

## WEYL'S CONSTRUCTION AND TENSOR POWER DECOMPOSITION FOR $G_2$

JING-SONG HUANG AND CHEN-BO ZHU

(Communicated by Roe Goodman)

ABSTRACT. Let  $V$  be the 7-dimensional irreducible representations of  $G_2$ . We decompose the tensor power  $V^{\otimes n}$  into irreducible representations of  $G_2$  and obtain all irreducible representations of  $G_2$  in the decomposition. This generalizes Weyl's work on the construction of irreducible representations and decomposition of tensor products for classical groups to the exceptional group  $G_2$ .

### 1. INTRODUCTION

In his book [W] H. Weyl studied two fundamental problems for classical groups: first, the polynomial invariants for an arbitrary number of variables for a standard classical group action; second, the decomposition of the tensor powers for such an action. We refer the study of the first problem as the invariant theory and the second as the decomposition of tensor powers. There remain the same problems for exceptional groups  $G_2$ ,  $F_4$ ,  $E_6$ ,  $E_7$  and  $E_8$ . G. Schwarz [S] has found an invariant theory for  $G_2$ . The aim of this paper is to find the decomposition of the tensor powers and Weyl's construction for  $G_2$ .

The exceptional group  $G_2$  can be realized as the group of automorphism of Cayley numbers. The minimal possible nontrivial representation  $V$  of  $G_2$  is 7-dimensional and can be realized to be  $G_2$ -action on the trace zero Cayley numbers. Based on the invariant theory for  $G_2$  due to Schwarz, we define a family of contraction and expansion operators. They are certain homomorphisms between various  $V^{\otimes n}$ 's which correspond to the  $G_2$ -invariants. We prove that the commutator algebra of  $G_2$  action on the tensor power  $V^{\otimes n}$  is generated by some compositions of contraction and expansion operators together with the permutation group  $S_n$ . We denote by  $V^{[n]}$  the intersection of kernels of all genuine contraction operators; then the commutator algebra of  $G_2$  action on the space  $V^{[n]}$  is just  $S_n$ . Denote by  $\mathbb{S}_\lambda$  the Schur functor for a Young diagram  $\lambda$ . We prove that any irreducible representation of  $G_2$  can be realized as the intersection  $\mathbb{S}_{[\lambda]}(V) = \mathbb{S}_\lambda(V) \cap V^{[n]}$ . We show that  $\mathbb{S}_{[\lambda]}V$  is non-zero if and only if  $\lambda$  is a partition of  $n$  into at most 2 parts. We also prove that the multiplicity of  $\mathbb{S}_{[\lambda]}(V)$  occurring in  $V^{[n]}$  is equal to the dimension of

---

Received by the editors March 25, 1997 and, in revised form, July 7, 1997.

1991 *Mathematics Subject Classification*. Primary 22E46, 13A50.

*Key words and phrases*. Cayley numbers, invariant theory, tensor power decomposition.

The first named author was partially supported by NSF Grant DMS 9306138 and RGC Competitive Earmarked Research Grant HKUST 588/94P.

$V_\lambda$ , where  $V_\lambda$  is the irreducible representation of  $S_n$  corresponding to the partition  $\lambda$ . More precisely, as a  $G_2 \times S_n$ -module,  $V^{[n]}$  is decomposed as follows:

$$V^{[n]} \cong \bigoplus S_{[\lambda]}(V) \otimes V_\lambda,$$

where the sum is over all partitions  $\lambda = (\lambda_1 \geq \lambda_2 \geq 0)$  of  $n$  into at most two parts.

The approach in this paper is in the same spirit of Weyl’s original work [W] and was explained in a much easier way in Fulton and Harris’ book [FH]. By explicitly making use of Schwarz’s invariant theory for  $G_2$ , we extend Weyl’s method of construction and tensor power decomposition to the exceptional group  $G_2$ . We remark that although the invariant theory for the 8-dimensional spin representation of  $\text{Spin}(7)$  is closely related to the  $G_2$  case we used here, the decomposition of the spin representation of  $\text{Spin}(n)$  should be done in a completely different fashion. One may use Howe’s theory of dual pairs [H, §4.3.5] to decompose all even powers, and then to describe the odd power by decomposing the tensor product of the spin representation with an even power.

In his Yale dissertation [Wo] Y.-K. Wong obtained the “First Fundamental Theorem for Covariants” for the 7-dimensional representation of  $G_2$  (and for the 8-dimensional representation of  $\text{Spin}(7)$  as well). In other words, he considered the decomposition of the whole polynomial algebra, as opposed to the individual tensor space which we investigate in this paper (Wong did the  $\text{Spin}(7)$  case as well).

## 2. INVARIANT THEORY OF $G_2$

In this section we recall the invariant theory of  $G_2$ . These results presented here are due to G. Schwarz [S].

It is well-known that up to isomorphism there is only one non-commutative, non-associative alternative algebra over  $\mathbb{C}$ , the 8-dimensional Cayley algebra. It can be constructed as follows. Let  $A_{\mathbb{R}}$  denote the set of ordered pairs of quaternions with coordinatewise addition and the following multiplication:

$$(a, b)(c, d) = (ac - \bar{d}b, da + b\bar{c}),$$

where  $a \mapsto \bar{a}$  is the usual conjugation of quaternions. Then  $A_{\mathbb{R}}$  is a central simple non-associate, non-commutative alternative algebra of dimension 8 over  $\mathbb{R}$ . If  $x = (a, b) \in A_{\mathbb{R}}$ , we define  $\bar{x} = (\bar{a}, -b)$  and  $\text{tr}(x) = \text{Re } a$ , the real part of  $a$ . We denote by  $A$  the complexification of  $A_{\mathbb{R}}$ .

The connected component of the automorphism group of  $A$  is isomorphic to the complex exceptional group  $G_2$ . Denote by  $V$  the subspace of  $A$  which consists of trace zero elements. The group  $G_2$  acts irreducibly and faithfully on  $V$ .

We now describe invariants. Let  $m \in \mathbb{N}$  and denote by  $mV = V^{\oplus m} = \bigoplus_{i=1}^m V$  the direct sum of  $m$ -copies of  $V$  and let  $(x_1, \dots, x_m) \in mV$  be arbitrary. We define the following three series of polynomial functions in  $\mathbb{C}[mV]^{G_2}$ :

$$\begin{aligned} \alpha_{ij} &= -\text{tr}(x_i x_j), \quad 1 \leq i, j \leq m; \\ \beta_{ijk} &= -\text{tr}(x_i(x_j x_k)), \quad 1 \leq i, j, k \leq m; \\ \gamma_{ijkl} &= \text{skew tr}(x_i(x_j(x_k x_l))), \quad 1 \leq i, j, k, l \leq m. \end{aligned} \tag{2.1}$$

Here the last function is skew symmetrized with respect to its arguments. We can identify above invariants as elements in  $(V^*)^{\otimes d}$ , for  $d = 2, 3, 4$ , where  $V^*$  denotes

the dual space of  $V$ . We write  $\alpha, \beta$  and  $\gamma$  for the corresponding elements. We have

$$(2.2) \quad \alpha \in \text{Sym}^2(V^*), \beta \in \wedge^3(V^*) \text{ and } \gamma \in \wedge^4(V^*).$$

Through the bilinear form  $\alpha$ , we may identify  $V^*$  with  $V$ . Thus we may also view these three invariants as elements in  $V^{\otimes d}$ , for  $d = 2, 3, 4$ . We shall use this identification if no confusion will arise.

The following is a theorem due to G. Schwarz [S].

**Theorem 2.1.** *The  $G_2$ -invariants  $\mathbb{C}[mV]^{G_2}$  are generated by  $\alpha_{ij}$ ,  $\beta_{ijk}$  and  $\gamma_{ijkl}$ .*

### 3. DECOMPOSITION OF TENSOR POWERS FOR $GL(V)$

Let  $S_n$  denote the symmetric group of  $n$  objects and  $\mathbb{C}S_n$  the group algebra of  $S_n$ . To a partition  $\lambda = (\lambda_1, \dots, \lambda_k)$  of  $n$  with  $\lambda_1 \geq \dots \geq \lambda_k \geq 1$ , we associate a Young diagram with  $\lambda_i$  boxes in the  $i$ th row, the rows of boxes lined up on the left. A Young tableau is a Young diagram with a numbering of boxes by  $1, \dots, n$ , such that the numbers are increased along each row and each column.

Given a Young tableau  $\lambda$ , let  $c_\lambda \in \mathbb{C}S_n$  be the Young symmetrizer corresponding to  $\lambda$ . Then some scalar multiple of  $c_\lambda$  is idempotent and the image of  $c_\lambda$  by right multiplication on  $\mathbb{C}S_n$  is an irreducible representation  $V_\lambda$  of  $S_n$ . Every irreducible representation of  $S_n$  can be obtained in this way for a unique partition. See for example [FH, Theorem 4.3].

Let  $V$  be a finite dimensional vector space. We consider the actions of  $GL(V)$  and  $S_n$  on  $V^{\otimes n}$ . The action of  $GL(V)$  is the diagonal action and  $S_n$  action is the permutations on the components of the tensor products.

We denote the image of a Young symmetrizer  $c_\lambda$  by  $\mathbb{S}_\lambda(V)$ :

$$\mathbb{S}_\lambda(V) = \text{Im} (c_\lambda : V^{\otimes n} \rightarrow V^{\otimes n}).$$

The functor  $V \rightsquigarrow \mathbb{S}_\lambda(V)$  is called the Schur functor.

**Theorem 3.1.** *As  $GL(V) \times S_n$  modules, we have the following canonical isomorphism:*

$$V^{\otimes n} \cong \bigoplus \mathbb{S}_\lambda(V) \otimes V_\lambda,$$

where the sum is over all partitions  $\lambda$  of  $n$  into at most  $\dim V$  parts.

For a proof of this theorem, see §6.2 of [FH].

### 4. CONTRACTIONS, EXPANSIONS AND COMMUTATOR ALGEBRA

From now on the vector space  $V$  is the 7-dimensional irreducible representation of  $G_2$ . We will use the three kinds of generators of  $G_2$ -invariants described in Section 2 to define the corresponding contraction and expansion operators.

Suppose that we are given  $\phi \in (V^*)^{\otimes d}$ , a  $d$ -multilinear functional on  $V$  that is either symmetric or skew-symmetric. Suppose further we are given an identification  $V^* \cong V$ . For any integer  $k$  such that  $0 \leq k \leq d$ , we define the operator  $CE_k(\phi)$  as follows:

$$(4.1) \quad \begin{aligned} CE_k(\phi) : V^{\otimes(d-k)} &\rightarrow V^{\otimes k}, \\ v_1 \otimes v_2 \otimes \dots \otimes v_{d-k} &\mapsto \phi((v_1, v_2, \dots, v_{d-k}, \underbrace{*, \dots, *}_k)) \in (V^*)^{\otimes k} \cong V^{\otimes k}. \end{aligned}$$

We say that  $CE_k(\phi)$  contracts at  $\{1, 2, \dots, d - k\}$  and then inserts at  $\{d - k + 1, d - k + 2, \dots, d\}$ .

We shall call  $CE_k(\phi)$  a contraction operator if  $0 \leq k \leq \frac{d}{2}$ , and an expansion operator if  $[\frac{d}{2}] < k \leq d$ . Furthermore a contraction operator  $CE_k(\phi)$  is termed genuine if  $k < \frac{d}{2}$ .

For each pair of tuples  $I = \{i_1 < i_2 < \dots < i_{d-k}\}$ ,  $J = \{j_1 < j_2 < \dots < j_k\}$  of integers between 1 and  $n$ , the operator  $CE_k(\phi)$  induces in an obvious way an operator  $CE_k(\phi)_{I,J}$  which contracts at  $I$  and then inserts at  $J$ :

$$CE_k(\phi)_{I,J} : V^{\otimes n} \rightarrow V^{\otimes(n-d+2k)}.$$

Recall the  $G_2$ -invariants  $\alpha \in \text{Sym}^2(V^*)$ ,  $\beta \in \wedge^3(V^*)$  and  $\gamma \in \wedge^4(V^*)$ . We also identify  $V^*$  with  $V$ , as before. We then have the following contraction operators of the first kind:

$$\begin{aligned} A_I &= CE_0(\alpha)_{I,\emptyset} : V^{\otimes n} \rightarrow V^{\otimes(n-2)}, & |I| &= 2, \\ B_J &= CE_0(\beta)_{J,\emptyset} : V^{\otimes n} \rightarrow V^{\otimes(n-3)}, & |J| &= 3, \\ C_K &= CE_0(\gamma)_{K,\emptyset} : V^{\otimes n} \rightarrow V^{\otimes(n-4)}, & |K| &= 4, \end{aligned}$$

and the second kind

$$\begin{aligned} B'_J &= CE_1(\beta)_J : V^{\otimes n} \rightarrow V^{\otimes(n-1)}, & J &= \{J_1, J_2\}, |J_1| = 2, |J_2| = 1, \\ C'_K &= CE_1(\gamma)_K : V^{\otimes n} \rightarrow V^{\otimes(n-2)}, & K &= \{K_1, K_2\}, |K_1| = 3, |K_2| = 1, \end{aligned}$$

and finally the third kind

$$C''_K = CE_2(\gamma)_K : V^{\otimes n} \rightarrow V^{\otimes n}, \quad K = \{K_1, K_2\}, |K_1| = 2, |K_2| = 2.$$

Note that strictly speaking, we also have the contraction operator

$$A'_I = CE_1(\alpha)_I : V^{\otimes n} \rightarrow V^{\otimes n}, \quad I = \{I_1, I_2\}, |I_1| = 2, |I_2| = 2.$$

But since it happens to be the identity operator, we shall not include it here.

We also have the following expansion operators of the first kind:

$$\begin{aligned} \Phi_I &= CE_2(\alpha)_{\emptyset,I} : V^{\otimes(n-2)} \rightarrow V^{\otimes n}, & |I| &= 2, \\ \Psi_J &= CE_3(\beta)_{\emptyset,J} : V^{\otimes(n-3)} \rightarrow V^{\otimes n}, & |J| &= 3, \\ \Theta_K &= CE_4(\gamma)_{\emptyset,K} : V^{\otimes(n-4)} \rightarrow V^{\otimes n}, & |K| &= 4, \end{aligned}$$

and the second kind

$$\begin{aligned} \Psi'_J &= CE_2(\beta)_J : V^{\otimes(n-1)} \rightarrow V^{\otimes n}, & J &= \{J_1, J_2\}, |J_1| = 1, |J_2| = 2, \\ \Theta'_K &= CE_3(\gamma)_K : V^{\otimes(n-2)} \rightarrow V^{\otimes n}, & K &= \{K_1, K_2\}, |K_1| = 1, |K_2| = 3. \end{aligned}$$

We summarize the contraction and expansion operators in the following table:

	Contractions	Expansions
First Kind	$A : V^{\otimes 2} \rightarrow \mathbb{C}$ $B : V^{\otimes 3} \rightarrow \mathbb{C}$ $C : V^{\otimes 4} \rightarrow \mathbb{C}$	$\Phi : \mathbb{C} \rightarrow V^{\otimes 2}$ $\Psi : \mathbb{C} \rightarrow V^{\otimes 3}$ $\Theta : \mathbb{C} \rightarrow V^{\otimes 4}$
Second Kind	$B' : V^{\otimes 2} \rightarrow V$ $C' : V^{\otimes 3} \rightarrow V$	$\Psi' : V \rightarrow V^{\otimes 2}$ $\Theta' : V \rightarrow V^{\otimes 3}$
Third Kind	$C'' : V^{\otimes 2} \rightarrow V^{\otimes 2}$	

Note that all expansion operators of the first and second kinds are injective. The contraction operators of the first and second kind are surjective. The contraction operator of third kind is neither surjective nor injective. Its kernel and image will be computed in the next section.

Define the commutator algebra

$$\begin{aligned} \mathcal{B} &= \text{Hom}_{G_2}(V^{\otimes n}, V^{\otimes n}) \\ &= \{\phi : V^{\otimes n} \rightarrow V^{\otimes n} \mid \phi(g \cdot v) = g \cdot \phi(v), \forall v \in V^{\otimes n}, g \in G_2\}, \end{aligned}$$

where  $g$  acts diagonally on  $V^{\otimes n}$ .

**Theorem 4.1.** *The commutator algebra  $\mathcal{B}$  is generated by all permutations in  $S_n$ , the contraction operators and expansion operators.*

*Proof.* The proof of this theorem follows the same line as the proof of Theorem 17.19 [FH, Appendix F.2] We denote by  $\text{Sym}^d$  the homogeneous polynomial functions of degree  $d$  on  $V$ , i.e.  $\text{Sym}^d = \text{Sym}^d(V^*)$ . For an  $m$ -tuple  $\mathbf{d} = (d_1, \dots, d_m)$  of non-negative integers, we denote by  $\text{Sym}^{\mathbf{d}} = \text{Sym}^{d_1}(V^*) \otimes \dots \otimes \text{Sym}^{d_m}(V^*)$  the polynomials on  $V^{\oplus m}$  which are homogeneous of degree  $d_i$  in  $i$ -th variable. We note that

$$\text{Sym}^k(V^{\oplus m})^* = \bigoplus_{\mathbf{d}} \text{Sym}^{\mathbf{d}},$$

where the sum is over all  $m$ -tuples  $\mathbf{d}$  with  $d_1 + \dots + d_m = k$ . By the standard technique of full polarization, we only need to examine the  $G_2$ -invariants in  $\text{Sym}^{\mathbf{d}}$ , where  $\mathbf{d} = (1, \dots, 1)$ , namely  $G_2$ -invariants in  $(V^*)^{\otimes m}$ . Now applying Schwarz's Theorem (Theorem 2.1), we see that they are all polynomials in  $\alpha(x^{(i)}, x^{(j)})$ ,  $\beta(x^{(i)}, x^{(j)}, x^{(k)})$  and  $\gamma(x^{(i)}, x^{(j)}, x^{(k)}, x^{(l)})$ , and are all linear combinations of products such as

$$\begin{aligned} \alpha(x^{\sigma(1)}, x^{\sigma(2)}) \dots \beta(x^{\sigma(k+1)}, x^{\sigma(k+2)}, x^{\sigma(k+3)}) \\ \dots \gamma(x^{\sigma(l+1)}, x^{\sigma(l+2)}, x^{\sigma(l+3)}, x^{\sigma(l+4)}) \dots, \end{aligned}$$

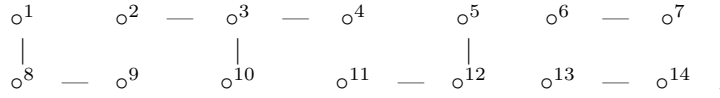
for some permutation  $\sigma$  of  $(1, \dots, m)$ . It is clear that 2 times the number of  $\alpha$ 's plus 3 times the number of  $\beta$ 's plus 4 times the number of  $\gamma$ 's is equal to  $m$ .

We need to reinterpret these invariants by means of the canonical isomorphism

$$(4.2) \quad (V^*)^{\otimes 2n} \cong (V^*)^{\otimes n} \otimes V^{\otimes n} \cong \text{Hom}(V^{\otimes n}, V^{\otimes n}) = \text{End}(V^{\otimes n}).$$

We claim that an invariant of the above form in  $(V^*)^{\otimes 2n}$  is taken by the isomorphism (4.2) to a composition of a permutation in  $S_n$  with some contractions, and then with some expansions and at last with another permutation in  $S_n$ . To unravel the

definitions and make things easier to see we will treat it pictorially rather than notationally. Suppose we are dealing with the case  $n = 7$ ; after a permutation  $\sigma$  for the 7 places and a permutation  $\tau$  of last 7 places, a  $G_2$ -invariant is  $\beta_{1,8,9} \cdot \gamma_{2,3,4,10} \cdot \beta_{5,11,12} \cdot \alpha_{6,7} \cdot \alpha_{13,14}$ . In a picture it looks as follows:



where the the positions connected by wedges represent invariants. Then the corresponding endomorphism is  $\tau \circ \Psi'_{\{1,8<9\}} \circ \Psi'_{\{5,11<12\}} \circ \Phi_{\{13<14\}} \circ A_{\{6<7\}} \circ C'_{\{2<3<4,10\}} \circ \sigma$ .  $\square$

*Remark 4.2.* It follows from the proof of Theorem 4.1 that each element in  $\mathcal{B}$  is a composition of following operations: first, a permutation; second, a composition of some contractions; and then a composition of some expansions; finally, another permutation.

5. TENSOR POWER DECOMPOSITION AND WEYL'S CONSTRUCTION

We put the standard Hermitian metric  $(\cdot, \cdot)$  on  $V$ . This extends to give a Hermitian metric on  $V^{\otimes n}$ . Note that this Hermitian metric is not  $G_2$ -invariant. The purpose of using this metric is for the convenience of using orthogonal complements for decompositions.

**Lemma 5.1.** *Among all contraction operators defined in the previous section, we have the following relations:*

- (a)  $\text{Ker}B'_J \subseteq \text{Ker}B_J$  and  $\text{Ker}B' |_{V^{\otimes 2}} \cong \text{Sym}^2V \oplus \mathfrak{g}$ . Here  $\mathfrak{g}$  denotes the adjoint representation of  $G_2$ .
- (b)  $\text{Ker}C'_K \subseteq \text{Ker}C_K$  and  $\text{Ker}C''_K \subseteq \text{Ker}C'_K$ .
- (c)  $\text{Ker}C'' |_{V^{\otimes 2}} = \text{Sym}^2V$  and  $C'' |_{\wedge^2 V}$  is an isomorphism.

*Remark.* Note that  $B' : V^{\otimes 2} \rightarrow V$  and  $B : V^{\otimes 3} \rightarrow \mathbb{C}$ . The formula  $\text{Ker}B'_J \subseteq \text{Ker}B_J$  should be understood as follows: we regard  $B'$  as a map  $V^{\otimes 2} \otimes V \rightarrow V \otimes V$  such that it is identity on the second factor  $V$ .

*Proof.* The inclusions given in (a) and (b) are obvious. We need only to prove  $\text{Ker}B' |_{V^{\otimes 2}} \cong \text{Sym}^2V \oplus \mathfrak{g}$  and formulae in (c). It follows from the fact that  $\beta$  is in  $\wedge^3V$  that  $\text{Sym}^2V$  is in the kernel of  $B'$ . We know that  $V \otimes V = \text{Sym}^2V \oplus \wedge^2V$  and  $\wedge^2V \cong \mathfrak{g} \oplus V$ . Note that  $B' : V \otimes V \rightarrow V$  is a non-zero  $G_2$ -equivariant map, since  $\beta$  is a  $G_2$ -invariant. Hence  $\text{Ker}B' = \text{Sym}^2V \oplus \mathfrak{g}$ . Since  $\gamma$  is in  $\wedge^4V$ ,  $\text{Sym}^2V$  is clearly contained in  $\text{Ker}C''$ .

It remains to prove that  $C''$  is an isomorphism when it is restricted to  $\wedge^2V$ . Following the notations in [FH], we decompose

$$V = W \oplus W^* \oplus \mathbb{C}$$

as  $SU(3)$ -modules. Here  $W$  is the 3-dimensional natural representation of  $SU(3)$  and  $W^*$  is the contragredient of  $W$ ,  $\mathbb{C}$  denotes the trivial representation of  $SU(3)$ . Now we fix a basis  $e_1, e_2, e_3$  for  $W$  and  $u$  a basis for  $\mathbb{C}$ . Let  $e_1^*, e_2^*, e_3^*$  be the dual basis for  $W^*$ . Then  $G_2$ -invariant  $\beta$  can be written as [FH, p. 359]

$$\sum_{i=1}^3 e_i \wedge u \wedge e_i^* + 2(e_1 \wedge e_2 \wedge e_3 + e_1^* \wedge e_2^* \wedge e_3^*).$$

Note that for an arbitrary  $n$ -dimensional vector space  $U$  with a nondegenerate quadratic form, there is a linear transformation (called *star map*)  $*$  :  $\wedge(U) \rightarrow \wedge(U)$ , which satisfies

$$* : \wedge^k(U) \rightarrow \wedge^{n-k}(U), \text{ and } ** = (-1)^{k(n-k)} \text{ on } \wedge^k(U).$$

Since  $\gamma \in \wedge^4 V$ , we have  $*\gamma \in \wedge^3 V$ . Note that since  $G_2$  fixes the quadratic form  $\alpha$  on  $V$ , the action of  $G_2$  commutes with the *star map*. Hence  $*\gamma$  is also a  $G_2$ -invariant. We have  $*\gamma = c\beta$  for some constant  $c$ . The map  $C''$  becomes

$$C'' : v \wedge w \mapsto *[(v \wedge w) \wedge (c\beta)], \text{ for any } v, w \in V.$$

An easy computation shows that  $C''$  is injective on  $\wedge^2 V$  and hence it is an isomorphism on  $\wedge^2 V$ .  $\square$

We define  $V^{[n]}$  ( $n \geq 4$ ) to be the intersection of all kernels of operators generated by first and second kind contractions:

$$V^{\otimes n} \rightarrow V^{\otimes(n-d)} \text{ for } d = 1, 2, 3, 4.$$

It follows from Lemma 5.1 that  $\text{Ker} B \supset \text{Ker} B'$ ,  $\text{Ker} C \supset \text{Ker} C'$  and  $C''$  is an isomorphism on  $\wedge^2 V \supset \text{Ker} B'$ . Thus  $V^{[n]}$  is actually intersection of the kernels of the contraction operators generated by  $A$ ,  $B'$  and  $C'$ . We make the convention that  $V^{[0]} = \mathbb{C}$ ,  $V^{[1]} = V$  and  $V^{[2]} = \text{Ker} A \cap \text{Ker} B'$ . For  $n = 3$ ,  $V^{[n]}$  is defined as the intersection of  $\text{Ker} A$ ,  $\text{Ker} B'$  and  $\text{Ker} C'$ , where  $A$  and  $B'$  are applied to any two factors of  $V^{\otimes 3}$ .

Clearly the actions of  $G_2$  and  $S_n$  both preserve  $V^{[n]}$ .

**Lemma 5.2.** *For  $n \geq 4$ , the tensor power  $V^{\otimes n}$  can be decomposed as follows:*

$$V^{\otimes n} = V^{[n]} \oplus \{ \Sigma[(\Phi_I(V^{\otimes(n-2)}) + \Psi'_J(V^{\otimes(n-1)}) + \Theta'_K(V^{\otimes(n-3)})] \},$$

the sum is over all pairs  $I = \{i < j\}$ , triples  $J = \{i, j < k\}$  and quadruplets  $K = \{i, j < k < l\}$  between 1 and  $n$ .

*Proof.* Let  $\alpha \in \text{Sym}^2 V$  be the  $G_2$ -invariant defined in §2. Let  $v, w \in V$  be any two vectors; then  $(\alpha, v \otimes w) = \alpha(v, w)$ . It follows that

$$(5.1) \quad \text{Ker}(A_I) = \text{Im}(\Phi_I)^\perp.$$

Similarly, we have

$$(5.2) \quad \text{Ker}(B'_J) = \text{Im}(\Psi'_J)^\perp \text{ and } \text{Ker}(C'_K) = \text{Im}(\Theta'_K)^\perp.$$

Hence the lemma follows from Lemma 5.1 and the formulae in (5.1) and (5.2).  $\square$

We denote by  $\mathbb{S}_{[\lambda]}(V)$  the intersection of  $\mathbb{S}_\lambda(V)$  and  $V^{[n]}$ . In other words, we have

$$\mathbb{S}_{[\lambda]}(V) = \text{Im}(c_\lambda : V^{[n]} \rightarrow V^{[n]}).$$

Let  $\mu_1, \mu_2$  be the fundamental weights of  $G_2$ . More precisely,  $\mu_1$  is the highest weight of the 7-dimensional irreducible representation, and  $\mu_2$  is the highest weight of the adjoint representation. We write  $\Gamma_{a,b}$  for the irreducible representation of  $G_2$  with highest weight  $a\mu_1 + b\mu_2$ . In this notation,  $\Gamma_{0,0} = \mathbb{C}$ ,  $\Gamma_{1,0} = V$  and  $\Gamma_{0,1} = \mathfrak{g}$ .

**Proposition 5.3.** *The space  $\mathbb{S}_{[\lambda]}(V)$  is nonzero if and only if the number of rows of the Young diagram of  $\lambda$  is at most 2.*

*Proof.* It is enough to show that if  $n \geq 3$ , then  $\wedge^3 V \otimes V^{\otimes(n-3)}$  is contained in the image of expansion operators on  $V^{\otimes(n-d)}$  for  $d = 1, 2, 3$ . Note that as  $G_2$ -module,  $\wedge^3 V \cong \Gamma_{2,0} \oplus V \oplus \mathbb{C}$ . Let us show  $\Gamma_{2,0} \otimes V^{\otimes(n-3)}$  is contained in  $\Psi'_{1,1<2}(V^{\otimes(n-1)})$  for example. Note that  $\Psi'$  maps  $V$  into  $\wedge^2 V$  and  $V \otimes V$  contains  $\Gamma_{2,0}$ . It follows  $\Psi'_{1,1<2}(V^{\otimes(n-1)}) = \Psi'(V) \otimes V \otimes V^{\otimes(n-3)}$ . Hence Lemma 5.2 implies that  $\mathbb{S}_{[\lambda]}(V) = 0$  provided  $\lambda_3 > 0$ .

Now for  $\lambda = (a + b, b)$  a partition of  $n$ , we show that the highest weight vector with weight equal to  $a\mu_1 + b\mu_2$  cannot be in the image of the expansion operators into  $V^{\otimes n}$ . This can be done by a mathematical induction on  $n$ . We show that for any pair  $I = \{i < j\}$ , triple  $J = \{i, j < k\}$  and quadruplet  $K = \{i, j < k < l\}$  between 1 and  $n$  a weight vector contained in

$$\Phi_I(V^{\otimes(n-2)}) + \Psi'_J(V^{\otimes(n-1)}) + \Theta'_K(V^{\otimes(n-3)})$$

has a weight equal to  $a\mu_1 + b\mu_2$  such that  $a + 2b < n$ . In other words, a highest weight vector of  $\Gamma_{a,b}$  with  $a + 2b = n$  cannot be in the image of expansion operators. Assume that this is true for any positive integer smaller than  $n$ . Now we show this is also true for  $n$ . We may assume  $W$  is an irreducible  $G_2$ -module contained in  $\Psi'_J(V^{\otimes(n-1)})$ ; the same argument will also work for  $\Phi_I(V^{\otimes(n-2)})$  and  $\Theta'_K(V^{\otimes(n-3)})$ . We have  $\Psi'_J(V^{\otimes(n-1)}) \cong \Psi'(V) \otimes V^{\otimes(n-2)}$  and  $\Psi'(V) \cong V$  is contained in  $\wedge^2 V$ . So  $W$  is contained in  $\Psi'(V) \otimes V^{\otimes(n-2)}$ . In particular it is contained in  $\Psi'(V) \otimes \Gamma_{a',b'}$  for some non-negative integers  $a', b'$  such that  $a' + 2b' = n - 2$ . For simplicity, we denote by  $(a, b)$  the weight  $a\mu_1 + b\mu_2$ . Note that the weight vectors of  $\Psi'(V) \cong V$  have weights equal to

$$(1, 0), (-1, 1), (1, -1), (-1, 1), (2, -1), (-2, 1).$$

So a highest weight of an irreducible representation contained in  $\Psi(V) \otimes \Gamma_{a',b'}$  has weight  $(a_1, b_1)$  equal to one of the following:

$$(a' + 1, b'), (a' - 1, b'), (a' + 1, b' - 1), \\ (a' - 1, b' + 1), (a' + 2, b' - 1), (a' - 2, b' + 1).$$

Then we have  $a_1 + 2b_1 \leq a' + 1 + 2b' = n - 1$ . It follows that the highest weight vector of  $\Gamma_{a,b}$  with  $a + 2b = n$  cannot be in the image of expansion operators. Hence the proposition is proved.  $\square$

We denote by  $\mathcal{A}$  the algebra of all endomorphisms of the space  $V^{\otimes n}$  which are  $\mathbb{C}$ -linear combinations of the operators of the form  $g \otimes \cdots \otimes g$ , for  $g \in G_2$ .

**Proposition 5.4.** *The algebra  $\mathcal{A}|_{V^{[n]}}$  is precisely the algebra of all endomorphisms of  $V^{[n]}$  commuting with all permutations in  $S_n$ .*

*Proof.* First, by the simplicity of the group  $G_2$  we know that  $\mathcal{A}$  is semisimple. In Theorem 4.1 we computed that the ring  $\mathcal{B}$  of the commutators of  $\mathcal{A}$  is the ring generated by permutations, contractions and expansions. By the general theory of semisimple algebras,  $\mathcal{A}$  must be the commutator algebra of  $\mathcal{B}$ .

Secondly, if  $E$  is an endomorphism of  $V^{[n]}$  commuting with all permutations of factors, then the endomorphism  $\tilde{E}$  of  $V^{\otimes n}$  which is  $E$  on the factor  $V^{[n]}$  and zero on the complementary summand is an endomorphism that commutes with all elements in the algebra  $\mathcal{B}$ . It follows that  $\tilde{E}$  is in  $\mathcal{A}$ , and so  $E$  is in  $\mathcal{A}|_{V^{[n]}}$ .  $\square$

**Theorem 5.5.** *If  $\lambda = (\lambda_1, \lambda_2)$  is a Young diagram, where  $\lambda_1 = a + b, \lambda_2 = b$ , then  $\mathbb{S}_{[\lambda]}(V)$  is the irreducible representation  $\Gamma_{a,b}$  of  $G_2$  with highest weight  $a\mu_1 + b\mu_2$ .*



*Proof.* The algebra  $\mathcal{A}|_{V^{[n]}}$  is the commutator algebra to the group algebra  $\mathbb{C}S_n$  which acts on the space  $V^{[n]}$ . It follows  $\mathbb{S}_{[\lambda]}(V)$  is an irreducible  $\mathcal{A}|_{V^{[n]}}$ -module. Hence it is an irreducible  $G_2$ -module. The highest weight of  $\mathbb{S}_{[\lambda]}(V)$  can be determined as follows: since the highest weight vector with weight equal to  $a\mu_1 + b\mu_2$  is contained in  $\mathbb{S}_{[\lambda]}(V)$  as shown in the proof of Proposition 5.3,  $\mathbb{S}_{[\lambda]}(V)$  has to be the irreducible representation  $\Gamma_{a,b}$ .  $\square$

**Theorem 5.6.** *We have the following spectral decomposition of  $V^{[n]}$  as  $G_2 \times S_n$  modules:*

$$V^{[n]} \cong \bigoplus \mathbb{S}_{[\lambda]}(V) \otimes V_\lambda,$$

where the sum is over all partitions  $\lambda = (\lambda_1 \geq \lambda_2 \geq 0)$  of  $n$  into at most two parts. Therefore, the multiplicity of  $\mathbb{S}_{[\lambda]}(V)$  occurring in  $V^{[n]}$  is equal to the dimension  $m_\lambda$  of the corresponding representation  $V_\lambda$  of  $S_n$ .

*Proof.* This follows from Proposition 5.3 and Theorem 5.5 and from the corresponding result for the general linear group  $GL(V)$  (Theorem 3.1).  $\square$

Here is the decomposition of  $V^{[n]}$  for  $n = 2, 3$  and  $4$ :

$$V^{[2]} = \Gamma_{2,0} \oplus \Gamma_{0,1}; \quad V^{[3]} = \Gamma_{3,0} \oplus 2\Gamma_{1,1}; \quad V^{[4]} = \Gamma_{4,0} \oplus 3\Gamma_{2,1} \oplus 2\Gamma_{0,2}.$$

For  $d = 0, 1, \dots, n$ , let  $V_{n-d}^{[n]}$  be the subspace of  $V^{\otimes n}$  consisting of images of all expansion operators of  $V^{[d]} \rightarrow V^{\otimes n}$ . More precisely,

$$V_{n-d}^{[n]} = \Sigma \Phi_I \circ \dots \circ \Psi'_J \circ \dots \circ \Theta'_K (V^{[d]}),$$

where the sum is over all possible compositions of expansion operators generated by  $\Phi$ ,  $\Psi'$  and  $\Theta'$  such that the image is in  $V^{\otimes n}$ . Since these expansion operators are all injective,  $V_{n-d}^{[n]}$  is isomorphic to the direct sum of several copies of  $V^{[d]}$ . Hence the irreducible  $G_2$ -module occurring in  $V_{n-d}^{[n]}$  as a summand is exactly the same as those occurring in  $V^{[d]}$ . It is clear that in the case  $d = n$  we have  $V_0^{[n]} = V_{n-n}^{[n]}$  is  $V^{[n]}$ .

**Theorem 5.7.** *The tensor power  $V^{\otimes n}$  decomposes into a direct sum*

$$V^{\otimes n} = V_0^{[n]} \oplus V_1^{[n]} \oplus \dots \oplus V_n^{[n]}.$$

*Proof.* It follows from Lemma 5.2 and by induction on  $n$  that  $V^{\otimes n}$  is equal to the sum of all  $V_{n-d}^{[n]}$  for  $d = 0, 1, \dots, n$ . We need to show it is a direct sum. This follows from Theorem 5.5 and Theorem 5.6 that  $V_i^{[n]}$  and  $V_j^{[n]}$  contain different irreducible  $G_2$ -modules for  $i \neq j$ . Hence the intersection of  $V_i^{[n]}$  and  $V_j^{[n]}$  is  $\{0\}$  when  $i \neq j$ .  $\square$

*Remark.* Recall that we have convention:  $V^{[0]} = V^{\otimes 0} = \mathbb{C}$  and  $V^{[1]} = V^{\otimes 1} = V$ . Then it is clear that

$$V^{\otimes 2} = V \otimes V \cong V^{[2]} \oplus V \oplus \mathbb{C},$$

which is exactly what Theorem 5.7 says for  $n = 2$ .

#### ACKNOWLEDGMENTS

We would like to thank W.-Y. Hsiang for initiating this project.

## REFERENCES

- [H] R. Howe, *Perspectives on invariant theory: Schur duality, multiplicity-free actions and beyond*, Israel Math. Conf. Proc. **8** (1995), 1-182. MR **96e**:13006
- [FH] W. Fulton and J. Harris, *Representation Theory*, GTM 129, Springer-Verlag, 1991. MR **93a**:20069
- [MP] W. G. McKay and J. Patera, *Tables of Dimensions, Indices and Branching Rules for Representations of Simple Lie Algebras*, Marcel Dekker, New York – Basel, 1981. MR **82i**:17008
- [S] G. Schwarz, *Invariant theory of  $G_2$  and  $Spin_7$* , Comment. Math. Helvetici **63** (1988), 624-663. MR **89k**:14080
- [W] H. Weyl, *The classical groups, Princeton Mathematical Series*, Princeton University Press, 1939; second edition, 1946; third edition, 1966. MR **1**:42c
- [Wo] Y.-K. Wong, *The First Fundamental Theorem of Covariants for  $G_2$  and  $Spin_7$* , Yale thesis, Yale University, 1995.

DEPARTMENT OF MATHEMATICS, THE HONG KONG UNIVERSITY OF SCIENCE AND TECHNOLOGY,  
CLEAR WATER BAY, KOWLOON, HONG KONG  
*E-mail address*: mahuang@uxmail.ust.hk

DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, KENT RIDGE, SINGAPORE 0511  
*E-mail address*: matzhucb@leonis.nus.sg