we define  $\tilde{D}_1 := D_1$ ,  $\tilde{D}_2 := (\varrho_2^1)^{-1}(D_1) \cap [D_2 + \varrho_3^2(A_3^1)]$  and, inductively,

$$\tilde{D}_{r+1} := (\varrho_{r+1}^r)^{-1}(\tilde{D}_r) \cap [D_{r+1} + \varrho_{r+2}^{r+1}(A_{r+2}^r)].$$

Again the first part of our lemma implies that every  $\tilde{D}_r \subseteq X_r$  is a Banach ball and we obviously have  $\varrho_{r+1}^r(\tilde{D}_{r+1}) \subseteq \tilde{D}_r$ . Proceeding by induction (on  $r$ ) we will show that for each  $r \geq 2$  and every  $m \geq r + 1$  there is  $\delta_m^r \in (0, 1)$  with

( $\star$ ) 
$$\delta_m^r D_r \subseteq \tilde{D}_r + \varrho_m^r(A_m^{r-1}).$$

$r = 2$ : Let  $m \geq 3$  and define  $\delta_m^2 = \varepsilon_m^1 \in (0, 1)$ . By ( $\alpha$ ) we have

$$\varepsilon_m^1 D_2 \subseteq \varrho_m^2(A_m^1) + (\varrho_2^1)^{-1}(D_1),$$

and since  $\varepsilon_m^1 < 1$ , this yields

$$\begin{aligned} \delta_m^2 D_2 &\subseteq \varrho_m^2(A_m^1) + \left[ (\varrho_2^1)^{-1}(D_1) \cap (D_2 + \varrho_m^2(A_m^1)) \right] \\ &\subseteq \varrho_m^2(A_m^1) + \left[ (\varrho_2^1)^{-1}(D_1) \cap (D_2 + \varrho_3^2(A_3^1)) \right] = \varrho_m^2(A_m^1) + \tilde{D}_2. \end{aligned}$$

$r \rightarrow r+1$ : Assume that  $(\star)$  holds for some  $r \geq 2$  and let  $m \geq r+2$ . Define  $\delta_m^{r+1} = \frac{1}{2}\varepsilon_m^r \delta_m^r \in (0, 1)$ . Using  $(\alpha)$  and then the induction hypothesis, we get

$$\begin{aligned} \delta_m^{r+1} D_{r+1} &\subseteq \frac{1}{2} \delta_m^r \varrho_m^{r+1}(A_m^r) + \frac{1}{2} (\varrho_{r+1}^r)^{-1} (\delta_m^r D_r) \\ &\subseteq \frac{1}{2} \varrho_m^{r+1}(A_m^r) + \frac{1}{2} (\varrho_{r+1}^r)^{-1} (\tilde{D}_r + \varrho_m^r(A_m^{r-1})). \end{aligned}$$

For  $x \in (\varrho_{r+1}^r)^{-1}(\tilde{D}_r + \varrho_m^r(A_m^{r-1}))$  there are  $y \in \tilde{D}_r$  and  $z \in A_m^{r-1}$  with  $\varrho_{r+1}^r(x) = y + \varrho_m^r(z)$ . Hence,  $\varrho_{r+1}^{r+1}(x + \varrho_m^{r+1}(z)) = y$ , which implies

$$x \in (\varrho_{r+1}^r)^{-1}(\tilde{D}_r) + \varrho_m^{r+1}(A_m^{r-1}).$$

We therefore get

$$\begin{aligned} \delta_m^{r+1} D_{r+1} &\subseteq \frac{1}{2} \varrho_m^{r+1}(A_m^r) + \frac{1}{2} (\varrho_{r+1}^r)^{-1}(\tilde{D}_r) + \frac{1}{2} \varrho_m^{r+1}(A_m^{r-1}) \\ &\subseteq \varrho_m^{r+1}(A_m^r) + (\varrho_{r+1}^r)^{-1}(\tilde{D}_r) \end{aligned}$$

by  $(\gamma)$ . This implies

$$\begin{aligned} \delta_m^{r+1} D_{r+1} &\subseteq \varrho_m^{r+1}(A_m^r) + \left[ (\varrho_{r+1}^r)^{-1}(\tilde{D}_r) \cap (D_{r+1} + \varrho_m^{r+1}(A_m^r)) \right] \\ &\subseteq \varrho_m^{r+1}(A_m^r) + \left[ (\varrho_{r+1}^r)^{-1}(\tilde{D}_r) \cap (D_{r+1} + \varrho_{r+2}^{r+1}(A_{r+2}^r)) \right] \\ &= \varrho_m^{r+1}(A_m^r) + \tilde{D}_{r+1}. \end{aligned}$$

This completes the induction step. To finish the proof, we will show that the sequence  $(\tilde{D}_r)_r$  satisfies Retakh's condition for  $\text{Proj}^1 \mathcal{X} = 0$  (2. in Theorem 1), i.e.

$$\varrho_{r+1}^r(X_{r+1}) \subseteq \varrho_m^r(X_m) + \tilde{D}_r \text{ for all } m \geq r+2.$$

Fix  $r \in \mathbb{N}$  and  $m \geq r+2$ . We know

$$\varrho_{r+1}^r(X_{r+1}) \subseteq \varrho^r(\text{Proj} \mathcal{X}) + D_r \subseteq \varrho_m^r(X_m) + D_r.$$

Multiplying with  $\delta_m^r$  we get

$$\varrho_{r+1}^r(X_{r+1}) \subseteq \varrho_m^r(X_m) + \delta_m^r D_r.$$

Using  $(\star)$  we conclude

$$\varrho_{r+1}^r(X_{r+1}) \subseteq \varrho_m^r(X_m) + \tilde{D}_r + \varrho_m^r(A_m^{r-1}) = \varrho_m^r(X_m) + \tilde{D}_r.$$

As we have noted above,  $\varrho_{r+1}^r(\tilde{D}_{r+1}) \subseteq \tilde{D}_r$  and each  $\tilde{D}_r$  is a bounded Banach ball. Thus, Retakh's and Palamodov's theorem implies the assertion.  $\square$

*Remark.* We do not know whether the existence of a sequence of Banach balls  $D_n$  satisfying only

$$(\beta) \quad \forall n \in \mathbb{N} \exists m > n \forall k > m \quad \varrho_m^n(X_m) \subseteq \varrho_k^n(X_k) + D_n$$

already implies  $\text{Proj}^1 \mathcal{X} = 0$ . If the sequence  $D_n$  satisfies condition  $(\alpha)$  of Theorem 1, one can endow  $X_n$  with the complete metrizable topology induced by the Minkowski functional of  $D_n$  and the Mittag-Leffler procedure (see e.g. [2, chap. 2, §3, théorème 1]) applies (because the spectral maps remain continuous) to get the hypothesis of Theorem 2. Condition  $(\alpha)$  is essentially equivalent to the continuity of the spectral maps with respect to the new topologies, thus, if  $(\alpha)$  is violated, there is no direct way to apply the Mittag-Leffler argument.

The above result and classical duality theory now provide a new characterization of weakly acyclic (LF)-spaces. We recall that an (LF)-space  $E = \text{ind}_n E_n$  (i.e.  $E$  is the union of the increasing sequence of Fréchet spaces  $(E_n)_n$  with continuous inclusions  $E_n \hookrightarrow E_{n+1}$ , endowed with the strongest locally convex topology such that all inclusions  $E_n \hookrightarrow E$  are continuous) is called weakly acyclic, if the projective spectrum of the duals with restrictions as spectral maps satisfies  $\text{Proj}^1 = 0$  (by [13, Lemma 4.1] this is equivalent to the original definition of Palamodov [8, §6], for more information see also [13], [14]). The equivalence of 1., 2. and 3. below is again due to Palamodov and Retakh.

**Theorem 3.** *Let  $E = \text{ind}_n E_n$  be an (LF)-space. The following conditions are equivalent.*

1.  $E$  is weakly acyclic.
2. For every  $n$  there exist an absolutely convex 0-neighbourhood  $U_n \subseteq E_n$  and an  $m > n$  such that

$$\begin{aligned} (\alpha) \quad & U_n \subseteq U_{n+1} \text{ and} \\ (\beta) \quad & \sigma(E_m, E'_m)|_{U_n} = \sigma(E_k, E'_k)|_{U_n} \text{ for all } k > m. \end{aligned}$$

3. For every  $n$  there exist an absolutely convex 0-neighbourhood  $U_n \subseteq E_n$  and an  $m > n$  such that

$$\begin{aligned} (\alpha) \quad & U_n \subseteq U_{n+1} \text{ and} \\ (\tilde{\beta}) \quad & \sigma(E_m, E'_m)|_{U_n} = \sigma(E, E')|_{U_n}. \end{aligned}$$

4. For every  $n$  there exist an absolutely convex 0-neighbourhood  $U_n \subseteq E_n$  and an  $m > n$  such that

$$(\tilde{\beta}) \quad \sigma(E_m, E'_m)|_{U_n} = \sigma(E, E')|_{U_n}.$$

*Proof.* The theorem follows immediately from Theorem 2 by the following observation: Let  $X \xrightarrow{i} Y \xrightarrow{j} Z$  be locally convex spaces with continuous inclusions, let  $h = j \circ i$  and  $U$  be an absolutely convex 0-neighbourhood in  $X$ . Then

$$\sigma(Y, Y')|_U = \sigma(Z, Z')|_U \text{ if and only if } i^t(Y') \subseteq h^t(Z') + U^0$$

(the polar taken in  $X'$ ). This can be proved by standard duality arguments using the theorem of bipolars.  $\square$

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FB IV – MATHEMATIK, UNIVERSITÄT TRIER, D – 54286 TRIER, GERMANY  
*E-mail address:* [wengen@uni-trier.de](mailto:wengen@uni-trier.de)