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A MULTIPLIER RELATION FOR CALDERÓN-ZYGMUND OPERATORS ON $L^1(\mathbb{R}^n)$

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ABSTRACT. A generalised integral is used to obtain a Fourier multiplier relation for Calderón-Zygmund operators on $L^1(\mathbb{R}^n)$. In particular we conclude that an operator in our class is injective on $L^1(\mathbb{R}^n)$ if it is injective on $L^2(\mathbb{R}^n)$.

1. INTRODUCTION

The Hilbert transform, defined almost everywhere (a.e.) for $f \in L^p(\mathbb{R})$, $1 \leq p < \infty$, by

$$Hf(x) = \lim_{\epsilon \to 0} \frac{1}{\pi} \int_{|y| \geq \epsilon} \frac{f(x-y)}{y} dy$$

is well known to be bounded on $L^p(\mathbb{R})$ for $1 < p < \infty$, and weak type (1,1). This is covered in Stein [2]. For $f \in L^2(\mathbb{R})$, the action of H can also be described by a Fourier multiplier, $\widehat{(Hf)}(\xi) = isign(\xi)\widehat{f}(\xi)$, $\widehat{\ }$ denoting the Fourier transform. This multiplier relation also holds for all $f \in L^1(\mathbb{R})$ such that $Hf \in L^1(\mathbb{R})$. This may be seen as follows; for the relevant background see Stein [4]. Recall that $\{f \in L^1(\mathbb{R}) : Hf \in L^1(\mathbb{R})\}$ is the real Hardy space $H^1(\mathbb{R})$, and H is bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$. (We may take $\|f\|_{H^1(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})} + \|Hf\|_{L^1(\mathbb{R})}$.)
A function $a : \mathbb{R}^n \longrightarrow \mathbb{R}$ is an $H^1(\mathbb{R}^n)$ atom if

(i) a is supported in a ball B,
(ii) $|a| \leq |B|^{-1}$, and
(iii) $\int a(x)dx = 0$.

If $f \in H^1(\mathbb{R})$, it can be shown that there exist non-negative constants $\{\lambda_k\}$ such that $\sum \lambda_k < \infty$, and $H^1(\mathbb{R})$ atoms $\{a_k\}$ such that $f = \sum \lambda_k a_k$ in $H^1(\mathbb{R})$ norm. This is the celebrated 'atomic decomposition of $H^1(\mathbb{R})$'. Since H is bounded from $H^1(\mathbb{R})$ to $L^1(\mathbb{R})$, $Hf = \sum \lambda_k H a_k$ in $L^1(\mathbb{R})$. On taking the Fourier transform of this expression we get the desired result, since each atom is in $L^2(\mathbb{R})$, and hence satisfies the multiplier relation. Observe that this implies that H is injective on $L^1(\mathbb{R})$.

The above discussion has its roots in Zygmund [7], where the analogue for the Fourier series is proved using the classical complex Hardy spaces. The analogue states that if f and its conjugate $\tilde{f}$ are in $L^1(\mathbb{T})$, then $c_k(\tilde{f}) = isign(k)c_k(f)$.

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Zygmund also describes a very different approach. He considers a generalised integral, referred to as integral B, with which the above multiplier relation for Fourier coefficients holds for all $f \in L^1(\mathbb{T})$.

The purpose of this paper is to deduce analogous $L^1(\mathbb{R}^n)$ results for a wide class of Calderón-Zygmund operators for which Hardy space techniques are not necessarily appropriate. The main conclusion is the following, which is Corollary 1 of section 5.

Theorem 1. *Let the operator T satisfy the conditions* (1), (2), *and* (3). *If $u \in L^1(\mathbb{R}^n)$ is such that $Tu \in L^1(\mathbb{R}^n)$, then*

$$\widehat{(Tu)}(\xi) = \mathfrak{m}(\xi)\widehat{u}(\xi)$$

for every $\xi \neq 0$, where $\mathfrak{m}$ is the Fourier multiplier corresponding to T.

The above shall be achieved by obtaining a multiplier relation on $L^1(\mathbb{R}^n)$ using a generalised integral. This was done for the Hilbert transform by Toland in [5], following the alternative approach in Zygmund.

It is worth remarking that the previous observations about H suggest we might try to characterise those Calderón-Zygmund operators T for which $\{f \in L^1(\mathbb{R}) : Tf \in L^1(\mathbb{R})\} = H^1(\mathbb{R})$. For some related results see Janson [1], and Uchiyama [6].

Finally, we would like to thank A. Carbery for suggesting numerous improvements to what would have followed.

2. The class of operators under study

Suppose $K : \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{C}$ satisfies

$$\int_{|x| \geq 2|y|} |K(x-y) - K(x)| dx \leq c \tag{1}$$

for all $y \neq 0$. Suppose T is bounded on $L^2(\mathbb{R}^n)$, commutes with translations and satisfies

$$Tf(x) = \int_{\mathbb{R}^n} K(y) f(x-y) dy \tag{2}$$

whenever $f \in \mathcal{S}(\mathbb{R}^n)$ with $x \notin supp(f)$. Such an operator is often referred to as a Calderón-Zygmund operator, with Calderón-Zygmund kernel K.

2.1. **Some useful properties of our class.**

(P1) For $0 < \alpha < \beta$, $\left|\int_{\alpha < |x| < \beta} K(x) dx\right|$ is bounded uniformly in α and β.
(P2) There is an $\mathfrak{m} \in L^\infty(\mathbb{R}^n)$ such that $\widehat{(Tf)}(\xi) = \mathfrak{m}(\xi)\widehat{f}(\xi)$ for $f \in L^2(\mathbb{R}^n)$.
(P3) $\mathfrak{m}$ is continuous on $\mathbb{R}^n \backslash \{0\}$.
(P4) T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and is weak type (1,1).

To see (P3), let a be a nonzero $H^1(\mathbb{R}^n)$ atom. Using (1) it can easily be shown (see [4]) that $Ta \in L^1(\mathbb{R}^n)$. So $\widehat{a}$ and $\widehat{Ta}$ are continuous. Since $a \in L^2(\mathbb{R}^n)$, $\widehat{Ta} = \mathfrak{m}\widehat{a}$ a.e. Therefore $\mathfrak{m}$ is continuous at every point for which $\widehat{a} \neq 0$. Choose any $\xi \in \mathbb{R}^n \backslash \{0\}$. For some $\eta \in \mathbb{R}^n \backslash \{0\}$, $\widehat{a}(\eta) \neq 0$. Let λ be a nonzero real number and ρ be an orthogonal matrix such that $\eta = \lambda\rho\xi$. Now $0 \neq \widehat{a}(\eta) = \int a(x) e^{2\pi i \lambda\rho\xi \cdot x} dx = \int a(x) e^{2\pi i \xi \cdot (\lambda\rho^{-1} x)} dx = \widehat{a_{\lambda,\rho}}(\xi)$, where $a_{\lambda,\rho}(x) = \lambda^{-n} a(\lambda^{-1}\rho x)$. Since $a_{\lambda,\rho}$ is an $H^1(\mathbb{R}^n)$ atom, $\mathfrak{m}$ is continuous at ξ and hence on $\mathbb{R}^n \backslash \{0\}$. We wish to thank F. Ricci for pointing out this simplification of the author's original argument.

For (P1) and (P4) see [2], and for (P2) see [3].

We shall impose one further condition on K, namely

$$|K(x)| \leq \frac{c}{|x|^n} \quad x \neq 0. \tag{3}$$

3. Realising the operators as principal values

Let $0 < \epsilon < R$ and

$$K_{\epsilon,R}(x) = \begin{cases} K(x) & \text{if } \epsilon \leq |x| \leq R \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1. *For $\xi \neq 0$, $\widehat{K_{\epsilon,R}}(\xi)$ converges as $R \to \infty$, and*

$$\widehat{K_\epsilon}(\xi) = \lim_{R\to\infty} \widehat{K_{\epsilon,R}}(\xi)$$

is bounded independently of ϵ.

Proof. It is well known (see [2]), that $\widehat{K_{\epsilon,R}}(\xi)$ is uniformly bounded in ϵ and R for each $\xi \in \mathbb{R}^n$. A similar argument shows that for fixed $\xi \neq 0$, $\widehat{K_{R,R'}}(\xi) \to 0$ as $R, R^{'} \to \infty$. Hence $\widehat{K_{\epsilon,R}}(\xi)$ converges to a bounded function as $R \to \infty$.

Lemma 2. *There exists a sequence $\{\epsilon_j\}$, converging to zero, for which $\{\widehat{K_{\epsilon_j}}(\xi)\}$ converges everywhere on $\mathbb{R}^n \backslash \{0\}$ to a bounded function.*

Proof. Fix $\xi \neq 0$. $\{\widehat{K_\epsilon}(\xi) : \epsilon > 0\}$ is bounded in $\mathbb{C}$, so there exists a sequence $\{\epsilon_j\}$ converging to zero such that $\{\widehat{K_{\epsilon_j}}(\xi)\}$ converges. Let $\xi^{'} \in \mathbb{R}^n \backslash \{0\}$. We shall show that $\{\widehat{K_{\epsilon_j}}(\xi^{'})\}$ is convergent also. Let

$$U_{j,l} = \{x \in \mathbb{R}^n : \min(\epsilon_j, \epsilon_l) \leq |x| \leq \max(\epsilon_j, \epsilon_l)\}.$$

Using spherical polar coordinates and (3),

$$\begin{aligned} &\left|(\widehat{K_{\epsilon_j}}(\xi) - \widehat{K_{\epsilon_j}}(\xi^{'})) - (\widehat{K_{\epsilon_l}}(\xi) - \widehat{K_{\epsilon_l}}(\xi^{'}))\right| \\ &\qquad = \left|\int_{U_{j,l}} K(x)(e^{2\pi i x\cdot\xi} - e^{2\pi i x\cdot\xi^{'}})dx\right| \leq c(|\xi| + |\xi^{'}|)\left|\int_{\epsilon_j}^{\epsilon_l} dt\right| \longrightarrow 0 \end{aligned}$$

as $j, l \to \infty$. So $\{\widehat{K_{\epsilon_j}}(\xi) - \widehat{K_{\epsilon_j}}(\xi^{'})\}_j$ converges, and hence $\{\widehat{K_{\epsilon_j}}(\xi^{'})\}_j$ converges.

Define $\widetilde{\mathfrak{m}} \in L^\infty(\mathbb{R}^n)$ by $\widetilde{\mathfrak{m}}(\xi) = \lim_{j\to\infty} \widehat{K_{\epsilon_j}}(\xi)$, $\xi \neq 0$. We now make some observations.

(i) By the Dominated Convergence Theorem (D.C.T.) and Plancherel's theorem

$$\|K_{\epsilon_j} * f - \mathcal{F}^{-1}(\widetilde{\mathfrak{m}}\widehat{f})\|_2 \longrightarrow 0 \text{ as } j \to \infty,$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform.

(ii) Fix $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \notin supp(f)$. There is a $J \in \mathbb{N}$ such that

$$Tf(x) = \int_{\mathbb{R}^n} K(x)f(x-y)dy = \int_{|y|\geq\epsilon_j} K(y)f(x-y)dy = K_{\epsilon_j} * f(x)$$

for $j \geq J$.

These observations allow us to define an operator $S : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ satisfying

(i) $\widehat{Sf} = \widetilde{\mathfrak{m}}\widehat{f}$, and

(ii) $Sf(x) = Tf(x)$ whenever $f \in \mathcal{S}(\mathbb{R}^n)$ and $x \notin supp(f)$.

Consequently $T-S$ has Calderón-Zygmund kernel 0, by (2). The fact that $T-S$ is bounded on $L^2(\mathbb{R}^n)$ and commutes with translations allows one to show that $T-S = \lambda I$, for some $\lambda \in \mathbb{C}$. This is equivalent to $\mathfrak{m}(\xi) = \widetilde{\mathfrak{m}}(\xi) + \lambda$. For our purposes we may suppose that $\lambda = 0$, i.e. $S = T$.

4. The generalised integral

For a set $E \subset \mathbb{R}^n$, $|E|$ shall denote its Lebesgue measure. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ have compact support, $t \in [0,1]^n$, and $m \in \mathbb{Z}$. Let

$$I_m(f)(t) \;=\; \frac{1}{2^{nm}} \sum_{k \in \mathbb{Z}^n} f\left(t + \frac{k}{2^m}\right) \quad \text{(a finite sum).}$$

Definition 1. For $I \in \mathbb{R}$, write $I = \#\int_{\mathbb{R}^n} f(x)dx$ (or more briefly $I = \#\int f$), if $I_m(f)(t) \to I$ in measure on $[0,1]^n$ as $m \to \infty$.

Observe that if $f \in C_c(\mathbb{R}^n)$, then $I_m(f)(t)$ is a Riemann partial sum. Hence $\#\int f = \int f$. From this we can deduce the following.

Lemma 3. *For $f \in L^1(\mathbb{R}^n)$ of compact support, $\#\int f = \int f$.*

This will follow as a corollary to a more interesting result.

Definition 2. Define for some measurable $f : \mathbb{R}^n \longrightarrow \mathbb{R}$,

$$\widetilde{I_m}(f)(t) \;=\; \frac{1}{2^{nm}} \sum_{k \in \mathbb{Z}^n} f\left(t + \frac{k}{2^m}\right) \quad t \in [0,1]^n$$

whenever the sum is absolutely convergent for a.e. $t \in [0,1]^n$. (So for $f \in L^1(\mathbb{R}^n)$ of compact support, $\widetilde{I_m} f = I_m f$.) Define $\widetilde{\#}\int f$ in analogy with $\#\int f$.

Lemma 4. *For $f \in L^1(\mathbb{R}^n)$, $\widetilde{\#}\int f = \int f$.*

Proof. We must first show that $I_m(f)$ is defined for $f \in L^1(\mathbb{R}^n)$. Let G be the set of lattice points in $[0, 2^m)^n$. Observe that

$$\begin{aligned}
&\sum_{k \in \mathbb{Z}^n} \frac{1}{2^{nm}} \int_{[0,1]^n} \left| f\left(t + \frac{k}{2^m}\right)\right| dt \\
&= \sum_{\gamma \in G} \sum_{k \in 2^m\mathbb{Z}^n + \{\gamma\}} \frac{1}{2^{nm}} \int_{[0,1]^n} \left| f\left(t + \frac{k}{2^m}\right)\right| dt \\
&= \sum_{\gamma \in G} \frac{1}{2^{nm}} \|f\|_1 = \|f\|_1 < \infty.
\end{aligned}$$

So by the Monotone Convergence Theorem, $\sum_{k\in\mathbb{Z}^n} \left|f\left(t + \frac{k}{2^m}\right)\right| < \infty$ a.e. $t \in [0,1]^n$ as required. Observe that we also have,

$$\int_{[0,1]^n} \left|\widetilde{I_m}(f)(t)\right| dt \le \|f\|_1. \tag{4}$$

Let $f \in L^1(\mathbb{R}^n)$, and $\alpha, \epsilon > 0$. Choose $f_1 \in C_c(\mathbb{R}^n)$, and $f_2 \in L^1(\mathbb{R}^n)$ such that $f = f_1 + f_2$ and $\|f_2\|_1 < \frac{\alpha}{4}\min(\epsilon, 1)$. By (4) and Chebychev's inequality,

$$\left|\left\{t \in [0,1]^n : \left|\widetilde{I_m}(f_2)(t)\right| \ge \frac{\alpha}{2}\right\}\right| \le \frac{2\|f_2\|_1}{\alpha} < \frac{\epsilon}{2}. \tag{5}$$

By the triangle inequality,

$$\left|\left\{t \in [0,1]^n : \left|\widetilde{I_m}(f)(t) - \int f\right| \geq \alpha\right\}\right|$$

$$\leq \left|\left\{t \in [0,1]^n : \left|\widetilde{I_m}(f_1)(t) - \int f_1\right| \geq \frac{\alpha}{4}\right\}\right| \tag{6}$$

$$+ \left|\left\{t \in [0,1]^n : \left|\widetilde{I_m}(f_2)(t)\right| \geq \frac{\alpha}{2}\right\}\right| \tag{7}$$

$$+ \left|\left\{t \in [0,1]^n : \left|\int f_2\right| \geq \frac{\alpha}{4}\right\}\right|. \tag{8}$$

Since $\|f_2\|_1 < \frac{\alpha}{4}$, the term (8) is zero. By (5) the term (7) is less than $\frac{\epsilon}{2}$. Since $f_1 \in C_c(\mathbb{R}^n)$, the remark preceding Lemma 3 implies that the term (6) can be made less than $\frac{\epsilon}{2}$ for sufficiently large m. This concludes the proof.

As will be seen, for our purposes it is more convenient to extend $\#\int$ to functions of non-compact support by the following limiting process, so I shall reject $\widetilde{I_m}$.
Let $\rho \in C_c^\infty(\mathbb{R}^n)$ satisfy

(i) $\rho(0) = 1$,
(ii) $0 \leq \rho(x) \leq 1$.

Let $\rho_N(x) = \rho(\frac{x}{N})$.

Definition 3. For $f : \mathbb{R}^n \longrightarrow \mathbb{R}$, we write $I = \#\int_{\mathbb{R}^n} f(x)dx$ (or $I = \#\int f$), if for every such ρ, $\#\int_{\mathbb{R}^n} \rho_N(x)f(x)dx$ converges to I as $N \to \infty$.

By Lemma 3 and the Dominated Convergence Theorem, $\#\int f = \int f$ for every $f \in L^1(\mathbb{R}^n)$.

In order to exploit the translation invariance of T, we shall need the following lemma.

Lemma 5. *Let $v \in C_c^1(\mathbb{R}^n)$, $u \in L^1(\mathbb{R}^n)$, and*

$$S_v(u)(x) = (Tvu)(x) - v(x)(Tu)(x).$$

S_v is bounded from $L^1(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$ for $1 < p < \frac{n}{n-1}$ when $n \geq 2$, and from $L^1(\mathbb{R}^n)$ to $L^\infty(\mathbb{R}^n)$ when $n = 1$.

Proof. ($n \geq 2$)

$$S_v(u)(x) = \int_{\mathbb{R}^n} (v(y) - v(x))u(y)K(x-y)dy.$$

By Minkowski's inequality for integrals, it is sufficient to show that

$$\sup_{y \in \mathbb{R}^n} \|(v(y) - v(\cdot))K(\cdot - y)\|_p < \infty \text{ for } 1 < p < \frac{n}{n-1}.$$

This is clear on observing that,

$$|(v(y) - v(x))K(x-y)| \leq \begin{cases} \frac{c\|\nabla v\|_\infty}{|x-y|^{n-1}} \in L^p(B(y;1)) & \text{for } p < \frac{n}{n-1}, \\ \frac{2c\|v\|_\infty}{|x-y|^n} \in L^p(B(y;1)^c) & \text{for } p > 1 \end{cases}$$

where $B(y;1)$ is the ball in $\mathbb{R}^n$ with centre y and radius 1, and $B(y;1)^c$ is its complement.

Lemma 6. *Suppose $\phi \in C_c^1(\mathbb{R}^n)$, $\alpha > 0$, and $0 < \epsilon < 1$. There is a constant $\kappa = \kappa(\phi, n)$ such that for $u \in L^1(\mathbb{R}^n)$ with $\|u\|_1 \leq \kappa\alpha\epsilon$,*

$$|\{t \in [0,1]^n : |I_m(\phi Tu)(t)| \geq \alpha\}| \leq \epsilon \text{ for all } m \in \mathbb{N}. \tag{9}$$

Proof. Let $t \in [0,1]^n$ and suppose N is chosen so that $supp(\phi) \in [-N,N]^n$. Let

$$A_{m,t} = \left\{ k \in \mathbb{Z}^n : t + \frac{k}{2^m} \in [-N,N]^n \right\}.$$

We shall dominate $I_m(\phi Tu)(t)$ by the sum of two terms, each of which will satisfy an expression of the form (9).

$$\begin{aligned} |I_m(\phi Tu)| \leq & \frac{1}{2^{nm}} \left| \sum_{k \in A_{m,t}} T(\phi u)\left(t + \frac{k}{2^m}\right) \right| \\ & + \frac{1}{2^{nm}} \left| \sum_{k \in A_{m,t}} S_\phi(u)\left(t + \frac{k}{2^m}\right) \right|, \end{aligned} \tag{10}$$

where S_ϕ is defined in Lemma 5. Let

$$v_k(x) = \phi\left(x + \frac{k}{2^m}\right) u\left(x + \frac{k}{2^m}\right).$$

Since T is linear and commutes with translations,

$$\frac{1}{2^{nm}} \sum_{k \in A_{m,t}} T(\phi u)\left(t + \frac{k}{2^m}\right) = T\left(\frac{1}{2^{nm}} \sum_{k \in A_{m,t}} v_k\right)(t). \tag{11}$$

Observe that for each m, $A_{m,t}$ is constant, say A_m, on $(0,1)^n$. Using this, (11), and the fact that T is weak type (1,1), we get for some constant c,

$$\begin{aligned} & \left| \left\{ t \in [0,1]^n : \left| \frac{1}{2^{nm}} \sum_{k \in A_{m,t}} T(\phi u)\left(t + \frac{k}{2^m}\right) \right| \geq \alpha \right\} \right| \\ & = \left| \left\{ t \in (0,1)^n : \left| T\left(\frac{1}{2^{nm}} \sum_{k \in A_{m,t}} v_k\right)(t) \right| \geq \alpha \right\} \right| \\ & \leq \frac{c}{\alpha} \left\| \frac{1}{2^{nm}} \sum_{k \in A_m} v_k \right\|_{L^1(\mathbb{R}^n)} \\ & \leq \frac{c 2^n N^n \|\phi\|_{L^\infty(\mathbb{R}^n)} \|u\|_{L^1(\mathbb{R}^n)}}{\alpha} < \epsilon \end{aligned}$$

provided $\|u\|_{L^1(\mathbb{R}^n)} \leq \frac{\alpha\epsilon}{c 2^n N^n \|\phi\|_{L^\infty(\mathbb{R}^n)}}$. This deals with the first term of (10) with $\kappa = \frac{1}{c 2^n N^n \|\phi\|_{L^\infty(\mathbb{R}^n)}}$. We now turn to the remaining term. Let

$$J_{m,\phi}(f)(t) = \frac{1}{2^{nm}} \sum_{k \in A_{m,t}} f\left(t + \frac{k}{2^m}\right) \quad \text{for } f \in L^p(\mathbb{R}^n),\ 1 \leq p \leq \infty.$$

By (4), $\|J_{m,\phi}(f)\|_{L^1([0,1]^n)} \leq \|f\|_{L^1(\mathbb{R}^n)}$, and by considering the number of elements of $A_{m,t}$, $\|J_{m,\phi}(f)\|_{L^\infty([0,1]^n)} \leq 2^n(N+1)^n \|f\|_{L^\infty(\mathbb{R}^n)}$. Therefore by the Riesz convexity theorem, $\|J_{m,\phi}(f)\|_{L^p([0,1]^n)} \leq (2^n(N+1)^n)^{\frac{1}{q}} \|f\|_{L^p(\mathbb{R}^n)}$ for $1 \leq p \leq \infty$.

Here, as usual, $\frac{1}{p} + \frac{1}{q} = 1$. By Lemma 4 and the composition of $J_{m,\phi}$ with S_ϕ,

$$u \longmapsto \frac{1}{2^{nm}} \left| \sum_{k \in A_{m,t}} S_\phi(u) \left(t + \frac{k}{2^m} \right) \right|$$

is bounded (independently of m), from $L^1(\mathbb{R}^n)$ to $L^p([0,1]^n)$ for $1 < p < \frac{n}{n-1}$. Let $\alpha > 0$ and $0 < \epsilon < 1$. By Chebyshev's inequality, there is a constant $\kappa = \kappa(\phi, n)$ such that

$$\left| \left\{ t \in [0,1]^n; \frac{1}{2^{nm}} \left| \sum_{k \in A_{m,t}} S_\phi(u) \left(t + \frac{k}{2^m} \right) \right| \geq \alpha \right\} \right| \leq \left(\frac{\|u\|_{L^1(\mathbb{R}^n)}}{\kappa\alpha} \right)^p < \epsilon^p < \epsilon$$

provided $\|u\|_{L^1(\mathbb{R}^n)} < \kappa\alpha\epsilon$. This deals with the second term in (10).

Lemma 7. *For $\phi \in C_c^1(\mathbb{R}^n)$, $T\phi \in L^\infty(\mathbb{R}^n)$.*

Proof. The proof of this is a simple consequence of the cancellation property (P1), and the size condition (3), for K.

Lemma 8. *If $\phi \in C_c^1(\mathbb{R}^n)$ and $u \in L^1(\mathbb{R}^n)$, then*

$$\# \int_{\mathbb{R}^n} \phi(x)\overline{(Tu)(x)}dx = \int_{\mathbb{R}^n} (T^*\phi)(x)\overline{u(x)}dx$$

where T^* is the L^2 adjoint of T, having Calderón-Zygmund kernel $K^*(x) = \overline{K(-x)}$. (Note that in general $Tu \notin L^1_{loc.}(\mathbb{R}^n)$.)

Proof. Let $u = v_j + w_j$ where $v_j \in C_c^1(\mathbb{R}^n)$ and $\|w_j\|_{L^1(\mathbb{R}^n)} \to 0$ as $j \to \infty$. Let $\alpha > 0$ and $0 < \epsilon < 1$. By the triangle inequality,

$$\left| \left\{ t \in [0,1]^n : \left| I_m \left(\phi\overline{Tu} \right)(t) - \int (T^*\phi)(x)\overline{u(x)}dx \right| \geq \alpha \right\} \right|$$

$$\leq \left| \left\{ t \in [0,1]^n : \left| I_m \left(\phi\overline{Tv_j} \right)(t) - \int (T^*\phi)(x)\overline{v_j(x)}dx \right| \geq \frac{\alpha}{3} \right\} \right| \tag{12}$$

$$+ \left| \left\{ t \in [0,1]^n : \left| \int (T^*\phi)(x)\overline{w_j(x)}dx \right| \geq \frac{\alpha}{3} \right\} \right| \tag{13}$$

$$+ \left| \left\{ t \in [0,1]^n : \left| I_m \left(\phi\overline{Tw_j} \right)(t) \right| \geq \frac{\alpha}{3} \right\} \right|. \tag{14}$$

By Lemma 6, there is an integer J such that

$$\left| \left\{ t \in [0,1]^n : \left| I_m \left(\phi\overline{Tw_j} \right)(t) \right| \geq \frac{\alpha}{3} \right\} \right| < \epsilon \ \ \forall m \in \mathbb{Z}, j \geq J.$$

So the term (14) is less than ϵ for $j \geq J$. By Lemma 7, $T^*\phi \in L^\infty(\mathbb{R}^n)$, and hence

$$\int (T^*\phi)(x)\overline{w_j(x)}dx \to 0$$

as $j \to \infty$, so increasing J if necessary we may suppose that

$$\left| \int (T^*\phi)(x)\overline{w_j(x)}dx \right| < \frac{\alpha}{3} \quad \forall j \geq J.$$

So for $j \geq J$, the term (13) is zero. As $v_j, \phi \in L^2(\mathbb{R}^n)$,

$$\int (T^*\phi)(x)\overline{v_j(x)}dx = \int \phi(x)\overline{(Tv_j)(x)}dx,$$

so term (12) now becomes

$$\left|\left\{t \in [0,1]^n : \left|I_m\left(\phi\overline{Tv_j}\right)(t) - \int \phi(x)\overline{(Tv_j)(x)}dx\right| \geq \frac{\alpha}{3}\right\}\right|.$$

Fix $j \geq J$. $\phi\overline{Tv_j} \in L^1(\mathbb{R}^n)$, so by Lemma 3 this term (12) tends to zero as $m \to \infty$. This completes the proof of Lemma 8.

5. The multiplier relation on $L^1(\mathbb{R}^n)$

Lemma 9. *If* $\psi_N^{(\xi)}(y) = \rho_N(y)e^{2\pi i\xi\cdot y}$, $\xi \neq 0$, *then*

$$T^*\psi_N^{(\xi)}(x) - \rho_N(x)\overline{\mathfrak{m}(-\xi)}e^{2\pi i\xi\cdot x} \longrightarrow 0$$

uniformly in x *as* $N \to \infty$.

Proof. Let $K^*(x) = \overline{K(-x)}$, and $\xi \neq 0$.

$$\begin{aligned}&T^*\psi_N^{(\xi)}(x) - \rho_N(x)\overline{\mathfrak{m}(-\xi)}e^{2\pi i\xi\cdot x}\\ &= \lim_{j\to\infty}\lim_{R\to\infty} e^{2\pi i\xi\cdot x}\int_{\epsilon_j\leq|y|\leq R} K^*(y)(\rho_N(x-y) - \rho_N(x))e^{-2\pi i\xi\cdot y}dy.\end{aligned}$$

By writing ρ as the inverse Fourier transform of $\widehat{\rho}$, and then by Fubini's theorem,

$$\begin{aligned}&\left|\int_{\epsilon_j\leq|y|\leq R} K^*(y)(\rho_N(x-y) - \rho_N(x))e^{-2\pi i\xi\cdot y}dy\right|\\ &= \left|\int_{\epsilon_j\leq|y|\leq R} K^*(y)\int_{\mathbb{R}^n}\widehat{\rho}(s)\left(e^{-2\pi i\frac{(x-y)}{N}\cdot s} - e^{-2\pi i\frac{x}{N}\cdot s}\right)e^{-2\pi i\xi\cdot y}dsdy\right|\\ &= \left|\int_{\mathbb{R}^n}\widehat{\rho}(s)e^{2\pi ix\cdot\frac{s}{N}}\left(\overline{\widehat{K_{\epsilon_j,R}}\left(\frac{s}{N}-\xi\right)} - \overline{\widehat{K_{\epsilon_j,R}}(-\xi)}\right)ds\right|\\ &\leq \int_{\mathbb{R}^n}|\widehat{\rho}(s)|\left|\widehat{K_{\epsilon_j,R}}\left(\frac{s}{N}-\xi\right) - \widehat{K_{\epsilon_j,R}}(-\xi)\right|ds\\ &\longrightarrow \int_{\mathbb{R}^n}|\widehat{\rho}(s)|\left|\mathfrak{m}\left(\frac{s}{N}-\xi\right) - \mathfrak{m}(-\xi)\right|ds\end{aligned}$$

as $R \to \infty$ and $j \to \infty$ by Lemmas 1, 2, and the D.C.T. The last expression tends to zero uniformly in x as $N \to \infty$, by the continuity of $\mathfrak{m}$ on $\mathbb{R}^n\backslash\{0\}$, (see (P3) of section 2.1), and the D.C.T.

Theorem 2. *Let* T *satisfy* (1), (2), *and* (3). *If* $u \in L^1(\mathbb{R}^n)$, *then*

$$\#\int_{\mathbb{R}^n}(Tu)(x)e^{2\pi i\xi\cdot x}dx = \mathfrak{m}(\xi)\widehat{u}(\xi)$$

for every $\xi \neq 0$.

Proof. If $u \in L^1(\mathbb{R}^n)$, and $\xi \neq 0$, then

$$\begin{aligned}
&\#\int_{\mathbb{R}^n} (Tu)(x)e^{2\pi i\xi\cdot x}\rho_N(x)dx \\
&= \#\int_{\mathbb{R}^n} (Tu)(x)\overline{e^{-2\pi i\xi\cdot x}\rho_N(x)}dx \\
&= \int_{\mathbb{R}^n} u(x)\overline{\left(T^*\psi_N^{(-\xi)}\right)(x)}dx \quad \text{(by Lemma 8)} \\
&\longrightarrow \int_{\mathbb{R}^n} u(x)e^{2\pi i\xi\cdot x}\mathfrak{m}(\xi)dx
\end{aligned}$$

as $N \to \infty$ by Lemma 9 and the D.C.T. Hence

$$\#\int_{\mathbb{R}^n} (Tu)(x)e^{2\pi i\xi\cdot x}dx = \mathfrak{m}(\xi)\widehat{u}(\xi).$$

Corollary 1. *Let T satisfy* (1), (2), *and* (3). *If $u \in L^1(\mathbb{R}^n)$ is such that $Tu \in L^1(\mathbb{R}^n)$, then*

$$\widehat{(Tu)}(\xi) = \mathfrak{m}(\xi)\widehat{u}(\xi), \quad \xi \neq 0.$$

Proof. Use Theorem 1 and the remark after Definition 3.

Corollary 2. *If T satisfies* (1), (2), *and* (3), *then T is injective on $L^1(\mathbb{R}^n)$ if and only if the zero set of $\mathfrak{m}$ has empty interior.*

Corollary 3. *Let T satisfy* (1) *and* (2). *Suppose K is homogeneous of degree $-n$ and $f \in L^1(\mathbb{R}^n)$ is non-negative. If $f \not\equiv 0$, then $Tf \notin L^1(\mathbb{R}^n)$.*

Proof. Use Corollary 1 and the fact that $\mathfrak{m}$ is homogeneous of degree 0.

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