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LINDELÖF PROPERTY AND ABSOLUTE EMBEDDINGS

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ABSTRACT. It is proved that a Tychonoff space is Lindelöf if and only if whenever a Tychonoff space Y contains two disjoint closed copies X_1 and X_2 of X, then these copies can be separated in Y by open sets. We also show that a Tychonoff space X is weakly C-embedded (relatively normal) in every larger Tychonoff space if and only if X is either almost compact or Lindelöf (normal almost compact or Lindelöf).

1. INTRODUCTION AND RESULTS

Notations and terminology follow [En]. Unless otherwise stated, all spaces are assumed to be Tychonoff (and T_1). By βX we denote the Stone-Cech extension of the space X. Recall that the Lindelöf number of a space X, denoted by $l(X)$, is the smallest cardinal τ such that every open cover of X has a subcover of size not greater than τ. If $l(X) \leq \omega$, then X is called Lindelöf. For a non-compact space X put $\lambda(X) = \min\{|A| : A \subset X \text{ and } \mathrm{Cl}_X A \text{ is not compact }\}$. $\lambda(X)$ might be called the index of boundedness of X, since $\lambda(X) \geq \omega_1$ if and only if X is ω-bounded.

Observe that if $\mu < \lambda(X)$, then each subset $A \subset X$ of cardinality $|A| \leq \mu$ has a complete accumulation point, so X is initially μ-compact (i.e., every open cover of X of size not bigger than μ has a finite subcover) [St, Theorem 2.2].

Two subsets A and B of a space X are said to be *completely separated* if there is a continuous function $f : X \to \mathbb{R}$ so that $f(A) = \{0\}$ and $f(B) = \{1\}$.

The main result of the paper is the following theorem, proving true the conjecture of Arhangel'skij [AT, Problem 2].

Theorem 1.1. *For a Tychonoff space X the following conditions are equivalent:*

1. *X is Lindelöf.*
2. *If a Tychonoff space Y contains two disjoint closed copies X_1 and X_2 of X, then these copies can be separated in Y by open sets.*

Hewitt was the first to consider the problem of whether a space X is C-embedded into every larger Tychonoff space Y (that is, every continuous function on X can be extended to a continuous function on Y). It was proved by Hewitt and Smirnov [He], [Sm] (see also [GJ, Ex. 6J]) that this is the case if and only if X is almost

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compact, i.e. $|\beta X \setminus X| \leq 1$, or, equivalently, for every two disjoint completely separated closed subsets of X at least one is compact.

Later, R.Blair proved that a space X is absolutely ν-embedded (that is, $\nu X \subset \nu Y$ whenever $X \subset Y$) if and only if X is either realcompact or almost compact [Bl]. R.Blair, A.Hager and D.Johnson proved that a space X is absolutely z-embedded (that is, each zero-set of X can be extended to a zero-set of Y whenever $X \subset Y$) if and only if X is either Lindelöf or almost compact. They also provided several interesting characterizations of such spaces [Bl], [BH], [HJ]. E.g. these are the spaces so that the $C(X)$ — ring of continuous functions on X — has no proper subalgebra which contains constant functions, separates points and closed sets, is closed under uniform convergence and is closed under inversion in $C(X)$ [HJ, Theorem 3]. Another result, which will be usefull for us, says that if every pair of disjoint non-compact zero-sets of X consists of Lindelöf sets, then X itself is either Lindelöf or almost compact (and hence absolutely z-embedded) [BH, Theorem 4.1].

Recently, the following weaker version of C-embedding has attracted attention, in particular in the theory of relative topological properties.

Definition 1.2 (Arhangel'skij)**.** A subspace X is weakly C-embedded into a space Y if every continuous real-valued function on the space X can be extended to a real-valued function on Y, which is continuous in all points of X.

Notice that, while C-embedding and C^*-embedding (if we assume that only bounded functions must have an extension) are, in general, different notions, weak C-embedding coincides with weak C^*-embedding. (Just consider an unbounded function as a function $f : X \to (-1, 1)$, extend it to the function $\tilde{f} : Y \to [-1, 1]$, which is continuous in all points of X and note that $\mathrm{Cl}_Y\, f^{-1}\{-1, 1\} \cap X = \emptyset$, so we can modify $\tilde{f}$ to have value 0 in all points y with $f(y) = \pm 1$ without any loss of continuity at the points of X.)

Every dense embedding into a Tychonoff space is a weak C-embedding [Ar], but in the case of non-dense embedding this concept is far from being trivial.

Arhangel'skij [Ar] observed that if X is almost compact or Lindelöf, then every embedding of X into a larger Tychonoff space is a weak C-embedding. Surprisingly, these are the only possibilities. Precisely, the following theorem, answering the question of Arhangel'skij [Ar, problem 15], holds.

Theorem 1.3. *A space X is weakly C-embedded into every larger Tychonoff space Y, containing X as a closed subspace, if and only if X is either almost compact, or Lindelöf.*

In [Ar] it was also asked to characterize the spaces which are (strongly) relatively normal in every larger Tychonoff space (Problems 49 and 50); see also [AT, Problems 3 and 4]. A subspace $X \subset Y$ is called *(strongly) relatively normal in* Y if every two disjoint closed in Y (closed in X) subsets of X can be separated by open in Y subsets (see [Ar] for a recent survey on relative topological properties).

In [AT] it was observed that all normal almost compact spaces and in [AG] that all Lindelöf spaces are strongly relatively normal in every larger Tychonoff space.

We prove here that "the converse" is true.

Theorem 1.4. *If X is a Tychonoff space, then the following conditions are equivalent:*

1. *X is either normal and almost compact or Lindelöf.*

2. *X is strongly normal in every Tychonoff space.*
3. *X is normal in every Tychonoff space.*
4. *X is normal in every Tychonoff space, containing X as a closed subspace.*
5. *X is normal and every pair of non-compact disjoint closed subsets of X consists of Lindelöf sets.*

2. Proofs

Lemma 2.1. *Let P and Q be disjoint closed sets of a space X. If any Tychonoff space Y, which contains X as a closed subspace, has two disjoint open sets separating P and Q, then either both P and Q are Lindelöf, or one of them is compact.*

Proof. Take two disjoint closed subsets P and Q of X and suppose that both are non-compact and P is not Lindelöf.

Lemma 2.2. $\lambda(Q) \geq l(P) > \omega$.

Proof of Lemma 2.2. Assuming the contrary, there is $A \subset Q$ so that $\mathrm{Cl}_Q A$ is non-compact and $|A| < l(P) = \tau$.

Since $|A| < \tau$, there is an open cover γ of the space P having no subcover of size $\leq |A|$. We may assume that γ has no subcover of size less than $|\gamma| = \tau'$. Consider $\alpha P = \mathrm{Cl}_{\beta X} P$ — a compactification of P. Then it is easy to see that there is a $Z \subset \alpha P$ such that Z can be separated from P in αP by a $G_{\tau'}$-set, but cannot be separated from P by any G_β-set for $\beta < \tau'$ [En, Ex. 3.12.24]. Since P is closed in X, $\alpha P \cap X = P$ and $Z \subset (\beta X \setminus X)$. Since $\mathrm{Cl}_Q A$ is not compact and Q is closed in X, there is a point $a \in \beta X \setminus X$ so that $a \in \mathrm{Cl}_{\beta X} A$. Consider a compactification C of the space X, obtained from βX by identifying the compact set $Z \cup \{a\}$ to a point b.

Let $S = P \cup Q \cup \{b\} \subset C$ and $T = S \times S \setminus \{(b, b)\}$. We claim that the closed subsets of T, $P \times \{b\}$ and $\{b\} \times Q$ cannot be separated in T by open sets. Indeed, let U be an open subset of T such that $\{b\} \times Q \subset U$. For every $x \in A \subset Q$ there exist open neighborhoods V_x and W_x of the points b and x in S such that $V_x \times W_x \subset U$. The set $G = \bigcap\{V_x : x \in A\}$ is a G_β set containing b with $\beta = |A| < \tau'$. Thus G meets P, say for instance in a point y. It follows that $\{y\} \times A \subset U$ and, since in S the point b is in the closure of A, we actually have $(y, b) \in \mathrm{Cl}_T\{y\} \times A \subset \mathrm{Cl}_T U$. Hence we cannot have $P \times \{b\} \cap \mathrm{Cl}_T U = \emptyset$.

As a consequence, we see that X is not normal in the quotient space of $X \oplus T$, obtained by identifying P with $P \times \{b\}$ and Q with $\{b\} \times Q$. This contradiction finishes the proof of the Lemma 2.2. □

Lemma 2.3. *$l(Q) \geq \lambda(Q)$, in particular Q is not Lindelöf.*

Proof of Lemma 2.3. If $l(Q) = \tau < \lambda(Q)$, then the closure in Q of any set of cardinality at most τ is compact. In particular, Q is initially τ-compact and hence compact since $l(Q) = \tau$. But this contradicts the assumption that Q is not compact. □

Lemma 2.4. $\lambda(P) \geq l(Q)$.

Proof of Lemma 2.4. Use the same argument as in the proof of Lemma 2.2. □

Lemma 2.5. *$\lambda(P) = l(P) = \lambda(Q) = l(Q) = \tau > \omega$ and τ is a regular cardinal.*

Proof of Lemma 2.5. $\lambda(P) = l(P) = \lambda(Q) = l(Q) = \tau > \omega$ follows from Lemmas 2.2–2.4. Suppose $cf(\tau) = \mu < \tau$, i.e. $\tau = \sup\{a_\alpha : a < \mu\}$. Since P is not Lindelöf, we can take an uncountable open cover γ of the space P, having no subcover of size less than $|\gamma|$. We claim that $|\gamma| = \tau$. Since $l(P) = \tau$, $|\gamma| \leq \tau$. If $|\gamma| < \tau$, then, as P is $|\gamma|$-initially compact, γ would have a finite subcover. So, $|\gamma| = \tau$. Well order γ as $\gamma = \{U_\alpha : \alpha < \tau\}$. For every $\alpha < \mu$ let $W_\alpha = \bigcup\{U_\beta : \beta < a_\alpha\}$. Now, $\{W_\alpha : \alpha < \mu\}$ is a cover of P of size less than τ, so it has a finite subcover. Hence γ has a subcover of size less than τ—a contradiction. $\square$

Now we continue the proof of Lemma 2.1.

Preliminary construction. By transfinite induction we shall construct in P something like an enlarged closed copy of the ordinal τ.

Since $l(P) = \lambda(P) = \tau$, the proof of Lemma 2.5 shows that there is an open cover γ of size τ having no subcover of size less than τ. Hence by the same argument as in the proof of Lemma 2.2, there is a compact $Z \subset \alpha P = \mathrm{Cl}_{\beta X} P$ such that Z can be separated from P in αP by a G_τ-set, but cannot be separated from P by any G_β-set for $\beta < \tau$.

By transfinite induction define for every $\alpha < \tau$ a compact subset $F_\alpha \subset P$ and an open in αP neighborhood $W_\alpha \supset Z$ in such a way that

(1) $F_\alpha \subset \bigcap\{\mathrm{Cl}_{\alpha P} W_\beta : \beta < \alpha\}$,
(2) $\mathrm{Cl}_{\alpha P} W_\alpha \cap \bigcup\{F_\beta : \beta \leq \alpha\} = \emptyset$, in particular all F_α are disjoint,
(3) if α is a non-limit ordinal, then F_α is a one-point-set $F_\alpha = \{x_\alpha\}$,
(4) if α is a limit ordinal, then

$$F_\alpha = \mathrm{Cl}_P \bigcup\{F_\beta : \beta < \alpha\} \setminus \bigcup\{F_\beta : \beta < \alpha\},$$

(5) $d(\bigcup\{F_\beta : \beta < \alpha\}) \leq \alpha$,
(6) for every $\alpha < \tau$, $\bigcup\{F_\beta : \beta \leq \alpha\}$ is a compact subset of P.

Take an arbitrary $x \in P$ as x_0, and let $F_0 = \{x_0\}$ and W_0 be an arbitrary neighborhood of Z so that $x_0 \notin \mathrm{Cl}_{\alpha P} W_0$.

Let $\alpha < \tau$ and suppose that for all $\beta < \alpha$ we have F_β and W_β satisfying conditions (1)–(6). Consider two possibilities:

I. $\alpha = \alpha' + 1$. Since Z cannot be separated from P by $G_{\alpha'}$-sets, there is a point $x_\alpha \in P \cap \bigcap\{W_\beta : \beta \leq \alpha'\}$. Let $F_\alpha = \{x_\alpha\}$. Clearly conditions (1),(3),(5),(6) are satisfied for α. From (6) it follows that there is an open neighborhood W_α of Z so that $\mathrm{Cl}_{\alpha P} W_\alpha \cap \bigcup\{F_\beta : \beta \leq \alpha\} = \emptyset$. Of course, even (2) is satisfied and we are done for α.

II. α is limit ordinal. Let $F'_\alpha = \mathrm{Cl}_P \bigcup\{F_\beta : \beta < \alpha\}$. Since (5) is true for each $\beta < \alpha$ and $\alpha < \lambda(P)$, it follows that F'_α is a compact subset of P. From (1), (2) and (6) for each $\beta < \alpha$ we have

$$F_\alpha = F'_\alpha \setminus \bigcup\{F_\beta : \beta < \alpha\} = F'_\alpha \cap \bigcap\{\mathrm{Cl}_{\alpha P} W_\beta : \beta < \alpha\}.$$

In particular, F_α is compact. Since F'_α is a compact subset of P, there is an open neighborhood W_α of Z so that $\mathrm{Cl}_{\alpha P} W_\alpha \cap F'_\alpha = \emptyset$. Now all conditions (1)–(6) are clearly satisfied and we are done for α.

Clearly, the W_α's may be chosen in such a way that in addition $Z = \bigcap\{\mathrm{Cl}_{\alpha P} W_\alpha : \alpha < \tau\}$. It then follows that $F = \bigcup\{F_\alpha : \alpha < \tau\}$ is closed in P.

The preliminary construction is now complete. We can think about F as a closed copy of τ in P where all non-limit ordinals are usual points, but all limit ordinals are enlarged to (maybe non-trivial) compact sets.

Next, apply the same method to get a similar "copy of τ" $G = \bigcup\{G_\alpha : \alpha < \tau\}$ in Q.

Now we are ready to embed X in a space Y in such a way that the sets P and Q cannot be separated by open sets in Y. Let $T = (\tau+1)\times(\tau+1)\setminus\{(\tau,\tau)\}$. We shall use the well known fact that closed copies of τ in T such as the diagonal $\Delta \subset T$ and $\tau \times \{\tau\}$ cannot be separated by open sets in T (see [Wa, Lemma 2.1.12]).

The idea of our construction is to "replace" here Δ and $\tau \times \{\tau\}$ by F and G and then "glue" the resulting space to X.

Let T' be the set of all points in $\tau \times \tau \setminus \Delta$ having both coordinates non-limit. The space Y we are looking for is $X \cup T'$ equipped with the following topology:

(1) All points of T' are isolated.
(2) All points $x \in X \setminus (F \cup G)$ have usual neighborhood base in X.
(3) All points $x \in \bigcup\{F_\alpha : \alpha \text{ is non-limit}\}$ have usual neighborhood base in X.
(4) Let $x \in F_\alpha$ and α be a limit ordinal. For every open in X a neighborhood W of x define a neighborhood W' of x in Y as

$$W' = W \cup \{(a,b) \in T' : F_a \subset W \text{ and } F_b \subset W\}.$$

Notice that, in the above formula, a and b are non-limit ordinals, so F_a and F_b are one-point-sets.

(5) Let $x \in G_\alpha$ and α be a non-limit ordinal. For every open in X neighborhood W of x and every $b < \tau$ define a neighborhood W'_b of x in Y as

$$W'_b = W \cup \{(\alpha, c) \in T' : c > b\}.$$

(6) Let $x \in G_\alpha$ and α be a limit ordinal. For every open in X neighborhood W of x and every $b < \tau$ define a neighborhood W'_b of x in Y as

$$W'_b = W \cup \{(a,c) \in T' : G_a \subset W \text{ and } c > b\}.$$

Notice that, in the above formula, a is a non-limit ordinal, so G_a is a one-point-set.

It is tedious but not difficult to realize that Y is a Tychonoff space and X is closed in Y.

We claim that F and G cannot be separated in Y by open sets. Take an arbitrary open set U of Y containing F. We shall prove that $\mathrm{Cl}_Y\, U \cap G \neq \emptyset$. Take for every *limit* $\alpha < \tau$ a point $x_\alpha \in F_\alpha$ and basic neighborhood $W'(x_\alpha) \subset U$. Observe that, by definition of F_α, the set $B_\alpha = \{b < \alpha : b \text{ is non-limit and } F_b \subset W\}$ is unbounded in α. For every α pick an arbitrary $f(\alpha) \in B_\alpha$. Since for each α $f(\alpha) < \alpha$, we can apply the Pressing-Down Lemma [Ku, Theorem 2.3] to find a non-limit $b < \tau$ so that $b = f(\alpha)$ for unboundedly (in τ) many α. We claim that for every $c < \tau$ there are $d > c$ and $\alpha \in f^{-1}(b)$ so that $(b,d) \in W'(x_\alpha)$. Indeed, there is $\alpha \in f^{-1}(b)$ with $\alpha > c$. Since B_α is unbounded in α, there is $d \in B_\alpha$ with $d > c$. Now both $b, d \in B_\alpha$, hence $(b,d) \in W'(x_\alpha)$ and our claim holds. Next, consider G_b and recall that, since b is a non-limit ordinal, G_b is a one-point-set, $G_b = \{g\}$. From the last claim it immediately follows that $g \in \mathrm{Cl}_Y\, U$, therefore $\mathrm{Cl}_Y\, U \cap G \neq \emptyset$ and the proof of Lemma 2.1 is complete. □

Lemma 2.6. *If X is weakly C-embedded in every larger Tychonoff space Y, containing X as a closed subspace, then for every two disjoint completely separated closed subsets P and Q of X either both are Lindelöf or one of them is compact.*

Proof. Let $f : X \to \mathbb{R}$ be a continuous function such that $f(P) = \{0\}$ and $f(Q) = \{1\}$. Suppose that X is a closed subspace of a space Y. By our assumption, there is a function $\tilde{f} : Y \to \mathbb{R}$, which is continuous at every point of X and satisfies $\tilde{f}|X = f$. Consequently, P and Q are separated by open sets in Y and hence, by Lemma 2.1, either both P and Q are Lindelöf or one of them is compact. □

Proof of Theorem 1.3. Sufficiency — the only part we need to prove — follows from Lemma 2.6 and the result of Blair, Hager and Johnson cited in the introduction. □

Proof of Theorem 1.4. $1 \Rightarrow 2$ was observed in [AT]. $5 \Rightarrow 1$ follows from the result of Blair, Hager and Johnson, cited in the introduction. $2 \Rightarrow 3 \Rightarrow 4$ is evident. $4 \Rightarrow 5$ follows from Lemma 2.1 and the fact that X is normal, being relatively normal in itself. □

Proof of Theorem 1.1. $1 \Rightarrow 2$ is easy. $2 \Rightarrow 1$ follows from Lemma 2.1 for $P = X_1$, $Q = X_2$ and $X \oplus X$ "as" X. □

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When this paper was already submitted we were informed that almost the same results were obtained independently by M.Matveev and O.Pavlov.

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