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FINITE FAMILIES WITH FEW SYMMETRIC DIFFERENCES

ALBERTO MARCONE, FRANCO PARLAMENTO, AND ALBERTO POLICRITI

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Abstract. We show that $2^{\lceil \log_2(m) \rceil}$ is the least number of symmetric differences that a family of m sets can produce. Furthermore we give two characterizations of the set-theoretic structure of the families for which that lower bound is actually attained.

1. Introduction

Throughout this paper $\mathbf{F}$ and $\mathbf{G}$ will always denote finite families of sets.

Definition 1.1. • Let $A\Delta B$ denote the *symmetric difference* between the sets A and B, defined as

$$A\Delta B = \{ x \mid (x \in A \wedge x \notin B) \vee (x \notin A \wedge x \in B) \}.$$

- For $\mathbf{F}$ and $\mathbf{G}$ families of sets and A a set let

$$\begin{aligned} \Delta\mathbf{F} &= \{ A\Delta B \mid A, B \in \mathbf{F} \}; \\ \bar{\Delta}\mathbf{F} &= \text{ the closure under } \Delta \text{ of } \mathbf{F}; \\ \mathbf{F}\Delta\mathbf{G} &= \{ A\Delta B \mid A \in \mathbf{F} \wedge B \in \mathbf{G} \}; \\ A\Delta\mathbf{F} &= \{ A\Delta B \mid B \in \mathbf{F} \}; \\ A \cap \mathbf{F} &= \{ A \cap B \mid B \in \mathbf{F} \}. \end{aligned}$$

Notice that $\emptyset \in \Delta\mathbf{F}$ for any $\mathbf{F}$, while $\emptyset \in \mathbf{F}\Delta\mathbf{G}$ if and only if $\mathbf{F} \cap \mathbf{G} \neq \emptyset$.

If $B \neq C$, then $A\Delta B \neq A\Delta C$: therefore the cardinality $|\Delta\mathbf{F}|$ of $\Delta\mathbf{F}$ is always greater than or equal to the cardinality $|\mathbf{F}|$ of $\mathbf{F}$. Hence m sets produce at least m symmetric differencies (and at most $m(m-1)/2 + 1$: this upper bound is attained for every m, e.g. by m pairwise disjoint sets).

Our first result, which will be proved in section 2, sharpens the above lower bound on $|\Delta\mathbf{F}|$ by showing that if $|\mathbf{F}| = m$ then $|\Delta\mathbf{F}| \geq 2^{\lceil \log_2(m) \rceil}$, i.e. that if $|\mathbf{F}| > 2^n$ then $|\Delta\mathbf{F}| \geq 2^{n+1}$. Since a family of subsets of a set with $n+1$ elements can produce at most 2^{n+1} symmetric differences our lower bound on $|\Delta\mathbf{F}|$ is optimal. Our result in particular entails that if $|\Delta\mathbf{F}| = |\mathbf{F}|$, then $|\mathbf{F}|$ is a power of 2 and we will also prove that if $|\mathbf{F}| > 2^n$ and $|\Delta\mathbf{F}| = 2^{n+1}$, then there exists $\mathbf{F}' \supseteq \mathbf{F}$ with $|\mathbf{F}'| = 2^{n+1}$ and $|\Delta\mathbf{F}'| = |\mathbf{F}'|$ (so that $\Delta\mathbf{F}' = \Delta\mathbf{F}$).

This shows that to describe the set-theoretic structure of the families $\mathbf{F}$ with as few as possible symmetric differences, i.e. such that the lower bound on $|\Delta\mathbf{F}|$

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is attained, it suffices to describe the set-theoretic structure of the families $\mathbf{F}$ such that $|\Delta\mathbf{F}| = |\mathbf{F}|$. The main goal of sections 3 and 4 is to shed some light on this set-theoretic structure and this will be accomplished using two different approaches. In section 3 we will focus on the Venn diagram of the family, while in section 4 we will concentrate on the way the elements of the family can be distinguished by way of a tree construction.

2. The least number of symmetric differences

The power set $\mathrm{Pow}(X)$ of a set X together with the operation Δ is a group with $\emptyset$ as the identity and the property that the inverse of any element is the element itself, namely a 2-group. Groups of this kind, also called Boolean groups, are well-known and easily seen to be abelian, finite whenever finitely generated, and, in that case, to have order a power of 2.

Proposition 2.1. *If $\mathbf{F}$ is finite, then $\bar{\Delta}\mathbf{F}$ is a finite Boolean group with the operation Δ, and $|\bar{\Delta}\mathbf{F}| = 2^n$ for some n.*

Finite Boolean groups provide the appropriate framework for the study of the operation Δ on finite families.

The following proposition collects a couple of elementary properties of cosets of subgroups of Boolean groups that will be useful in the sequel.

Proposition 2.2. *Let G be a Boolean group and H be a subgroup of G.*

1) *Let $g, g' \in G$. g and g' are in the same coset of H if and only if $gg' \in H$.*
2) *If $g \in G$, then $H \cup (gH)$, i.e. the union of H with the coset of H containing g, is a subgroup of G.*

Proof. 1) g and g' are in the same coset of H if and only if for some $h \in H$ $g' = gh$ if and only if $gg' \in H$.

2) Every subset of G is closed under inverses. By 1) the product of two elements of gH belongs to H, as does the product of two elements of H, while the product of an element of H with an element of gH belongs to gH. Therefore $H \cup (gH)$ is closed under products. $\square$

The above facts about Boolean groups allow us to establish the lower bound on $|\Delta\mathbf{F}|$ stated in the introduction.

Theorem 2.3. *For any n and any family $\mathbf{F}$ with $|\mathbf{F}| > 2^n$ we have $|\Delta\mathbf{F}| \geq 2^{n+1}$. Hence for every m the least number of symmetric differences that m sets can produce is $2^{\lceil \log_2(m) \rceil}$.*

Proof. Let $\mathbf{G} = \bar{\Delta}\mathbf{F}$, which by Proposition 2.1 is a finite Boolean group with the operation Δ. Moreover $|\mathbf{G}| = 2^{n+1+k}$ for some $k \geq 0$. Let h be maximal such that there exists a subgroup $\mathbf{H} \leq \mathbf{G}$ of order 2^h such that $\mathbf{H} \cap \Delta\mathbf{F} = \{\emptyset\}$.

We claim that $h \leq k$. In fact if $h > k$, then $\mathbf{H}$ has at most 2^n cosets and, since $|\mathbf{F}| > 2^n$, one of them contains at least two distinct elements of $\mathbf{F}$ whose symmetric difference, by Proposition 2.2.1, would be a nonempty element of $\mathbf{H} \cap \Delta\mathbf{F}$.

A consequence of the claim is that $\mathbf{H}$ has at least 2^{n+1} cosets. If one of them has empty intersection with $\Delta\mathbf{F}$, then the union of this coset with $\mathbf{H}$ is a subgroup (by Proposition 2.2.2) of order 2^{h+1} which intersects $\Delta\mathbf{F}$ only in $\emptyset$, contradicting the maximality of h. Hence every coset of $\mathbf{H}$ contains at least one element of $\Delta\mathbf{F}$ and $|\Delta\mathbf{F}| \geq 2^{n+1}$. $\square$

We now study the families for which the lower bound on the cardinality of the family of the symmetric differences is attained, i.e. families $\mathbf{F}$ such that for some n, $|\mathbf{F}| > 2^n$ and $|\Delta\mathbf{F}| = 2^{n+1}$.

Theorem 2.4. *Let $\mathbf{F}$ be a nonempty finite family of sets. The following are equivalent:*

i) $|\Delta\mathbf{F}|$ *is the least number of symmetric differences* $|\mathbf{F}|$ *sets can produce;*
ii) $|\mathbf{F}| > |\Delta\mathbf{F}|/2$ *and* $\Delta\mathbf{F}$ *is a group under* Δ*;*
iii) *there exists* $\mathbf{F}' \supseteq \mathbf{F}$ *such that* $|\mathbf{F}| > |\mathbf{F}'|/2$ *and* $|\mathbf{F}'| = |\Delta(\mathbf{F}')|$.

Proof. For $|\mathbf{F}| = 1$ the theorem holds since i), ii), and iii) are all true. So let n be such that $2^n < |\mathbf{F}| \leq 2^{n+1}$.

To prove that i) implies ii) notice that if $|\Delta\mathbf{F}|$ is the least number of symmetric differences $|\mathbf{F}|$ sets can produce then, since, as pointed out in the introduction, the lower bound in Theorem 2.3 is optimal, $|\Delta\mathbf{F}| = 2^{n+1}$ and hence $|\mathbf{F}| > |\Delta\mathbf{F}|/2$. Therefore we need only to show that $\Delta\mathbf{F}$ is closed under symmetric differences.

Let $\mathbf{G} = \bar{\Delta}\mathbf{F}$, so that by Proposition 2.1 $|\mathbf{G}| = 2^{n+1+k}$ for some $k \geq 0$. If $k = 0$, then $\mathbf{G} = \Delta\mathbf{F}$, and we are done; so we assume that $k \geq 1$.

We first show by induction on i, $1 \leq i \leq k$, that for every $X \in \mathbf{G} \setminus (\Delta\mathbf{F})$ there exists a subgroup $\mathbf{H}$ of $\mathbf{G}$ of order 2^i such that $X \in \mathbf{H}$ and $\mathbf{H} \cap \Delta\mathbf{F} = \{\emptyset\}$. For $i = 1$ simply take $\mathbf{H} = \{\emptyset, X\}$. Assume the property holds for $i < k$ and let $\mathbf{H}$ be of order 2^i, with $X \in \mathbf{H}$ and $\mathbf{H} \cap \Delta\mathbf{F} = \{\emptyset\}$. $\mathbf{H}$ has $2^{n+1+k-i} > 2^{n+1}$ cosets, and since $|\Delta\mathbf{F}| = 2^{n+1}$ there are cosets of $\mathbf{H}$ which do not contain any element of $\Delta\mathbf{F}$. If $Y\Delta\mathbf{H}$ is one of them, then necessarily $Y\Delta\mathbf{H} \neq \mathbf{H}$, and then $\mathbf{H} \cup (Y\Delta\mathbf{H})$ is (by Proposition 2.2.2) a subgroup of $\mathbf{G}$ of order 2^{i+1} which contains X and no element of $\Delta\mathbf{F}$ but $\emptyset$.

Suppose now that A, B are such that $A\Delta B \notin \Delta\mathbf{F}$. Let $\mathbf{H}$ be a subgroup of $\mathbf{G}$ of order 2^k such that $A\Delta B \in \mathbf{H}$ and $\mathbf{H} \cap \Delta\mathbf{F} = \{\emptyset\}$. By Proposition 2.2.1 A and B belong to the same coset of $\mathbf{H}$, and by the proof of Theorem 2.3 if $|\Delta\mathbf{F}| = 2^{n+1}$, then at most one of A, B is in $\Delta\mathbf{F}$. Hence if $A, B \in \Delta\mathbf{F}$, then $A\Delta B \in \Delta\mathbf{F}$ and $\Delta\mathbf{F}$ is closed under Δ.

To prove that ii) implies iii), assume ii) holds, fix any $A \in \mathbf{F}$, and let $\mathbf{F}' = A\Delta(\Delta\mathbf{F})$. $\mathbf{F}' \supseteq \mathbf{F}$ holds because $A' = A\Delta(A\Delta A')$ for every $A' \in \mathbf{F}$, and hence $\Delta\mathbf{F} \subseteq \Delta(\mathbf{F}')$. Since $\Delta\mathbf{F}$ is a group $\mathbf{F}'$ is a coset of $\Delta\mathbf{F}$ within $\bar{\Delta}\mathbf{F}$, so that $|\mathbf{F}'| = |\Delta\mathbf{F}|$, and $\Delta(\mathbf{F}') \subseteq \Delta\mathbf{F}$. Therefore $\Delta(\mathbf{F}') = \Delta\mathbf{F}$ and hence $|\Delta(\mathbf{F}')| = |\mathbf{F}'|$.

iii) implies i) is immediate because iii) implies that $|\mathbf{F}'| = |\Delta(\mathbf{F}')| = 2^{n+1}$ and $\Delta\mathbf{F} \subseteq \Delta(\mathbf{F}')$. Hence $|\Delta\mathbf{F}| = 2^{n+1}$ since by Theorem 2.3, $|\Delta\mathbf{F}| \geq 2^{n+1}$. □

Remark 2.5. The proof that ii) implies iii) of Theorem 2.4 shows that if $\mathbf{F}$ produces as few as possible symmetric differences, then either $|\bar{\Delta}\mathbf{F}| = 2^{n+1}$ or $|\bar{\Delta}\mathbf{F}| = 2^{n+2}$. Nevertheless $|\bar{\Delta}\mathbf{F}|$ being small is not equivalent to $|\Delta\mathbf{F}|$ being least: $\mathbf{F} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2,3,4\}\}$ is a family of $2^2 + 1$ sets such that $|\bar{\Delta}\mathbf{F}| = 2^4$ but $|\Delta\mathbf{F}| = 2^3 + 3$.

Remark 2.6. In ii) of Theorem 2.4 both conditions are necessary: any family of 4 sets producing 7 symmetric differences shows that $|\mathbf{F}| > |\Delta\mathbf{F}|/2$ alone does not suffice to ensure that $|\Delta\mathbf{F}|$ is least, while $\mathbf{F} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2,3,4\}\}$ is a family of $2^2 + 2$ sets such that $\Delta\mathbf{F}$ is a group of order 2^4 under Δ, and hence shows that the closure of $\Delta\mathbf{F}$ under Δ does not suffice to ensure that $|\Delta\mathbf{F}|$ is least.

3. Venn diagrams

Theorem 2.4 shows that if a family has few symmetric differences then it is contained in some $\mathbf{F}$ satisfying $|\Delta\mathbf{F}| = |\mathbf{F}|$. In this and in the next section we will study families satisfying $|\Delta\mathbf{F}| = |\mathbf{F}|$.

A special case is of course offered by the families $\mathbf{F}$ such that $\mathbf{F} = \Delta\mathbf{F}$, i.e. such that $(\mathbf{F}, \Delta)$ is a Boolean group; a condition which is clearly equivalent to $|\mathbf{F}| = |\Delta\mathbf{F}|$ and $\emptyset \in \mathbf{F}$. A typical example of a family $\mathbf{F}$ satisfying the equality $\mathbf{F} = \Delta\mathbf{F}$ is the power set $\mathrm{Pow}(X)$ of any given set X. On the other hand every finite Boolean group G is easily seen to be isomorphic to a power set with the operation of symmetric difference. Indeed if X is a minimal set of generators of G, every product of elements of X equals the product of the elements of X which actually occur in it an odd number of times, so that to every element of G corresponds a unique subset of X. Furthermore the product of two elements in G is exactly the product of the elements in the symmetric difference of their corresponding subsets of X.

Therefore from the algebraic point of view there are no solutions to the equation $\mathbf{F} = \Delta\mathbf{F}$ but the power sets. Moreover, since $|\mathbf{F}| = |\Delta\mathbf{F}|$ implies $\Delta\mathbf{F} = \Delta(\Delta\mathbf{F})$, from the same algebraic point of view the families $\mathbf{F}$ such that $|\mathbf{F}| = |\Delta\mathbf{F}|$ are again just the power sets. But this is far from being true as far as the set-theoretic structure of $\mathbf{F}$ is concerned.

In this section we explore the Venn diagrams that families satisfying $|\mathbf{F}| = |\Delta\mathbf{F}|$ can have. We will give first a characterization of the families satisfying $\mathbf{F} = \Delta\mathbf{F}$, and then show how to characterize the remaining solutions to the equation $|\mathbf{F}| = |\Delta\mathbf{F}|$ by making use of the operation, to be defined below, of forming a *Venn variant* of a family of sets.

Definition 3.1.
- Let $\mathcal{G}_n$ be the family of sets of cardinality n of the form
$$\{\{s_1\} \cup A_1, \dots, \{s_n\} \cup A_n\}$$
where the s_i are all distinct and $\{s_1, \dots, s_n\} \cap (A_1 \cup \ldots \cup A_n) = \emptyset$.
- Let $\mathcal{D}_n = \left\{ \bar{\Delta}\mathbf{G} \mid \mathbf{G} \in \mathcal{G}_n \right\}$.

We will use the following set-theoretic notion, which has been introduced in [1].

Definition 3.2. S is a *differentiating set* for the family $\mathbf{F}$ if for every $A, B \in \mathbf{F}$ with $A \neq B$ we have $A \cap S \neq B \cap S$, i.e. $(A\Delta B) \cap S \neq \emptyset$. S is a *minimal differentiating set* for $\mathbf{F}$ if it is a differentiating set and for every $s \in S$, $S \setminus \{s\}$ is not a differentiating set for $\mathbf{F}$.

In [1] it is shown that if $|\mathbf{F}| = m$, then $\mathbf{F}$ has a differentiating set S of cardinality $m - 1$.

Theorem 3.3.
1) *If $\mathbf{F} \in \mathcal{D}_n$, then $|\mathbf{F}| = 2^n$ and $\mathbf{F} = \Delta\mathbf{F}$;*
2) *If $\mathbf{F} = \Delta\mathbf{F}$, then for every minimal differentiating set S for $\mathbf{F}$ the map $A \mapsto A \cap S$ is a bijection between $\mathbf{F}$ and $\mathrm{Pow}(S)$;*
3) *$\mathbf{F} = \Delta\mathbf{F}$ if and only if for some n, $\mathbf{F} \in \mathcal{D}_n$.*

Proof. 1) Let $\mathbf{G} = \{\{s_1\} \cup A_1, \dots, \{s_n\} \cup A_n\} \in \mathcal{G}_n$ and $\mathbf{F} = \bar{\Delta}\mathbf{G} \in \mathcal{D}_n$. Then $\emptyset \in \mathbf{F}$ and obviously $\mathbf{F} = \Delta\mathbf{F}$. Furthermore the fact that the s_i are all distinct and the condition $\{s_1, \dots, s_n\} \cap (A_1 \cup \ldots \cup A_n) = \emptyset$ ensure that no element of $\mathbf{G}$ can be generated from the others by means of Δ. Hence they are a minimal set of generators of $\mathbf{F}$, which therefore has cardinality 2^n.

2) Assume that $\mathbf{F} = \Delta\mathbf{F}$ so that $\mathbf{F}$ is closed under Δ. In general for $\mathbf{F}$ closed under Δ, from the identity $A\Delta(A\Delta(B\Delta C)) = B\Delta C$, it follows that for every $A \in \mathbf{F}$ we have $\Delta\mathbf{F} = A\Delta\mathbf{F}$. Let S be a minimal differentiating set for $\mathbf{F}$. For $X \in \Delta\mathbf{F}$ let $\pi(X) = X \cap S$. Since S is a differentiating set for $\mathbf{F}$, π is one-to-one. In fact if $(A\Delta B) \cap S = (A\Delta C) \cap S$ then $(B\Delta C) \cap S = \emptyset$; therefore if for $A, B, C \in \mathbf{F}$ with $B \neq C$ we had $\pi(A\Delta B) = \pi(A\Delta C)$, S would fail to be a differentiating set for $\mathbf{F}$. Furthermore, from the identity $(X \cap S)\Delta(Y \cap S) = (X\Delta Y) \cap S$ and the closure of $\Delta\mathbf{F}$ under Δ, it follows that the range of π is closed under Δ as well. Finally, by the minimality of S, for every $s \in S$ there exist $X \in \Delta\mathbf{F}$ such that $\pi(X) = X \cap S = \{s\}$. Thus every subset of S, being the symmetric difference of the singletons of its elements, is in the range of π. π is therefore a bijection between $\Delta\mathbf{F}$ and $\mathrm{Pow}(S)$.
3) The "if" part follows immediately from 1). For the "only if" part let $S = \{s_1, \ldots, s_n\}$ be a minimal differentiating set for $\mathbf{F}$ as in the proof of 2). If $X_1 = \{s_1\} \cup A_1, \ldots, X_n = \{s_n\} \cup A_n$ are the elements of $\Delta\mathbf{F}$ that intersect S in singletons, we have
$$\Delta\mathbf{F} = \bar{\Delta}\{\{s_1\} \cup A_1, \ldots, \{s_n\} \cup A_n\}$$
and $\{s_1, \ldots, s_n\} \cap (A_1 \cup \ldots \cup A_n) = \emptyset$. Thus $\mathbf{F} \in \mathcal{D}_n$.

□

Remark 3.4. $\mathcal{D}_n$ does not exhaust all the families $\mathbf{F}$ with 2^n elements such that $|\mathbf{F}| = |\Delta\mathbf{F}|$; for example the family $\mathbf{F} = \{\{1\}, \{2\}\}$ is such that $\Delta\mathbf{F} = \{\emptyset, \{1, 2\}\}$, so that $|\mathbf{F}| = |\Delta\mathbf{F}|$, although $\mathbf{F} \neq \Delta\mathbf{F}$.

We now prove that if $|\Delta\mathbf{F}|$ is minimal then the size of any minimal differentiating set for $\mathbf{F}$ is minimal.

Proposition 3.5. *If $|\mathbf{F}| > 2^n$, $|\Delta\mathbf{F}| = 2^{n+1}$ and S is a minimal differentiating set for $\mathbf{F}$, then $|S| = n + 1$.*

Proof. If S is a minimal differentiating set for $\mathbf{F}$, then $|S| \geq n+1$. For every $s \in S$, $\{s\} \in S \cap \Delta\mathbf{F}$. By Theorem 2.4 $\Delta\mathbf{F}$ is closed under Δ, and therefore $S \cap \Delta\mathbf{F}$ is closed under Δ. It follows that $\mathrm{Pow}(S) = S \cap \Delta\mathbf{F}$ and thus $|S| \leq n + 1$. □

Remark 3.6. The converse of Proposition 3.5 is false even in the special case $|\mathbf{F}| = 2^{n+1}$: $\mathbf{F}$ can be such that $|\mathbf{F}| = 2^{n+1}$ and have only minimal differentiating sets of the smallest possible size, i.e. $n + 1$, but fail to satisfy $|\Delta\mathbf{F}| = 2^{n+1}$. An example is provided by $\mathbf{F} = \{\emptyset, \{1\}, \{2\}, \{1, 2, 3\}\}$ which is a family of 2^2 sets generating $2^2 + 3$ symmetric differences, although its only minimal differentiating set is $\{1, 2\}$.

Starting with families in $\mathcal{D}_n$, new families satisfying the equality $|\mathbf{F}| = |\Delta\mathbf{F}|$ can be obtained by the operation of making what we call Venn variants.

Definition 3.7.
- Let $V(\mathbf{F})$ denote the *Venn diagram* of $\mathbf{F}$, namely the partition induced on $\bigcup\mathbf{F}$ by the equivalence relation $\sim_{\mathbf{F}}$ defined by
$$x \sim_{\mathbf{F}} y \text{ if and only if } \forall A \in \mathbf{F}(x \in A \longleftrightarrow y \in A).$$
- For $v \in V(\mathbf{F})$ we let
$$\mathbf{F}_v = \{\, A \setminus v \mid A \in \mathbf{F} \wedge v \subseteq A \,\} \cup \{\, A \cup v \mid A \in \mathbf{F} \wedge A \cap v = \emptyset \,\}$$
and call $\mathbf{F}_v$ the *Venn variant* of $\mathbf{F}$ determined by v.

The following proposition is immediate from the definition.

Proposition 3.8. *For all $v \in V(\mathbf{F})$, $|\mathbf{F}_v| = |\mathbf{F}|$ and $\Delta(\mathbf{F}_v) = \Delta\mathbf{F}$, i.e. a Venn variant of $\mathbf{F}$ has the same cardinality and produces the same collection of symmetric differences as $\mathbf{F}$.*

Definition 3.9. $\mathcal{V}_n$ is the least family which contains $\mathcal{D}_n$ and is closed under formation of Venn variants, namely

- if $\mathbf{F} \in \mathcal{D}_n$, then $\mathbf{F} \in \mathcal{V}_n$;
- if $\mathbf{F} \in \mathcal{V}_n$ and $v \in V(\mathbf{F})$, then $\mathbf{F}_v \in \mathcal{V}_n$.

By Theorem 3.3 and iterated application of Proposition 3.8 we obtain:

Proposition 3.10. *If $\mathbf{F} \in \mathcal{V}_n$, then $|\mathbf{F}| = |\Delta\mathbf{F}| = 2^n$.*

The analysis we will now carry out will show that not even $\mathcal{V}_n$ exhausts the families of cardinality 2^n which have only 2^n symmetric differencies.

Definition 3.11. For $v, v' \in V(\mathbf{F})$ we say that v is *opposite* to v' in $\mathbf{F}$ if

$$\{ A \in \mathbf{F} \mid v \subseteq A \} = \{ A \in \mathbf{F} \mid A \cap v' = \emptyset \}$$

or, equivalently, if

$$\{ A \in \mathbf{F} \mid v \subseteq A \} = \mathbf{F} \setminus \{ A \in \mathbf{F} \mid v' \subseteq A \} .$$

Proposition 3.12. *Let $\mathbf{F}$ be a family and $v, v', v'' \in V(\mathbf{F})$:*

1) *v is not opposite to v in $\mathbf{F}$;*
2) *if v is opposite to v' in $\mathbf{F}$, then v' is opposite to v in $\mathbf{F}$;*
3) *if v' and v'' are opposite to v in $\mathbf{F}$, then $v' = v''$;*
4) *if no element of $V(\mathbf{F})$ is opposite to v in $\mathbf{F}$, then $V(\mathbf{F}_v) = V(\mathbf{F})$, so that for every $u \in V(\mathbf{F})$ $\mathbf{F}_{vu}$ is defined; furthermore $\mathbf{F}_{vv} = \mathbf{F}$;*
5) *if v and v' are opposite in $\mathbf{F}$, then*
$$V(\mathbf{F}_v) = (V(\mathbf{F}) \setminus \{v, v'\}) \cup \{v \cup v'\},$$
so that for every $u \in V(\mathbf{F}) \setminus \{v, v'\}$ $\mathbf{F}_{vu}$ is defined; furthermore $\mathbf{F}_v = \mathbf{F}_{v'}$;
6) *if $v \in V(\mathbf{F})$, then either v (if v has no opposite in $\mathbf{F}$) or $v \cup v'$ (if v and v' are opposite in $\mathbf{F}$) has no opposite in $\mathbf{F}_v$;*
7) *for any $v \in V(\mathbf{F})$, if v' and v'' are not opposite in $\mathbf{F}$ then v' and v'' are not opposite in $\mathbf{F}_v$.*

Proof. 1) and 2) are immediate.

3) Let $A \in \mathbf{F}$. $v' \subseteq A$ is equivalent (since v and v' are opposite) to $v \cap A = \emptyset$ which is equivalent (since v and v'' are opposite) to $v'' \subseteq A$. Hence
$$\{ A \in \mathbf{F} \mid v' \subseteq A \} = \{ A \in \mathbf{F} \mid v'' \subseteq A \} ,$$
which entails $v' = v''$.

4) It suffices to show that $\sim_{\mathbf{F}}$ and $\sim_{\mathbf{F}_v}$ are the same equivalence relation. Let $x, y \in \bigcup\mathbf{F}$. If $x \sim_{\mathbf{F}} y$, then either $x, y \in v$ or $x, y \notin v$, from which it follows immediately that $x \sim_{\mathbf{F}_v} y$. Conversely let us assume that $x \nsim_{\mathbf{F}} y$ so that at most one of x, y is in v. If $x, y \notin v$ let $A \in \mathbf{F}$ be such that $x \in A$ and $y \notin A$; then one of $A \setminus v$ and $A \cup v$ is in $\mathbf{F}_v$ and witnesses that $x \nsim_{\mathbf{F}_v} y$. Now suppose $x \in v$, and hence $y \notin v$. Since v has no opposite element in $\mathbf{F}$, there is $B \in \mathbf{F}$ such that either $x, y \in B$ or $x, y \notin B$. In the former case $B \setminus v$ contains y but does not contain x; in the latter case $B \cup v$ contains x but does not contain y. In both cases we have that $x \nsim_{\mathbf{F}_v} y$.

$\mathbf{F}_{vv} = \mathbf{F}$ follows immediately from the definitions.

5) The proof of 4) shows that if $u \in V(\mathbf{F})$, $u \neq v$ and $u \neq v'$, then u is a $\sim_{\mathbf{F}_v}$ equivalence class, namely an element of $V(\mathbf{F}_v)$. On the other hand if $x \in v$ and $y \in v'$, then $x \sim_{\mathbf{F}_v} y$ so that $v \cup v'$ replaces both v and v' in $V(\mathbf{F}_v)$.
6) Let $\bar{v}$ be either v (if v has no opposite in $\mathbf{F}$) or $v \cup v'$ (if v and v' are opposite in $\mathbf{F}$). Suppose $u \in V(\mathbf{F}_v)$ is opposite to $\bar{v}$ in $\mathbf{F}_v$. By 1) $u \neq \bar{v}$ and hence by 4) or 5) $u \in V(\mathbf{F})$. Since $u \neq v$ let $A \in \mathbf{F}$ be such that either $u \subseteq A$ and $v \not\subseteq A$ or $u \not\subseteq A$ and $v \subseteq A$. In the former case $A \cup v \in \mathbf{F}_v$ and contains both u and $\bar{v}$; in the latter case $A \setminus v \in \mathbf{F}_v$ and contains neither u nor $\bar{v}$. In both cases u and $\bar{v}$ are not opposite in $\mathbf{F}_v$.
7) If v is either v' or v'', then the conclusion follows immediately from 6). Otherwise $v', v'' \in V(\mathbf{F})$ and, since v' and v'' are not opposite in $\mathbf{F}$, there exists $A \in \mathbf{F}$ such that either $v' \subseteq A$ and $v'' \subseteq A$ or $v' \cap A = \emptyset$ and $v'' \cap A = \emptyset$. Then either $A \cup v$, if $v \cap A = \emptyset$, or $A \setminus v$, if $v \subseteq A$, witnesses that v' and v'' are not opposite in $\mathbf{F}_v$. □

As an immediate consequence we have the following:

Corollary 3.13. *If $V(\mathbf{F})$ has no pair of opposite elements, then for every $v \in V(\mathbf{F})$, $V(\mathbf{F}_v)$ has no pair of opposite elements.*

The following proposition relates the absence of opposite elements in the Venn diagram of a family with the Venn diagram of the family of its symmetric differences.

Proposition 3.14. *$V(\mathbf{F})$ has no pair of opposite elements if and only if $V(\mathbf{F}) = V(\Delta\mathbf{F})$.*

Proof. Let $x \sim_{\mathbf{F}}^{-} y$ stand for $\forall A \in \mathbf{F}(x \in A \longleftrightarrow y \notin A)$. As it is easy to check $x \sim_{\Delta\mathbf{F}} y$ if and only if either $x \sim_{\mathbf{F}} y$ or $x \sim_{\mathbf{F}}^{-} y$, so that $V(\Delta\mathbf{F})$ is the partition induced on $\bigcup \mathbf{F}$ by the equivalence relation $x \sim_{\mathbf{F}} y \vee x \sim_{\mathbf{F}}^{-} y$. If $V(\mathbf{F})$ has no pair of opposite elements, $\sim_{\mathbf{F}}^{-}$ is the empty relation and $\sim_{\Delta\mathbf{F}}$ coincides with $\sim_{\mathbf{F}}$ so that $V(\mathbf{F}) = V(\Delta\mathbf{F})$. Conversely if $V(\mathbf{F}) = V(\Delta\mathbf{F})$, then $\sim_{\mathbf{F}} = \sim_{\Delta\mathbf{F}}$ and $\sim_{\mathbf{F}}^{-}$ must be empty, which entails that in $V(\mathbf{F})$ there are no pairs of opposite elements. □

Proposition 3.15. *If $\mathbf{F} \in \mathcal{V}_n$, then $V(\mathbf{F})$ has no pair of opposite elements.*

Proof. If $\mathbf{F} \in \mathcal{D}_n$, then $\mathbf{F} = \Delta\mathbf{F}$; thus $V(\mathbf{F}) = V(\Delta\mathbf{F})$ so that by the previous proposition $V(\mathbf{F})$ has no pair of opposite elements. Since the families in $\mathcal{V}_n$ are obtained from those in $\mathcal{D}_n$ by iterating the operation of Venn variant, by Corollary 3.13 their Venn diagrams have no pairs of opposite elements. □

Remark 3.16. If $\mathbf{F} = \{\{1\}, \{2\}, \{1,3\}, \{2,3\}\}$, then $\Delta\mathbf{F} = \{\emptyset, \{1,2\}, \{3\}, \{1,2,3\}\}$ so that $|\mathbf{F}| = |\Delta\mathbf{F}|$. However $\{1\}$ and $\{2\}$ are elements of $V(\mathbf{F})$ which are opposite in $\mathbf{F}$, so that $\mathbf{F} \notin \mathcal{V}_2$ by Proposition 3.15.

The following proposition shows that the families in $\mathcal{V}_n$ are precisely those which have the least possible number of symmetric differences and, at the same time, have no pair of opposite elements in their Venn diagram.

Proposition 3.17.

$$\mathcal{V}_n = \{\, \mathbf{F} \mid |\mathbf{F}| = |\Delta\mathbf{F}| = 2^n \wedge V(\mathbf{F}) \textit{ has no pair of opposite elements} \,\}.$$

Proof. Propositions 3.10 and 3.15 show that $\mathcal{V}_n$ is included in the set on the right hand side of the equality.

To prove the reverse inclusion assume that $|\mathbf{F}| = |\Delta\mathbf{F}| = 2^n$ and $V(\mathbf{F})$ has no pair of opposite elements. Let S be a minimal differentiating set for $\mathbf{F}$. Since S is a differentiating set for $\mathbf{F}$, the map $\lambda : \mathbf{F} \to \mathrm{Pow}(S)$ defined by $\lambda(A) = A \cap S$ is one-to-one. Since $|\mathbf{F}| = |\Delta\mathbf{F}|$, $\Delta\mathbf{F} = \Delta(\Delta\mathbf{F})$, and, by Theorem 3.3.2, $|\Delta\mathbf{F}| = |\mathrm{Pow}(S)|$. Therefore λ is actually a bijection. Thus there exists a unique $A_0 \in \mathbf{F}$ such that $A_0 \cap S = \emptyset$. Let $v_1, \dots, v_k \in V(\mathbf{F})$ be such that $A_0 = v_1 \cup \dots \cup v_k$. Since $V(\mathbf{F})$ has no pair of opposite elements $\mathbf{G} = \mathbf{F}_{v_1 \dots v_k}$ is well defined. Clearly $\emptyset \in \mathbf{G}$ and by Proposition 3.8 $|\mathbf{G}| = |\Delta\mathbf{G}|$. As noticed earlier these conditions entail $\mathbf{G} = \Delta\mathbf{G}$. Thus by Theorem 3.3 we have that $\mathbf{G} \in \mathcal{D}_n$. Finally, since by Proposition 3.12.4

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_{v_1 \dots v_k v_k v_{k-1} \dots v_1}\\ &= \mathbf{G}_{v_k \dots v_1},\end{aligned}$$

we have that $\mathbf{F} \in \mathcal{V}_n$. □

By Proposition 3.12.6 and 7, eliminating pairs of opposite elements is simply a matter of iterating the operation of making Venn variants.

Definition 3.18. If $(v_1, v_1'), \dots, (v_k, v_k')$ are all the pairs of opposite elements in $\mathbf{F}$, we let

$$\mathbf{F}^* = \mathbf{F}_{v_1 \dots v_k}.$$

Note that $\mathbf{F}^*$ depends neither on the order in which $v_1, \dots, v_k$ are taken (since, in general, $\mathbf{F}_{vu} = \mathbf{F}_{uv}$ as long as both Venn variants are legal), nor on which element of the pair (v_i, v_i') is used to make the Venn variant (by Proposition 3.12.5). As an immediate consequence of Proposition 3.12.6 and 7 we have the following:

Proposition 3.19. *$V(\mathbf{F}^*)$ has no pair of opposite elements.*

We can now state our characterization of the finite families satisfying $|\mathbf{F}| = |\Delta\mathbf{F}|$.

Theorem 3.20. $|\mathbf{F}| = |\Delta\mathbf{F}| = 2^n$ *if and only if* $\mathbf{F}^* \in \mathcal{V}_n$.

Proof. Since by Proposition 3.8 $|\mathbf{F}| = |\mathbf{F}^*|$ and $\Delta\mathbf{F} = \Delta(\mathbf{F}^*)$, the if part follows from Proposition 3.10, while the only if part follows from Propositions 3.19 and 3.17. □

4. Trees describing a family

In this section we provide a different characterization of the familes $\mathbf{F}$ satisfying $|\mathbf{F}| = |\Delta\mathbf{F}|$. This characterization is based on the analysis of how elements of $(\bigcup\mathbf{F}) \setminus (\bigcap\mathbf{F})$ discriminate between the sets in $\mathbf{F}$.

Definition 4.1. Given a family $\mathbf{F}$ and an element $x \in (\bigcup\mathbf{F}) \setminus (\bigcap\mathbf{F})$, we let

$$\mathbf{F}_x = \{\, A \in \mathbf{F} \mid x \in A \,\} \qquad \text{and} \qquad \mathbf{F}_{\bar{x}} = \{\, A \in \mathbf{F} \mid x \notin A \,\}.$$

We begin with some simple facts that will turn out to be useful.

Proposition 4.2. *For any $x \in (\bigcup\mathbf{F}) \setminus (\bigcap\mathbf{F})$ we have*

1) $\Delta\mathbf{F} = (\Delta\mathbf{F}_x) \cup (\Delta\mathbf{F}_{\bar{x}}) \cup (\mathbf{F}_x \Delta \mathbf{F}_{\bar{x}})$;
2) $[(\Delta\mathbf{F}_x) \cup (\Delta\mathbf{F}_{\bar{x}})] \cap (\mathbf{F}_x \Delta \mathbf{F}_{\bar{x}}) = \emptyset$;
3) $|\Delta\mathbf{F}| \geq 2\max(|\mathbf{F}_x|, |\mathbf{F}_{\bar{x}}|)$.

Proof. 1) is immediate. 2) follows from the fact that for every $A \in (\Delta\mathbf{F}_x) \cup (\Delta\mathbf{F}_{\bar{x}})$ we have $x \notin A$ while for every $B \in \mathbf{F}_x \Delta \mathbf{F}_{\bar{x}}$ we have $x \in B$. 3) is a consequence of 1) and 2) together with $|\mathbf{F}\Delta\mathbf{G}| \geq \max(|\mathbf{F}|, |\mathbf{G}|)$. □

Proposition 4.3. *If* $|\mathbf{F}| = |\Delta\mathbf{F}|$ *and* $x \in (\bigcup \mathbf{F}) \setminus (\bigcap \mathbf{F})$*, then* $|\Delta\mathbf{F}_x| = |\mathbf{F}_x| = |\Delta\mathbf{F}_{\bar{x}}| = |\mathbf{F}_{\bar{x}}| = |\mathbf{F}|/2$ *and* $\Delta\mathbf{F}_x = \Delta\mathbf{F}_{\bar{x}}$.

Proof. $|\mathbf{F}_x| = |\mathbf{F}_{\bar{x}}| = |\mathbf{F}|/2$ follows immediately from the hypothesis and Proposition 4.2.3. Since $|\mathbf{F}_x \Delta \mathbf{F}_{\bar{x}}| \geq |\mathbf{F}|/2$ by Proposition 4.2.2 we must have $\Delta\mathbf{F}_x = \Delta\mathbf{F}_{\bar{x}}$ and $|\Delta\mathbf{F}_x| = |\Delta\mathbf{F}_{\bar{x}}| = |\mathbf{F}|/2$. ☐

Definition 4.4. A tree *describing* $\mathbf{F}$ is a binary tree satisfying the following properties:

1. the nodes are pairwise different subsets of $\mathbf{F}$,
2. the root is $\mathbf{F}$,
3. the leaves are singleton subsets of $\mathbf{F}$,
4. the children of any internal node $\mathbf{G} \subseteq \mathbf{F}$ are $\mathbf{G}_x$ and $\mathbf{G}_{\bar{x}}$ for some $x \in (\bigcup \mathbf{G}) \setminus (\bigcap \mathbf{G})$.

The *height* $h(\mathbf{F})$ of a family $\mathbf{F}$ is the height of the highest tree describing $\mathbf{F}$.

The next theorem gives another characterization of the finite families satisfying $|\mathbf{F}| = |\Delta\mathbf{F}|$.

Theorem 4.5. $|\mathbf{F}| = |\Delta\mathbf{F}|$ *if and only if* $h(\mathbf{F}) = \log_2(|\mathbf{F}|)$.

Proof. By induction on $|\mathbf{F}|$.

If $|\mathbf{F}| = 1$ the only tree describing $\mathbf{F}$ consists only of the root, and hence the theorem holds.

If $|\mathbf{F}| > 1$ consider any tree describing $\mathbf{F}$ and let $x \in (\bigcup \mathbf{F}) \setminus (\bigcap \mathbf{F})$ be such that $\mathbf{F}_x$ and $\mathbf{F}_{\bar{x}}$ are the children of the root. The subtrees lying above these nodes describe respectively $\mathbf{F}_x$ and $\mathbf{F}_{\bar{x}}$; if $|\mathbf{F}| = |\Delta\mathbf{F}|$ by Proposition 4.3 and the inductive hypothesis we have that their heights are $\log_2(|\mathbf{F}|) - 1$ and hence that the original tree has height $\log_2(|\mathbf{F}|)$.

If $h(\mathbf{F}) = \log_2(|\mathbf{F}|)$, then both $\mathbf{F}_x$ and $\mathbf{F}_{\bar{x}}$ have height less than or equal to $\log_2(|\mathbf{F}|) - 1$. Hence their size is bounded by $2^{\log_2(|\mathbf{F}|)-1} = |\mathbf{F}|/2$. From this we can conclude that $|\mathbf{F}_x| = |\mathbf{F}_{\bar{x}}| = |\mathbf{F}|/2$ and that $h(\mathbf{F}_x) = h(\mathbf{F}_{\bar{x}}) = \log_2(|\mathbf{F}|) - 1$. By inductive hypothesis we have that $|\mathbf{F}_x| = |\Delta\mathbf{F}_x| = |\mathbf{F}_{\bar{x}}| = |\Delta\mathbf{F}_{\bar{x}}| = |\mathbf{F}|/2$.

We begin by showing that $|\mathbf{F}| = |\Delta\mathbf{F}|$ holds if $|\mathbf{F}| = 4$. Let $\mathbf{F} = \{A, B, C, D\}$ and notice that $C\Delta D = A\Delta B$; otherwise $A\Delta B\Delta C\Delta D \neq \emptyset$ and there exists $x \in (\bigcup \mathbf{F}) \setminus (\bigcap \mathbf{F})$ belonging to either exactly one or exactly three elements of $\mathbf{F}$: in both cases we could construct a tree describing $\mathbf{F}$ of height 3. Therefore we have also $B\Delta C = A\Delta D$ and $B\Delta D = A\Delta C$, so that $\Delta\mathbf{F} = \{A\Delta A, A\Delta B, A\Delta C, A\Delta D\}$ and hence $|\mathbf{F}| = |\Delta\mathbf{F}|$.

We now turn to the general case. Let α be a sequence indexing a node in a tree describing $\mathbf{F}$, and let β be a sequence obtained from α replacing zero or more characters x by $\bar{x}$, and zero or more characters $\bar{x}$ by x. For example: $\alpha = x\bar{y}z\bar{w}$ and $\beta = \bar{x}\bar{y}zw$.

Let also denote by $\bar{\gamma}$ the sequence obtained from γ replacing each x by $\bar{x}$ and each $\bar{x}$ by x.

The following equality will be proved by induction on $|\mathbf{F}|$:

$$\mathbf{F}_\alpha \Delta \mathbf{F}_\beta = \mathbf{F}_{\bar{\alpha}} \Delta \mathbf{F}_{\bar{\beta}}.$$

The base case is immediate.

The inductive step is proved by a further induction on $\log_2(|\mathbf{F}|) - |\alpha|$.

The base case corresponds to the case in which $|\alpha| = \log_2(|\mathbf{F}|)$ and is the most complex. In this case, let $\mathbf{F}_\alpha = \{A\}$, $\mathbf{F}_\beta = \{B\}$, $\mathbf{F}_{\bar\alpha} = \{A'\}$, and $\mathbf{F}_{\bar\beta} = \{B'\}$; we must prove that $A\Delta B = A'\Delta B'$.

If $\alpha = \bar\beta$ the result is obvious. Otherwise let x be an element occurring both in α and in β (the argument for the case where only elements of the form $\bar x$ occur both in α and in β is analogous). We can assume without loss of generality that $\alpha = x\alpha_1$ and $\beta = x\beta_1$. By inductive hypothesis on the cardinality of the family,

$$\mathbf{F}_\alpha \Delta \mathbf{F}_\beta = \mathbf{F}_{x\alpha_1} \Delta \mathbf{F}_{x\beta_1} = \mathbf{F}_{x\bar{\alpha_1}} \Delta \mathbf{F}_{x\bar{\beta_1}}.$$

Moreover

$$\mathbf{F}_{\bar\alpha} = \mathbf{F}_{\bar x\bar{\alpha_1}} = \{A'\} \implies \mathbf{F}_{\bar{\alpha_1}} = \{A', A''\} \text{ with } x \in A'',$$
$$\mathbf{F}_{\bar\beta} = \mathbf{F}_{\bar x\bar{\beta_1}} = \{B'\} \implies \mathbf{F}_{\bar{\beta_1}} = \{B', B''\} \text{ with } x \in B''.$$

From this it follows that $\mathbf{F}_{x\bar{\alpha_1}} = \{A''\}$ and $\mathbf{F}_{x\bar{\beta_1}} = \{B''\}$. Hence $\mathbf{F}_\alpha \Delta \mathbf{F}_\beta = \mathbf{F}_{x\bar{\alpha_1}} \Delta \mathbf{F}_{x\bar{\beta_1}}$ implies that $A\Delta B = A''\Delta B''$ and it suffices to prove $A''\Delta B'' = A'\Delta B'$.

To this end consider the family $\mathbf{G} = \{A', B', A'', B''\}$. We show that $h(\mathbf{G}) = \log_2(|\mathbf{G}|) = 2$: let $\mathbf{H}$ be the first common ancestor of $\mathbf{F}_{\bar{\alpha_1}} = \{A', A''\}$ and $\mathbf{F}_{\bar{\beta_1}} = \{B', B''\}$, and let $\mathbf{H}_y$ and $\mathbf{H}_{\bar y}$ be the children of $\mathbf{H}$. Since y discriminates between $\{A', A''\}$ and $\{B', B''\}$, either y appears in $\bar{\alpha_1}$ or it appears in $\bar{\beta_1}$. Assuming, without loss of generality, that the former is the case, we have that for some ξ, $\mathbf{F}_{\bar\alpha} = \mathbf{F}_{\xi\bar x y}$ and hence $\mathbf{F}_\xi = \{A', A'', B', B''\} = \mathbf{G}$. If we had a tree describing $\mathbf{G}$ of height greater than two, we could graft such a tree in place of $\mathbf{F}_\xi$ in a tree describing $\mathbf{F}$ and produce a tree describing $\mathbf{F}$ of height greater than $\log_2(|\mathbf{F}|)$, contradicting the hypothesis.

Hence $h(\mathbf{G}) = 2$ and by the case $|\mathbf{F}| = 4$ considered above, $A''\Delta B'' = A'\Delta B'$, which concludes the base case.

For the inductive step pick $x \in \bigcup(\mathbf{F}_\alpha \Delta \mathbf{F}_\beta) \setminus \bigcap(\mathbf{F}_\alpha \Delta \mathbf{F}_\beta)$ and notice that

$$\begin{aligned}
\mathbf{F}_\alpha \Delta \mathbf{F}_\beta &= (\mathbf{F}_\alpha \Delta \mathbf{F}_\beta)_x \cup (\mathbf{F}_\alpha \Delta \mathbf{F}_\beta)_{\bar x} \\
&= [(\mathbf{F}_{\alpha x} \Delta \mathbf{F}_{\beta\bar x}) \cup (\mathbf{F}_{\alpha\bar x} \Delta \mathbf{F}_{\beta x})] \cup [(\mathbf{F}_{\alpha x} \Delta \mathbf{F}_{\beta x}) \cup (\mathbf{F}_{\alpha\bar x} \Delta \mathbf{F}_{\beta\bar x})] \\
&= [(\mathbf{F}_{\bar\alpha\bar x} \Delta \mathbf{F}_{\bar\beta x}) \cup (\mathbf{F}_{\bar\alpha x} \Delta \mathbf{F}_{\bar\beta\bar x})] \cup [(\mathbf{F}_{\bar\alpha\bar x} \Delta \mathbf{F}_{\bar\beta\bar x}) \cup (\mathbf{F}_{\bar\alpha x} \Delta \mathbf{F}_{\bar\beta x})] \\
&= (\mathbf{F}_{\bar\alpha} \Delta \mathbf{F}_{\bar\beta})_x \cup (\mathbf{F}_{\bar\alpha} \Delta \mathbf{F}_{\bar\beta})_{\bar x} \\
&= \mathbf{F}_{\bar\alpha} \Delta \mathbf{F}_{\bar\beta}
\end{aligned}$$

where the third equality has been obtained by induction hypothesis.

If $\alpha = \beta = x$, the equality we just proved shows that $\Delta\mathbf{F}_x = \Delta\mathbf{F}_{\bar x}$. Since we already have $|\Delta\mathbf{F}_x| = |\mathbf{F}|/2$, Proposition 4.2.1 entails that our thesis $|\mathbf{F}| = |\Delta(\mathbf{F})|$ will follow from

$$|\mathbf{F}_x \Delta \mathbf{F}_{\bar x}| = |\mathbf{F}|/2.$$

Since $|\Delta\mathbf{F}_x| = |\mathbf{F}_x|$ and $|\Delta\mathbf{F}_{\bar x}| = |\mathbf{F}_{\bar x}|$ for any $A \in \mathbf{F}_x$ and $B \in \mathbf{F}_{\bar x}$, we have $\Delta\mathbf{F}_x = \{\, A\Delta A' \mid A' \in \mathbf{F}_x \,\}$ and $\Delta\mathbf{F}_{\bar x} = \{\, B\Delta B' \mid B' \in \mathbf{F}_{\bar x} \,\}$. Fix $B \in \mathbf{F}_{\bar x}$ and consider the function $\varphi : \mathbf{F}_x \to \mathbf{F}_x \Delta \mathbf{F}_{\bar x}$ defined by $\varphi(A') = A'\Delta B$.

Clearly φ is injective. φ is also surjective: for any $A'\Delta B' \in \mathbf{F}_x \Delta \mathbf{F}_{\bar x}$, since $B\Delta B' \in \Delta\mathbf{F}_{\bar x} = \Delta\mathbf{F}_x$ there exists $A'' \in \mathbf{F}_x$ such that $B\Delta B' = A'\Delta A''$. Hence

$$A'\Delta B' = A''\Delta B = \varphi(A'').$$

Hence φ is a bijection between $\mathbf{F}_x$ and $\mathbf{F}_x \Delta \mathbf{F}_{\bar x}$: we have $|\mathbf{F}_x \Delta \mathbf{F}_{\bar x}| = |\mathbf{F}_x| = |\mathbf{F}|/2$ and the proof is complete. ∎

Remark 4.6. The generalization of Theorem 4.5 stating that for any family $\mathbf{F}$, $|\Delta\mathbf{F}| = 2^{\lceil \log_2(|\mathbf{F}|)\rceil}$ if and only if $h(\mathbf{F}) = \lceil \log_2(|\mathbf{F}|)\rceil$ is false. The "only if" direction follows easily by Theorems 2.4 and 4.5, but the "if" direction does not hold. A counterexample is provided by $\mathbf{F} = \{\emptyset, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3,4\}\}$, which is a family of 2^2+1 sets generating 2^3+3 symmetric differences, although $h(\mathbf{F}) = \lceil \log_2(|\mathbf{F}|)\rceil = 3$. Notice also that this family has only minimal differentiating sets of size 3 (see Remark 3.6).

The result in this section can be interpreted in the following, playful, way. A family of sets $\mathbf{F}$ is given and players I and II have full knowledge of $\mathbf{F}$. Player I picks $X \in \mathbf{F}$. Player II has to discover X, by asking, one after another, questions of the form "does a belong to X?". Player I has to give correct yes/no answers. Player II asks only questions whose answer he/she cannot recover from his/her knowledge of $\mathbf{F}$ and from the answers to the previous questions. When $\mathbf{F}$ is finite, player II will always discover X after asking at most $|\mathbf{F}| - 1$ questions. In general, the number of questions II has to ask depends on X as well as on the sequence of the questions asked. For $\mathbf{F}$ finite, Theorem 4.5 says that the number of questions player II has to ask depends neither on X nor on the sequence of the questions asked if and only if $|\mathbf{F}| = |\Delta\mathbf{F}|$. The results in Section 3 then give a way of constructing plays of that sort.

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(A. Marcone) Dipartimento di Matematica, Università di Torino, via Carlo Alberto 10, 10123 Torino, Italy

E-mail address: marcone@dm.unito.it

Current address: Dipartimento di Matematica e Informatica, Università di Udine, viale delle Scienze, 33100 Udine, Italy

E-mail address: marcone@dimi.uniud.it

(F. Parlamento and A. Policriti) Dipartimento di Matematica e Informatica, Università di Udine, viale delle Scienze, 33100 Udine, Italy

E-mail address: parlamen@dimi.uniud.it

E-mail address: policrit@dimi.uniud.it