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LENGTHS OF RADII UNDER CONFORMAL MAPS OF THE UNIT DISC

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ABSTRACT. If $E_f(R)$ is the set of endpoints of radii which have length greater than or equal to $R > 0$ under a conformal map f of the unit disc, then $\operatorname{cap} E_f(R) = O(R^{-1/2})$ as $R \to \infty$ for the logarithmic capacity of $E_f(R)$. The exponent $-1/2$ is sharp.

Suppose $\mathbf{D}$ is the unit disc in the plane. A well-known theorem by Beurling [Beu] states that if $f : \mathbf{D} \to \mathbf{C}$ is conformal, then the set of radii whose images under f have infinite length has vanishing logarithmic capacity. We give a quantitative version of this statement which is asymptotically sharp and improves an estimate by Pommerenke [Pom, p. 215].

Theorem. *There exists a universal constant $K > 0$ with the following property.*

Suppose $f : \mathbf{D} \to \mathbf{C}$ is a conformal map with $f'(0) = 1$. If $E_f(R)$ is the set of all $\zeta \in \partial\mathbf{D}$ with $\operatorname{length} f\big([0,\zeta)\big) \geq R > 0$, *then* $\operatorname{cap} E_f(R) \leq K/\sqrt{R}$.

On the other hand, there exist functions f, e.g. the Koebe function, for which $\operatorname{cap} E_f(R) \geq \frac{1}{2\sqrt{R}}$ *for large R.*

Our theorem implies that $\operatorname{cap} E_f(R) = O(R^{-1/2})$ as $R \to \infty$ for all conformal maps f of $\mathbf{D}$ and that $1/2$ is the best possible constant in this statement.

1. Notation and auxiliary results

A curve $\gamma : I \to \mathbf{C}$ is a continuous mapping of an interval $I \subseteq \mathbf{R}$. It is understood that a curve is locally rectifiable. If we speak of a curve in an open set Ω, then we allow the endpoints of the curve to lie on the boundary of the set. A curve γ in Ω connects two sets $A, B \subseteq \overline{\Omega}$, if γ has one endpoint in A and one in B. We denote by $\operatorname{length}(\gamma) \in [0,\infty]$ the euclidean length of γ.

For the proof of our theorem we need the following version of the Gehring-Hayman theorem (cf. [GH], [Pom, p. 88]).

Theorem A. *There is a universal constant $C > 0$ with the following property. Suppose $f : \mathbf{D} \to \mathbf{C}$ is conformal, γ is a curve in $\mathbf{D}$ with endpoints 0 and $\zeta \in \partial\mathbf{D}$, and $[0,\zeta)$ is the radius of $\mathbf{D}$ with endpoint ζ. Then*

$$\operatorname{length} f\big([0,\zeta)\big) \leq C \operatorname{length}(f \circ \gamma).$$

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The modulus $\operatorname{mod}\Gamma \in [0,\infty]$ of a family Γ of curves in an open set Ω is defined as

$$\operatorname{mod}\Gamma = \inf_{\rho} \int_{\Omega} \rho(z)^2 \, dm_2(z).$$

Here m_2 is two-dimensional Lebesgue measure and the infimum is taken over all Borel measurable densities $\rho : \Omega \to [0,\infty]$ that satisfy $\int_\gamma \rho(z)\,|dz| \geq 1$ for all $\gamma \in \Gamma$, where $|dz|$ means integration with respect to euclidean arc-length. The notation for the modulus should indicate which reference set Ω we consider, but we will suppress this, since it will be clear from the context which Ω we mean. If Γ_1 is a curve family in some open set Ω_1, $f : \Omega_1 \to \Omega_2$ is a conformal map and Γ_2 is the curve family in Ω_2 consisting of the curves $f \circ \gamma$, $\gamma \in \Gamma_1$, then $\operatorname{mod}\,\Gamma_1 = \operatorname{mod}\,\Gamma_2$. See [Pom, Ch. 9] for basic properties of the modulus.

We denote the logarithmic capacity of a Borel set $E \subseteq \mathbf{C}$ by $\operatorname{cap} E$. For the definition of the logarithmic capacity see [Pom, Ch. 9].

The following statement which relates the concepts of modulus and capacity is needed in the proof of the theorem. It is part of Pfluger's theorem (cf. [Pom, p. 212]).

Theorem B. *Suppose E is a Borel subset of $\partial\mathbf{D}$ and $\Gamma_E(\epsilon)$ is the family of all curves γ in $\Omega = \{z \in \mathbf{D} : \epsilon < |z| < 1\}$ that connect $\{z \in \mathbf{D} : |z| = \epsilon\}$ and E. Then for sufficiently small $\epsilon > 0$*

$$\operatorname{cap}\, E \leq \frac{1+\epsilon}{\sqrt{\epsilon}} \exp\!\left(-\frac{\pi}{\operatorname{mod}\,\Gamma_E(\epsilon)}\right).$$

The next lemma states a standard modulus estimate. The constant 2π in this inequality is crucial to get the right asymptotic behavior in the theorem. The usefulness of modulus estimates with sharp constants is well-known and dates back to Ahlfors's distortion theorem (cf. [Ahl]).

Lemma. *Suppose $\Omega \subseteq \mathbf{C}$ is a region and Γ is a family of curves in Ω which have one endpoint in a compact set $M \subseteq \overline{\Omega}$. Suppose M is contained in a disc of diameter $\delta > 0$ centered at the origin. If $L \geq \delta$ and $\operatorname{length}\gamma \geq L$ for all $\gamma \in \Gamma$, then*

$$\operatorname{mod}\Gamma \leq \frac{2\pi}{\log(1 + L/\delta)}.$$

This lemma and its proof are similar to Lem. 3.2 in [BKR].

Proof of the lemma. In addition to our assumptions on M we may assume that there exists at least one rectifiable curve in Ω which connects a point in Ω to a point in M. For otherwise it is easy to see that $\operatorname{mod}\Gamma = 0$. (Consider test functions ρ which are equal to $\epsilon > 0$ on $B \cap \Omega$ where B is some open disc containing M and 0 elsewhere. Let ϵ tend to 0.)

For $w \in \Omega$ define $l(w) = \inf_\gamma \operatorname{length}(\gamma)$, where the infimum is taken over all curves in Ω connecting w and M. The additional assumption on M implies that $l(w) < \infty$ for all $w \in \Omega$. The function l is continuous on Ω and satisfies $l(w) \geq |w| - \delta/2$ for $w \in \Omega$. Moreover, if $\gamma : [0, t_0] \to \mathbf{C}$ is a curve in Ω parameterized with respect to arc-length and if $\gamma(0) \in M$, then $l(\gamma(t)) \leq t$ for $t \in (0, t_0]$.

Define $\rho : \Omega \to [0, \infty)$ by

$$\rho(w) = \begin{cases} \dfrac{1}{(\log(1+L/\delta))(\delta + l(w))} & \text{if} \quad l(w) \le L, \\ 0 & \text{otherwise.} \end{cases}$$

Obviously, the function ρ is Borel measurable and we claim that $\int_\gamma \rho(w)\,|dw| \ge 1$ for all $\gamma \in \Gamma$.

To see this let $\gamma \in \Gamma$ be arbitrary. We may assume that $\gamma : I \to \mathbf{C}$ has an arc-length parametrization with $I = [0, \text{length}(\gamma)]$ and that $\gamma(0) \in M$. We have $l(\gamma(s)) \le s$ for all $s \in I \backslash \{0\}$. By assumption length$(\gamma) \ge L$ and so

$$\int_\gamma \rho(w)\,|dw| \ge \frac{1}{\log(1+L/\delta)} \int_0^L \frac{ds}{\delta + l(\gamma(s))} \ge \frac{1}{\log(1+L/\delta)} \int_0^L \frac{ds}{\delta + s} = 1.$$

Therefore, if $L \ge \delta$

$$\begin{aligned} \operatorname{mod}\Gamma &\le \int_\Omega \rho(w)^2\, dm_2(w) \\ &= \frac{1}{[\log(1+L/\delta)]^2} \int_{\{w\in\Omega : l(w)\le L\}} \frac{dm_2(w)}{(\delta + l(w))^2} \\ &\le \frac{1}{[\log(1+L/\delta)]^2} \int_{\{w\in\mathbf{C} : |w|\le L+\delta/2\}} \frac{dm_2(w)}{(\delta/2 + |w|)^2} \\ &= \frac{2\pi}{\log(1+L/\delta)} + 2\pi \frac{\log 2 - 1 + \delta/(2L+2\delta)}{[\log(1+L/\delta)]^2} \\ &\le \frac{2\pi}{\log(1+L/\delta)}. \end{aligned}$$

The lemma follows. □

2. Proof of the theorem

The idea of the proof is essentially the same as in [Pom, p. 215–216]. A limiting argument is employed in Pfluger's theorem which is related to the concept of reduced extremal distance (cf. [Ahl]). The new ingredients in our proof are the more refined modulus estimate of the lemma and the use of the Gehring-Hayman theorem. The proof will show that for the constant K in the theorem we can take $K = \sqrt{2C}$ where C is the constant in the Gehring-Hayman theorem.

We use the notation of the theorem and may assume $f(0) = 0$. Let $\epsilon \in (0,1)$ be arbitrary. Let $\Gamma_1(\epsilon)$ be the family of all curves in $\{z \in \mathbf{D} : \epsilon < |z| < 1\}$ connecting $\{z \in \mathbf{D} : |z| = \epsilon\}$ and $E_f(R)$. We leave it to the reader to show that the set $E_f(R)$ is a countable intersection of open subsets of $\partial\mathbf{D}$. Hence it is a Borel set.

Suppose $\gamma \in \Gamma_1(\epsilon)$ and let $z_0 \in \mathbf{D}$, $|z_0| = \epsilon$, and $\zeta \in E_f(R)$ be the endpoints of γ. Let $[0, z_0]$ be the line segment with endpoints 0 and z_0. If we join $[0, z_0]$ and γ, then we get a curve $\tilde{\gamma}$ in $\mathbf{D}$ connecting 0 and ζ. By the Gehring-Hayman theorem and by definition of $E_f(R)$

$$\text{length}(f \circ \tilde{\gamma}) \ge (1/C)\,\text{length}\, f\big([0,\zeta)\big) \ge R/C.$$

By Koebe's distortion theorem (cf. [Pom, p. 9]), $|f'(z)| \le (1+5\epsilon)$ if $|z| \le \epsilon$ and $\epsilon > 0$ is sufficiently small. It follows that for small ϵ

$$\text{length}(f \circ \gamma) \ge R/C - (\epsilon + 5\epsilon^2) =: L.$$

We now apply the lemma for the region $\Omega = f\big(\mathbf{D}\backslash\{z \in \mathbf{D} : |z| \le \epsilon\}\big)$, the compact set $M = f\big(\{z \in \mathbf{D} : |z| = \epsilon\}\big) \subseteq \overline{\Omega}$ and the curve family $\Gamma_2(\epsilon) = \{f \circ \gamma : \gamma \in \Gamma_1(\epsilon)\}$. By Koebe's distortion theorem M is contained in a disc centered at the origin of diameter $\delta = 2\epsilon(1 + 3\epsilon)$ for small $\epsilon > 0$. It follows that for small $\epsilon > 0$

$$\operatorname{mod}\Gamma_1(\epsilon) = \operatorname{mod}\Gamma_2(\epsilon) \le \frac{2\pi}{\log\big(\frac{R/C+\epsilon+\epsilon^2}{2\epsilon(1+3\epsilon)}\big)}.$$

Hence Pfluger's theorem implies

$$\operatorname{cap} E_f(R) \le \liminf_{\epsilon\to 0} \frac{(1+\epsilon)(2+6\epsilon)^{1/2}}{(R/C+\epsilon+\epsilon^2)^{1/2}} = \frac{\sqrt{2C}}{\sqrt{R}}.$$

The first part of the theorem follows.

For the second part consider the Koebe function $f(z) = z/(1-z)^2$, $z \in \mathbf{C}\backslash\{1\}$. If $R > 1/4$ there exists $\phi \in (0, \pi)$ such that $R = 1/\big(4\sin^2(\phi/2)\big)$. Since length $f\big([0,\zeta)\big) \ge |f(\zeta)|$ for $\zeta \in \partial\mathbf{D}$, we have

$$A = \{e^{i\alpha} : \alpha \in [-\phi, \phi]\} \subseteq E_f(R).$$

Since the capacity of the circular arc A is $\operatorname{cap} A = \sin(\phi/2)$ (cf. [Pom, p. 207]) we obtain $\operatorname{cap} E_f(R) \ge \frac{1}{2\sqrt{R}}$. The theorem follows. □

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References

[Ahl] L.V. Ahlfors, *Conformal invariants: Topics in geometric function theory*, McGraw-Hill, New York, 1973. MR **50:**10211

[Beu] A. Beurling, *Ensembles exceptionnels*, Acta Math. **72** (1940), 1–13. MR **1:**226a

[BKR] M. Bonk, P. Koskela and St. Rohde, *Conformal metrics on the unit ball in euclidean space*, to appear in Proc. London Math. Soc.

[GH] F.W. Gehring and W.K. Hayman, *An inequality in the theory of conformal mapping*, J. Math. Pures Appl. (9) **41** (1962), 353–361. MR **26:**6381

[Pom] Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer, Berlin, 1992. MR **95b:**30008

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