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A κ-NORMAL, NOT DENSELY NORMAL TYCHONOFF SPACE

WINFRIED JUST AND JAMAL TARTIR

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ABSTRACT. We give an example of a κ-normal space which is not densely normal.

In [2] Arhangelskii introduced the definition of a densely normal topological space and noted that every densely normal space is κ-normal. The definition of a κ-normal topological space was introduced by Stchepin in [1]. Problem 25 of [2] asked whether every κ-normal space is densely normal. Here we show that the answer is negative.[1]

Definition 1. A space X is κ-normal if whenever E, F are disjoint canonical closed subsets of X there exist disjoint open subsets of X, U and V, such that $E \subseteq U$ and $F \subseteq V$.

Recall that a canonical closed set is a set which is equal to the closure of its interior.

One version of relative normality is the idea of X being normal on a subspace Y, which was introduced by Arhangelskii in [2]. If A and Y are subsets of a space X, A is concentrated on Y if $A \subseteq \overline{A \cap Y}$. A space X is normal on a subspace Y if whenever E and F are disjoint closed subsets of X concentrated on Y, then there are disjoint open $U, V \subseteq X$ such that $E \subseteq U$ and $F \subseteq V$.

Definition 2. A space X is densely normal if there exists a dense subspace Y of X such that X is normal on Y.

Theorem 1. *There is a Tychonoff space which is κ-normal but not densely normal.*

Let $C_{\mathbf{R}}$ denote the Cantor set. For each bounded $I \subseteq \mathbf{R}$, let $l(I)$ denote the infimum of I and let $r(I)$ denote the supremum of I. Let $D \subseteq \mathbf{R}\backslash(\mathbf{Q} \cup C_{\mathbf{R}})$ be a countable dense subset of $\mathbf{R}$. Let $X = \mathbf{R}\backslash[D \cup (C_{\mathbf{R}} \cap \mathbf{Q})]$ and let $\tau_{\mathbf{R}}$ be the subspace topology on X inherited from $\mathbf{R}$. We will define a topology τ on X such that $\tau_{\mathbf{R}} \subseteq \tau$. In order to distinguish $(X, \tau_{\mathbf{R}})$ from (X, τ) we will write $X_{\mathbf{R}}$ when considering X as a subspace of $\mathbf{R}$. For each $x \in X$ we will describe the basic open neighborhoods of x and define a local base $\mathcal{B}_x$ at x. If $x \in \mathbf{Q}$, an open neighborhood of x is any element of $\tau_{\mathbf{R}}$ that contains x. Thus, if $x \in \mathbf{Q}$, let $\mathcal{B}_x = \{U \subseteq X\colon U$ is open in $X_{\mathbf{R}}, x \in U\}$. We must go to some length to describe $\mathcal{B}_x$ for $x \notin \mathbf{Q}$.

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[1]Subsequent to our result, O. Pavlov has given another example of a κ-normal, not densely normal Tychonoff space. However, this construction is very different from ours.

Let $C_0 = [0, \frac{1}{3}]$ and let $C_1 = [\frac{2}{3}, 1]$. Let $q_0 = \frac{1}{6}$ and let $q_1 = \frac{5}{6}$. Also, let $I_0 = I_1 = (0, 1)$. Define open intervals I_{00}, I_{01}, I_{10}, I_{11} of $X_{\mathbf{R}}$ as follows. Let $I_{00}, I_{01} \subseteq (\frac{1}{9}, \frac{2}{9}) \cap X$ such that $l(I_{00}) < l(I_{01}) < \frac{1}{6} < r(I_{00}) < r(I_{01})$ and $l(I_{00}), r(I_{00}), l(I_{01}), r(I_{01}) \in D$. Choose $I_{10}, I_{11} \subseteq (\frac{7}{9}, \frac{8}{9}) \cap X$ such that $l(I_{10}) < l(I_{11}) < \frac{5}{6} < r(I_{10}) < r(I_{11})$ and $l(I_{10}), r(I_{10}), l(I_{11}), r(I_{11}) \in D$. Let $n \in \omega \backslash \{0\}$ and let $s \in {}^n\{0, 1\}$, and suppose C_s has been defined. Suppose $C_s = [a, b]$, and let $C_{s\frown 0} = [a, a + \frac{b-a}{3}]$ and $C_{s\frown 1} = [b - \frac{b-a}{3}, b]$. Also, let q_s be the midpoint of C_s and choose open intervals of $X_{\mathbf{R}}$, $I_{s\frown 0}$ and $I_{s\frown 1}$, such that $I_{s\frown 0}, I_{s\frown 1} \subseteq C_s \backslash (C_{s\frown 0} \cup C_{s\frown 1})$ and $l(I_{s\frown 0}) < l(I_{s\frown 1}) < q_s < r(I_{s\frown 0}) < r(I_{s\frown 1})$ and $l(I_{s\frown 0}), r(I_{s\frown 0}), l(I_{s\frown 1}), r(I_{s\frown 1}) \in D$. For each $f \in {}^\omega\{0, 1\}$, let $x_f = \lim_{n\to\infty} q_{f|_n}$ and let $\mathbf{C} = \{x_f : f \in {}^\omega\{0, 1\}\} \cap X$.

Note that $\mathbf{C}$ is perfect, $|\, \mathbf{C} \,| = 2^{\aleph_0}$, and $\mathbf{C}$ with the subspace topology inherited from $\mathbf{R}$ is a G_δ subspace of $\mathbf{R}$. Hence, $\mathbf{C}$ is a Polish space and the Baire Category Theorem applies.

Now partition $\mathbf{C}$ into two disjoint sets A_0 and A_1 (which are not of first Baire category in $\mathbf{C}$) as follows. Let $\{I_n\}_{n\in\omega}$ be an indexing of all open intervals of $X_{\mathbf{R}}$ with endpoints in D such that $I_n \cap \mathbf{C} \neq \emptyset$. For each $n \in \omega$, let $\{\mathcal{K}^n_\alpha\}_{\alpha<2^{\aleph_0}}$ be an indexing of all countable collections of closed nowhere dense subsets of $I_n \cap \mathbf{C}$. For each $\alpha < 2^{\aleph_0}$ and for each $n \in \omega$, choose

$$\begin{aligned} x_{(\alpha,n,0)}, x_{(\alpha,n,1)} \in (\mathbf{C} \cap I_n) \backslash [(\bigcup \mathcal{K}^n_\alpha) \cup (\bigcup_{\beta<\alpha} \bigcup_{j\in\omega} \{x_{(\beta,j,0)}, x_{(\beta,j,1)}\}) \\ \cup (\bigcup_{k<n} \{x_{(\alpha,k,0)}, x_{(\alpha,k,1)}\})] \end{aligned}$$

such that $x_{(\alpha,n,0)} \neq x_{(\alpha,n,1)}$. Let $A_0 = \{x_{(\alpha,n,0)} : \alpha < 2^{\aleph_0}, n \in \omega\}$ and let $A_1 = \mathbf{C} \backslash A_0$.

Let $\{\langle U_\alpha, V_\alpha\rangle\}_{\alpha<2^{\aleph_0}}$ be an indexing of all pairs of disjoint open subsets of $X_{\mathbf{R}}$ with $\langle U_0, V_0\rangle$ chosen such that $|\, cl_{X_{\mathbf{R}}}(U_0) \cap cl_{X_{\mathbf{R}}}(V_0) \,| = 2^{\aleph_0}$ and $[cl_{X_{\mathbf{R}}}(U_0) \cup cl_{X_{\mathbf{R}}}(V_0)] \cap \mathbf{C} = \emptyset$. Let $z_0 \in [cl_{X_{\mathbf{R}}}(U_0) \cap cl_{X_{\mathbf{R}}}(V_0)]$. For $1 \leq \alpha < 2^{\aleph_0}$, if $|\, cl_{X_{\mathbf{R}}}(U_\alpha) \cap cl_{X_{\mathbf{R}}}(V_\alpha)| \leq \aleph_0$, let $z_\alpha = z_0$; otherwise choose $z_\alpha \in [cl_{X_{\mathbf{R}}}(U_\alpha) \cap cl_{X_{\mathbf{R}}}(V_\alpha)] \backslash \bigcup_{\beta<\alpha}\{z_\beta\}$. The construction of $\{z_\alpha\}_{\alpha<2^{\aleph_0}}$ and the neighborhoods of the elements of $\{z_\alpha\}_{\alpha<2^{\aleph_0}}$ will result in the following property which will be used to prove that (X, τ) is κ-normal:

† Whenever E, F are disjoint canonical closed subsets of X, $|\, cl_{X_{\mathbf{R}}}(int_X E) \cap cl_{X_{\mathbf{R}}}(int_X F) \,| \leq \aleph_0$.

Now we describe the basic open neighborhoods of elements of $X \backslash \mathbf{Q}$. This construction will prevent the separation of A_0 and A_1 by disjoint open subsets of X. For each $x \in X \backslash \mathbf{Q}$ and for each $n \in \omega$, we will define a set I^x_n and let $\mathcal{B}_x = \{\{x\} \cup \bigcup_{n\geq k} I^x_n : k \in \omega\}$.

<u>Case 1</u> : $x \notin (\{z_\alpha\}_{\alpha\, <2^{\aleph_0}} \cup A_0 \cup A_1)$.

Let $\{q_n\}_{n\in\omega} \subseteq \mathbf{Q} \cap X$ such that $q_n \to x$. Also, let $\{I^x_n\}_{n\in\omega}$ be a sequence of pairwise disjoint open intervals of $X_{\mathbf{R}}$ with endpoints in D such that $q_n \in I^x_n$.

<u>Case 2</u> : $x \in \{z_\alpha\}_{\alpha<2^{\aleph_0}} \backslash (A_0 \cup A_1)$.

Then $x = z_\beta$ for some $\beta < 2^{\aleph_0}$. Choose a sequence $\{q_n\}_{n\in\omega} \subseteq \mathbf{Q} \cap X$ such that $q_n \to x$ and $q_n \in U_\beta$ if n is even and $q_n \in V_\beta$ if n is odd. Also, let $\{I^x_n\}_{n\in\omega}$ be a sequence of disjoint open intervals of $X_{\mathbf{R}}$ with endpoints in D such that $q_n \in I^x_n$.

<u>Case 3</u> : $x \in A_0 \backslash \{z_\alpha\}_{\alpha<2^{\aleph_0}}$. $[x \in A_1 \backslash \{z_\alpha\}_{\alpha<2^{\aleph_0}}]$.

Then there exists $f \in {}^\omega\{0, 1\}$ such that $x = x_f$. Let $I^x_0 = ((0, 1) \cap X_{\mathbf{R}}) \backslash (C_0 \cup C_1)$. For $n \in \omega \backslash \{0\}$, let $I^x_n = I_{f|_n \frown 0}$ $[I^x_n = I_{f|_n \frown 1}]$.

<u>Case 4</u> : $x \in A_0 \cap \{z_\alpha\}_{\alpha<2^{\aleph_0}}$ $[x \in A_1 \cap \{z_\alpha\}_{\alpha<2^{\aleph_0}}]$.

Then $x = x_f$ for some $f \in {}^\omega\{0,1\}$ and $x = z_\beta$ for some $\beta < 2^{\aleph_0}$.

<u>Subcase 4a</u> : $I_{f|_n{}^\frown 0} \cap U_\beta \neq \emptyset$ $[I_{f|_n{}^\frown 1} \cap U_\beta \neq \emptyset]$ for infinitely many $n \in \omega$ and $I_{f|_n{}^\frown 0} \cap V_\beta \neq \emptyset$ $[I_{f|_n{}^\frown 1} \cap V_\beta \neq \emptyset]$ for infinitely many $n \in \omega$.

For each $n \in \omega$, let $I_n^x = I_{f|_n{}^\frown 0}$ $[I_n^x = I_{f|_n{}^\frown 1}]$.

<u>Subcase 4b</u> : $I_{f|_n{}^\frown 0} \cap U_\beta \neq \emptyset$ $[I_{f|_n{}^\frown 1} \cap U_\beta \neq \emptyset]$ for infinitely many $n \in \omega$ and $I_{f|_n{}^\frown 0} \cap V_\beta = \emptyset$ $[I_{f|_n{}^\frown 1} \cap V_\beta = \emptyset]$ for all but finitely many $n \in \omega$.

Choose a sequence $\{q_n\}_{n\in\omega} \subseteq \mathbf{Q} \cap V_\beta$ such that $q_n \to x$. Also, choose a sequence $\{I_n^\beta\}_{n\in\omega}$ of pairwise disjoint open intervals in $X_{\mathbf{R}}$ with endpoints in D such that $q_n \in I_n^\beta$ and $I_n^\beta \cap (\bigcup_{m\in\omega} I_{f|_m{}^\frown 0}) = \emptyset$ $[I_n^\beta \cap (\bigcup_{m\in\omega} I_{f|_m{}^\frown 1}) = \emptyset]$. Let $N_3 \subseteq \{3^n : n \in \omega\}$ be such that $| N_3 |= \aleph_0$ and $| \{n \in \omega : I_{f|_n{}^\frown 0} \cap U_\beta \neq \emptyset\} \backslash N_3 |= \aleph_0$ $[| \{n \in \omega : I_{f|_n{}^\frown 1} \cap U_\beta \neq \emptyset\} \backslash N_3 |= \aleph_0]$. For each $n \in \omega$, if $n \in N_3$, let $I_n^x = I_n^\beta$; otherwise, let $I_n^x = I_{f|_n{}^\frown 0}$ $[I_n^x = I_{f|_n{}^\frown 1}]$.

<u>Subcase 4c</u> : $I_{f|_n{}^\frown 0} \cap V_\beta \neq \emptyset$ $[I_{f|_n{}^\frown 1} \cap V_\beta \neq \emptyset]$ for infinitely many $n \in \omega$ and $I_{f|_n{}^\frown 0} \cap U_\beta = \emptyset$ $[I_{f|_n{}^\frown 1} \cap U_\beta = \emptyset]$ for all but finitely many $n \in \omega$.

Choose a sequence $\{q_n\}_{n\in\omega} \subseteq \mathbf{Q} \cap U_\beta$ such that $q_n \to x$. Also, choose a sequence $\{I_n^\beta\}_{n\in\omega}$ of pairwise disjoint open intervals in $X_{\mathbf{R}}$ with endpoints in D such that $q_n \in I_n^\beta$ and $I_n^\beta \cap (\bigcup_{m\in\omega} I_{f|_m{}^\frown 0}) = \emptyset$ $[I_n^\beta \cap (I_{f|_m{}^\frown 1}) = \emptyset]$. Let $N_2 \subseteq \{2^n : n \in \omega\}$ be such that $| N_2 |= \aleph_0$ and $| \{n \in \omega : I_{f|_n{}^\frown 0} \cap V_\beta \neq \emptyset\} \backslash N_2 |= \aleph_0$ $[| \{n \in \omega : I_{f|_n{}^\frown 1} \cap V_\beta \neq \emptyset\} \backslash N_2 |= \aleph_0]$. For each $n \in \omega$, if $n \in N_2$, let $I_n^x = I_n^\beta$; otherwise, let $I_n^x = I_{f|_n{}^\frown 0}$ $[I_n^x = I_{f|_n{}^\frown 1}]$.

<u>Subcase 4d</u> : $I_{f|_n{}^\frown 0} \cap U_\beta = \emptyset$ $[I_{f|_n{}^\frown 1} \cap U_\beta = \emptyset]$ for all but finitely many $n \in \omega$ and $I_{f|_n{}^\frown 0} \cap V_\beta = \emptyset$ $[I_{f|_n{}^\frown 1} \cap V_\beta = \emptyset]$ for all but finitely many $n \in \omega$.

Choose a sequence $\{q_n\}_{n\in\omega} \subseteq \mathbf{Q} \cap X$ such that $q_n \to x$ and $q_n \in U_\beta$ if n is even and $q_n \in V_\beta$ if n is odd. Also, let $\{I_n^\beta\}_{n\in\omega}$ be a sequence of pairwise disjoint open intervals of $X_{\mathbf{R}}$ with endpoints in D such that $q_n \in I_n^\beta$ and such that $(\bigcup_{n\in\omega} I_n^\beta) \cap (\bigcup_{n\in\omega} I_{f|_n{}^\frown 0}) = \emptyset$ $[(\bigcup_{n\in\omega} I_n^\beta) \cap (\bigcup_{n\in\omega} I_{f|_n{}^\frown 1}) = \emptyset]$. If n is a power of 2 or a power of 3, let $I_n^x = I_n^\beta$. Otherwise, let $I_n^x = I_{f|_n{}^\frown 0}$ $[I_n^x = I_{f|_n{}^\frown 1}]$.

Let τ be the topology on X generated by $\bigcup_{x\in X} \mathcal{B}_x$. Note that (X,τ) is Hausdorff since τ is stronger than the subspace topology on X inherited from $\mathbf{R}$. Also, note that (X,τ) is Tychonoff since τ has a base of clopen sets.

Lemma 1. *(X,τ) is κ-normal.*

Proof. Let E, F be disjoint canonical closed subsets of X. Note that $cl_X(int_{X_{\mathbf{R}}}(E)) \subseteq cl_X(int_X(E)) = cl_X(int_X(E) \cap \mathbf{Q}) = cl_X(int_{X_{\mathbf{R}}}(E) \cap \mathbf{Q}) \subseteq cl_X(int_{X_{\mathbf{R}}}(E))$.

Therefore, $E = cl_X(E_0)$ and $F = cl_X(F_0)$ where $E_0 = int_{X_{\mathbf{R}}}(E)$ and $F_0 = int_{X_{\mathbf{R}}}(F)$. Let $A = cl_{X_{\mathbf{R}}}(E_0) \cap cl_{X_{\mathbf{R}}}(F_0)$. By the construction of $\{z_\alpha\}_{\alpha<2^{\aleph_0}}$, and cases 2 and 4 of the definition of τ, $| A | \leq \aleph_0$. Enumerate $A = \{a_k\}_{k\in\omega}$ and note that $A \cap \mathbf{Q} = \emptyset$. For each $x \in E \cup F$, we need to define W_x open in X with $x \in W_x$. First, let $k \in \omega$ and consider a_k. If $a_k \in E$, choose $n_k \in \omega$ such that $(\{a_k\} \cup \bigcup_{i\geq n_k} I_i^{a_k}) \cap (F \cup \bigcup\{W_{a_n} : n < k \text{ and } a_n \in F\}) = \emptyset$ and such that $\{a_k\} \cup \bigcup_{i\geq n_k} I_i^{a_k} \subseteq (a_k - \frac{1}{k}, a_k + \frac{1}{k})$. If $a_k \in F$, choose $n_k \in \omega$ such that $(\{a_k\} \cup \bigcup_{i\geq n_k} I_i^{a_k}) \cap (E \cup \bigcup\{W_{a_n} : n < k \text{ and } a_n \in E\}) = \emptyset$, and such that $\{a_k\} \cup \bigcup_{i\geq n_k} I_i^{a_k} \subseteq (a_k - \frac{1}{k}, a_k + \frac{1}{k})$. Let $W_{a_k} = \{a_k\} \cup \bigcup_{i\geq n_k} I_i^{a_k}$. Now suppose that $x \in E\backslash A$ $[x \in F\backslash A]$. Then there exists $\delta > 0$ such that $(x-\delta, x+\delta) \cap F = \emptyset$ $[(x-\delta, x+\delta) \cap E = \emptyset]$. Choose $n_\delta \in \omega\backslash\{0\}$ such that $\frac{1}{n_\delta} < \frac{\delta}{2}$. Then for each $n > n_\delta$ such that $a_n \in F$ $[a_n \in E]$, we have $(x - \frac{\delta}{2}, x + \frac{\delta}{2}) \cap W_{a_n} = \emptyset$. For each $n < n_\delta$ such that $a_n \in F$ $[a_n \in E]$, there exists $\delta_n > 0$ such that $(x-\delta_n, x+\delta_n) \cap W_{a_n} = \emptyset$. Let $\delta_x = min\{\frac{\delta}{2}, min\{\delta_n : n < n_\delta \text{ and } a_n \in F\ [a_n \in E]\}\}$. Choose an interval I_x which is open in $X_{\mathbf{R}}$ with endpoints in D such that $I_x \subseteq (x - \frac{\delta_x}{4}, x + \frac{\delta_x}{4})$. Let $W_x = I_x$. Let $U = \bigcup_{x\in E} W_x$ and let $V = \bigcup_{x \in F} W_x$. To see that $U \cap V = \emptyset$, choose $s \in E$ and $t \in F$ and consider $W_s \cap W_t$.

Case 1 : $s, t \in A$.

Then there exists $n_s, n_t \in \omega$ such that $s = a_{n_s}$ and $t = a_{n_t}$. Then $W_{a_{n_s}} \cap W_{a_{n_t}} = \emptyset$ by design.

Case 2 : Exactly one of s and t is an element of A.

Without loss of generality suppose $t \in A$. Then W_s is chosen such that $W_s \cap W_t = \emptyset$.

Case 3 : $s, t \notin A$.

Without loss of generality suppose $\delta_s > \delta_t$. If $W_s \cap W_t \neq \emptyset$, then $(s - \frac{\delta_s}{4}, s + \frac{\delta_s}{4}) \cap (t - \frac{\delta_t}{4}, t + \frac{\delta_t}{4}) \neq \emptyset$, which implies $t \in (s - \delta_s, s + \delta_s)$ and contradicts the choice of δ_s. Hence, $W_s \cap W_t = \emptyset$.

Therefore, X is κ-normal. $\square$

Lemma 2. *(X, τ) is not densely normal.*

Proof. Let Y be dense in X. For each $n \in \omega \backslash \{0\}$ and for each $s \in {}^n\{0,1\}$ choose $y_{s\frown 0} \in (I_{s\frown 0} \backslash I_{s \frown 1}) \cap Y$, and choose $y_{s\frown 1} \in (I_{s\frown 1} \backslash I_{s\frown 0}) \cap Y$. Let $Y_0 = \{y_{f\restriction_n^\frown 0} : f \in {}^\omega\{0,1\} \text{ and } x_f \in \mathbf{C}\}$ and let $Y_1 = \{y_{f\restriction_n^\frown 1} : f \in {}^\omega\{0,1\} \text{ and } x_f \in \mathbf{C}\}$.

Claim : $cl_X(Y_0) = Y_0 \cup A_0$ and $cl_X(Y_1) = Y_1 \cup A_1$.

Proof. By design, $Y_0 \cup A_0 \subseteq cl_X(Y_0)$. To see that $cl_X(Y_0) \subseteq Y_0 \cup A_0$, let $y \in cl_X(Y_0) \backslash Y_0$. Note that $y \in cl_{X_{\mathbf{R}}}(Y_0)$ which implies that $y \in cl_{X_{\mathbf{R}}}(\{q_{f\restriction_n} : f \in {}^\omega\{0,1\}, n \in \omega\})$. Let $B_0^0 = \{q_{f\restriction_n} : f \in {}^\omega\{0,1\}, n \in \omega, \text{ and } f(0) = 0\}$ and let $B_0^1 = \{q_{f\restriction_n} : f \in {}^\omega\{0,1\}, n \in \omega, \text{ and } f(0) = 1\}$. Then $y \in cl_{X_{\mathbf{R}}}(B_0^0)$ or $y \in cl_{X_{\mathbf{R}}}(B_0^1)$. If $y \in cl_{X_{\mathbf{R}}}(B_0^0)$, let $B_0 = B_0^0$; otherwise, let $B_0 = B_0^1$. For $k \in \omega\backslash\{0\}$, let $B_k^0 = \{q_{f\restriction_n} \in B_{k-1} : f \in {}^\omega\{0,1\}, n \in \omega, \text{ and } f(k) = 0\}$ and let $B_k^1 = \{q_{f\restriction_n} \in B_{k-1} : f \in {}^\omega\{0,1\}, n \in \omega, \text{ and } f(k) = 1\}$. Then $y \in cl_{X_{\mathbf{R}}}(B_k^0) \cup cl_{X_{\mathbf{R}}}(B_k^1)$. If $y \in cl_{X_{\mathbf{R}}}(B_k^0)$, let $B_k = B_k^0$; otherwise, let $B_k = B_k^1$. Now define $f_y : \omega \to \{0,1\}$ by $f_y(k) = i$ where $B_k = B_k^i$. Then $y \in \bigcap_{k\in\omega} cl_{X_{\mathbf{R}}}(B_k) = cl_{X_{\mathbf{R}}}(\{q_{f_y\restriction_n}\}_{n\in\omega}) = \{x_f\}$. Hence, $y \in A_0 \cup A_1$ but $y \notin A_1$ by the construction of Y_0 and Y_1. Therefore, $y \in A_0$ as desired. Similarly, $cl_X(Y_1) = Y_1 \cup A_1$. $\square$

To see that X is not normal on Y, suppose that U, V are open subsets of X such that $cl_X(Y_0) \subseteq U$ and $cl_X(Y_1) \subseteq V$. Let $D_0^k = \{x \in A_0 : \{x\} \cup \bigcup_{n\geq k} I_n^x \subseteq U\}$. Note that $A_0 = \bigcup_{k\in\omega} D_0^k \subseteq \bigcup_{k\in\omega} cl_{X_{\mathbf{R}}}(D_0^k)$. Recall that A_0 is constructed such that A_0 is not of first Baire category in $\mathbf{C}$. So there exists $k_0 \in \omega$ such that $int_{X_{\mathbf{R}}}(cl_{X_{\mathbf{R}}}(D_0^{k_0})) \neq \emptyset$. Let $f_0 \in {}^\omega\{0,1\}$ such that $x_{f_0} \in D_0^{k_0}$, and let L_0 be an open interval of $X_{\mathbf{R}}$ such that $x_{f_0} \in L_0 \subseteq cl_{X_{\mathbf{R}}}(D_0^{k_0})$ and such that L_0 has endpoints in D. Let $D_1^k = \{x_f \in A_1 \cap L_0 : \{x_f\} \cup \bigcup_{n\geq k} I_n^{x_f} \subseteq L_0 \cap V\}$. Note that $A_1 \cap L_0 = \bigcup_{k\in\omega} D_1^k$. Since A_1 is not of first Baire category in $\mathbf{C}$, there exists $k_1 \in \omega$ such that $int_{X_{\mathbf{R}}}(cl_{X_{\mathbf{R}}}(D_1^{k_1})) \neq \emptyset$. Note that $cl_{X_{\mathbf{R}}}(D_1^{k_1}) \subseteq cl_{X_{\mathbf{R}}}(D_0^{k_0})$. Let $k = max\{k_0, k_1\}$. Choose $g_1 \in {}^\omega\{0,1\}$ such that $x_{g_1} \in D_1^{k_1}$. Also, choose L_1 to be an open interval of $X_{\mathbf{R}}$ with endpoints in D such that $x_{g_1} \in L_1 \subseteq cl_{X_{\mathbf{R}}}(D_1^{k_1}) \subseteq cl_{X_{\mathbf{R}}}(D_0^{k_0})$ and such that $x_f \in L_1$ implies that $f\restriction_{k+4} = g_1\restriction_{k+4}$. Now choose $g_0 \in {}^\omega\{0,1\}$ such that $x_{g_0} \in D_0^{k_0} \cap L_1$. Then $g_0|_{k+4} = g_1|_{k+4}$. Choose $m \in \omega, k \leq m \leq k+4$, such that m is not a power of 2 and m is not a power of 3. Since $x_{g_0} \in D_0^{k_0}$, $x_{g_1} \in D_1^{k_1}$, and $k \leq m$, $\{x_{g_0}\} \cup \bigcup_{n \geq m} I_n^{x_{g_0}} \subseteq U$ and $\{x_{g_1}\} \cup \bigcup_{n \geq m} I_n^{x_{g_1}} \subseteq V$. Also, $I_m^{x_{g_0}} = I_{g_0|_m^\frown 0}$ and $I_m^{x_{g_1}} = I_{g_1|_m^\frown 1}$ since m is not a power of 2 or a power of 3. Since $g_0|_m = g_1|_m$, $I_{g_0|_m^\frown 0} \cap I_{g_1|_m^\frown 1} \neq \emptyset$. This implies that $U \cap V \neq \emptyset$. Hence, X is not normal on Y. $\square$

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Department of Mathematics, Ohio University, Athens, Ohio 45701
E-mail address: Just@bing.math.ohiou.edu

Current address, J. Tartir: Department of Mathematics and Statistics, Catholic University of America, Washington, DC 20064
E-mail address: Jt944485@oak.cats.ohiou.edu