PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 3, March 1999, Pages 695–700
S 0002-9939(99)04765-6

THE STRUCTURE OF SOME VIRTUALLY FREE PRO-p GROUPS

CLAUS SCHEIDERER

(Communicated by Ronald M. Solomon)

Abstract. We prove two conjectures on pro-p groups made by Herfort, Ribes and Zalesskii. The first says that a finitely generated pro-p group which has an open free pro-p subgroup of index p is a free pro-p product $H_0 * (S_1 \times H_1) * \cdots * (S_m \times H_m)$, where the H_i are free pro-p of finite rank and the S_i are cyclic of order p. The second says that if F is a free pro-p group of finite rank and S is a finite p-group of automorphisms of F, then $\mathrm{Fix}(S)$ is a free factor of F. The proofs use cohomology, and in particular a "Brown theorem" for profinite groups.

1. The results

Let p be a prime number, and let G be a finitely generated pro-p group which contains an open free pro-p subgroup F of index p. We shall prove the following structure theorem:

1.1. Theorem. *G is isomorphic to a free pro-p product*

$$H_0 * (S_1 \times H_1) * \cdots * (S_m \times H_m)$$

where $m \geq 0$, the S_i are cyclic groups of order p and the H_i are free pro-p groups of finite rank.

This theorem was conjectured by Herfort, Ribes and Zalesskii in their recent preprint [5]. A proof is given there in the case where the rank of F is at most two. Theorem 1.1 is analogous to a similar theorem for discrete groups, due to Dyer and Scott: A group which contains a free normal subgroup of index p is a free product of a free group and groups of the form $\mathbb{Z}/p\mathbb{Z} \times H_\lambda$ where the H_λ are free groups ([1], Thm. 1). The proof of this latter theorem is rather straightforward if one uses the Bass-Serre theory of groups acting on trees, and in particular the fact that every finite extension of a free group is the fundamental group of a suitable graph of finite groups.

Unfortunately it seems that no profinite analog of Bass-Serre theory is available which would be flexible enough to allow such applications. Various authors have worked on such a profinite theory. One has profinite notions of trees, of graphs of groups and of their fundamental groups; and many of the results from discrete

Received by the editors July 1, 1997.
1991 *Mathematics Subject Classification.* Primary 20E18; Secondary 20E34, 20E36, 20E06.
Key words and phrases. Pro-p groups, virtually free groups, group cohomology, Brown theorem.

©1999 American Mathematical Society

Bass-Serre theory have been transferred to a profinite setting. See in particular [2] and [11]. These methods seem to work best for the class of pro-p groups. However what is still missing is something like Stallings' theory of ends, as an instrument for splitting up a given group as an amalgamated product.

Therefore other ideas have to be found, in order to attack such questions. While the proof in [5] (for $\mathrm{rk}(F) \le 2$) splits into many case distinctions and uses extensive explicit calculations, we propose here a completely different approach based on group cohomology. As one can see below, this allows a very short and natural proof of the general case. The principal tool is a profinite "Brown theorem", which expresses high-degree cohomology of suitable profinite groups in terms of their finite subgroups. It is used here as a means for detecting torsion elements of G in the cohomology of G. The general result in this direction is in [8]. However, the special case considered in [7], ch. 12, is more easily accessible and suffices for the purpose of this note.

From Theorem 1.1 we get similar applications as in the discrete case, cf. [1]. For example, given a free pro-p group F of finite rank, there is a normal form for the action of an automorphism $\alpha \in \mathrm{Aut}(F)$ of order p, which is completely analogous to the discrete one ([1], Thm. 3). One can copy the proof from [1] once Theorem 1.1 is known. Alternatively one may also obtain the pro-p case as a corollary to the discrete case, since 1.1 shows that there are Φ, a free discrete group, and $\beta \in \mathrm{Aut}(\Phi)$ of order p such that F and α are the pro-p completions of Φ and β.

Another application is the following (cf. [1], Thm. 2, for the case of discrete groups):

1.2. Theorem. *Let F be a free pro-p group of finite rank, and let S be a finite p-group of automorphisms of F. Then* $\mathrm{Fix}(S)$*, the subgroup of F consisting of the elements fixed by S, is a free factor of F. In particular,* $\mathrm{rk}\,\mathrm{Fix}(S) \le \mathrm{rk}(F)$.

Here rk denotes the rank of a (free) pro-p group. Note that the last inequality is strict if S doesn't consist of the identity alone. The fact that $\mathrm{rk}\,\mathrm{Fix}(S) \le \mathrm{rk}(F)$ was conjectured by Herfort, Ribes and Zalesskii in [4] for S cyclic, and was proved there if $\mathrm{rk}(F) = 2$. In fact, it was also conjectured in [4] that $\mathrm{rk}\,\mathrm{Fix}(\alpha) \le \mathrm{rk}(F)$ should hold for every automorphism α of F whose order (as a "super-natural number") is p^∞. Whether or not this last conjecture is true seems still unknown at present. It is remarkable that, on the other hand, the fixgroup of an automorphism α of finite order prime to p has always infinite rank (if $\alpha \ne \mathrm{id}$ and $\mathrm{rk}(F) > 1$), as shown by Herfort and Ribes [3]. This sharply contrasts the case of discrete free groups.

Another application which may perhaps claim some interest is the following. We say that a pro-p group is *virtually free of finite rank* if it has an open subgroup which is a free pro-p group of finite rank.

1.3. Corollary. *Let G be a pro-p group which is virtually free of finite rank.*

a) *G has only finitely many conjugacy classes of finite subgroups.*
b) *If S is a finite subgroup of G, then the centralizer and the normalizer of S are again virtually free of finite rank.*

The proofs of the two theorems and of the corollary will be given in the next section.

2. The proofs

If G, H are pro-p groups, then $G * H$ denotes their free pro-p product [6]. We write $H^n(G) := H^n(G, \mathbb{Z}/p\mathbb{Z})$. Observe that $H^n(G * H) = H^n(G) \oplus H^n(H)$ for all $n \geq 1$ [6]. The rank $\mathrm{rk}(F)$ of a free pro-p group F is the cardinality of a minimal set of generators of F; equivalently $\mathrm{rk}(F) = \dim H^1(F)$, where dim always means dimension over $\mathbb{Z}/p\mathbb{Z}$. If S is a subgroup of G, then $N_G(S)$, resp. $C_G(S)$, will denote the normalizer, resp. the centralizer, of S in G.

As usual, $\mathrm{cd}_p(G)$ denotes the cohomological p-dimension of a profinite group G ([10], I.3). Recall that a pro-p group G is free pro-p if and only if $\mathrm{cd}_p(G) \leq 1$ (*loc. cit.* I.4.2, Cor. 2).

We say that a map $f\colon A \to \prod_{i\in I} A_i$ between sets is a *dense embedding* if f is injective and for every finite subset J of I the induced map $A \to \prod_{i\in J} A_i$ is surjective. In particular, if the index set I is finite this means that f is bijective.

The principal tool for our proof is the following result. Let G be a profinite group which has an open subgroup H such that $\mathrm{cd}_p(H) = d < \infty$.

2.1. Theorem. *If G contains no subgroup isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$, then the natural restriction map*

$$H^n(G, A) \longrightarrow \prod_{\substack{|S|=p \\ S \text{ mod conj.}}} H^n\big(N_G(S), A\big) \tag{1}$$

is a dense embedding for every $n > d$ and every finite discrete p-primary G-module A. Here the direct product is taken over a set of representatives S of the conjugacy classes of subgroups of G of order p.

This is Corollary 12.19 from [7]. One can remove the condition $\mathbb{Z}/p\mathbb{Z}\times\mathbb{Z}/p\mathbb{Z} \not\subset G$. Since the formulation becomes more technical then, and since we have no need for this greater generality here, we refer the interested reader to [8]. It should be remarked that there is a precise description of the image of (1); see [7], Theorem 12.17.

We will now simultaneously prove Theorems 1.1 and 1.2. Let G be a finitely generated pro-p group which has an open subgroup F of index p, which is a free pro-p group. It is clear that $r := \mathrm{rk}(F)$ is finite (cf. also [10], I.4.2, exerc. 4b). We will assume that G contains an element of order p, i.e. that the extension $1 \to F \to G \to G/F \to 1$ splits, since otherwise G is itself a free pro-p group [9]. Thus we get short exact sequences

$$0 \to H^n(G/F) \to H^n(G) \to H^{n-1}\big(G/F,\, H^1(F)\big) \to 0$$

($n \geq 1$) from the Hochschild-Serre spectral sequence. Since $\dim H^1(F) = r$, we see

$$\dim H^n(G) \;\leq\; 1 + r \tag{2}$$

for all n.

Applying Theorem 2.1 we find that the restriction map

$$H^n(G) \longrightarrow \prod_{\substack{|S|=p \\ S \text{ mod conj.}}} H^n\big(N_G(S)\big) \tag{3}$$

is a dense embedding for $n \geq 2$. If $S \subset G$ is a subgroup of order p, then $N_G(S) = C_G(S) = S \times C_F(S)$. So by the Künneth formula,

$$H^n\big(N_G(S)\big) \;=\; H^n(S) \otimes H^0\big(C_F(S)\big) \;\oplus\; H^{n-1}(S) \otimes H^1\big(C_F(S)\big).$$

This gives

$$\dim H^n\big(N_G(S)\big) = 1 + \operatorname{rk} C_F(S) \quad \text{for } n \geq 1. \tag{4}$$

Putting together (2), (3) and (4) we conclude

2.2. Lemma. *G has only finitely many conjugacy classes of subgroups of order p, and the map* (3) *is bijective for $n \geq 2$.*

In the following let $S_1, \dots, S_m$ be representatives of these conjugacy classes, and write $H_i := C_F(S_i)$ for their centralizers. We have

$$m + \sum_{i=1}^{m} \operatorname{rk}(H_i) = \dim H^n(G) \leq 1 + r \quad \text{for } n \geq 2.$$

In particular we see that $m \leq 1 + r$, and that $H_i = C_F(S_i)$ has rank at most r. (At this point an obvious induction argument already yields part of Theorem 1.2, namely that $\operatorname{rk}\operatorname{Fix}(S) \leq \operatorname{rk}(F)$.)

The homomorphism $G \to G/F \cong \mathbb{Z}/p\mathbb{Z}$ defines an element $\zeta \in H^1(G)$. Let $\beta\colon\ H^1(G) \to H^2(G)$ be the Bockstein, i.e. the connecting map coming from the extension $0 \to \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \to 0$ of discrete G-modules. By a theorem of Serre ([9], Prop. 5), cup-product $H^n(G) \to H^{n+2}(G)$ with $\beta(\zeta)$ is surjective for $n = 1$ (and bijective for $n \geq 2$).

Consider the commutative diagram

$$\begin{array}{ccc} H^1(G) & \longrightarrow & \bigoplus_i H^1(S_i \times H_i) \\ {\scriptstyle \cup\beta(\zeta)}\downarrow & & \downarrow{\scriptstyle \cup \operatorname{res}\beta(\zeta)} \\ H^3(G) & \xrightarrow{\sim} & \bigoplus_i H^3(S_i \times H_i) \end{array} \tag{5}$$

in which the horizontal arrows are restriction maps. The vertical arrows are surjective by Serre's result, the lower horizontal map is bijective by Lemma 2.2.

The right vertical map is in fact bijective. This follows already from the fact that it is surjective and both groups have the same finite order. More conceptually, one sees it by decomposing both groups à la Künneth and expressing the effect of the map in terms of these decompositions. In any case we can conclude:

2.3. Lemma. *The restriction maps*

$$\rho_G\colon\ H^1(G) \longrightarrow \bigoplus_i H^1(S_i \times H_i) \quad \textit{and} \quad \rho_F\colon\ H^1(F) \longrightarrow \bigoplus_i H^1(H_i)$$

are surjective.

Indeed, the first assertion follows from (5), the second from the first and from the commutative diagram

$$\begin{array}{ccc} H^1(G) & \longrightarrow & H^1(F) \\ {\scriptstyle \rho_G}\downarrow & & \downarrow{\scriptstyle \rho_F} \\ \bigoplus_i H^1(S_i \times H_i) & \longrightarrow & \bigoplus_i H^1(H_i) \end{array}$$

in which all maps are restriction maps. Note that the lower horizontal map is surjective.

The following lemma is obvious:

2.4. Lemma. *Let A be a subgroup of $H^1(F)$. Then there exists a closed subgroup K of F such that the restriction $H^1(F) \to H^1(K)$ maps A isomorphically onto $H^1(K)$.*

The kernel of the restriction map $H^1(G) \to H^1(F)$ is generated by ζ. Since $\rho_G(\zeta) \neq 0$, this restriction maps $\ker(\rho_G)$ injectively into $\ker(\rho_F)$. Therefore, using Lemma 2.4, we find closed subgroups H and K of F such that

(a) res: $H^1(G) \to H^1(H)$ restricts to an isomorphism $\ker(\rho_G) \to H^1(H)$, and
(b) res: $H^1(F) \to H^1(K)$ restricts to an isomorphism $\ker(\rho_F) \to H^1(K)$.

This means that the two restriction maps $H^1(G) \to H^1(H) \oplus \bigoplus_i H^1(S_i \times H_i)$ and $H^1(F) \to H^1(K) \oplus \bigoplus_i H^1(H_i)$ are bijective. Therefore each of the natural homomorphisms

$$H * (S_1 \times H_1) * \cdots * (S_m \times H_m) \to G \quad \text{and} \quad K * H_1 * \cdots * H_m \to F \tag{6}$$

induces a surjection between the respective H^1-groups (with coefficients $\mathbb{Z}/p\mathbb{Z}$). Since the induced maps between the H^2-groups are bijective as well (cf. Lemma 2.2 for the first case), we conclude that the maps (6) are bijective, by the following well known

2.5. Lemma. *Let $f\colon H \to G$ be a homomorphism of pro-p groups. If $H^1(G) \to H^1(H)$ is bijective and $H^2(G) \to H^2(H)$ is injective, then f is an isomorphism.* (*Compare* [6]*, Satz* 4.3.)

This proves Theorem 1.1, and also Theorem 1.2 in the case where S is cyclic of order p (apply 1.1 to the semi-direct product of F with S). To establish the general case of 1.2, choose a series $1 = S_0 \subset S_1 \subset \cdots \subset S_r = S$ of normal subgroups of S with S_i of order p^i. From what we know it follows that $\mathrm{Fix}(S_{i+1})$ is a free factor of $\mathrm{Fix}(S_i)$ for $0 \leq i < r$. By induction, therefore, $\mathrm{Fix}(S)$ is a free factor of F.

We conclude by proving Corollary 1.3. For b) it suffices to treat the centralizer. Let F be an open normal subgroup of G which is free pro-p. Then $C_F(S)$ has finite rank ($\leq \mathrm{rk}(F)$) by Theorem 1.2.

In order to prove a), it suffices to show: G has up to conjugation only finitely many subgroups of order p^n, for every $n \geq 1$. The case $n = 1$ follows easily from Lemma 2.2. To do the induction step from p^{n-1} to p^n, it suffices to show the following. Fix a subgroup T of G of order p^{n-1}. Then the subgroups S of G of order p^n which contain T lie in finitely many G-conjugacy classes.

The last assertion in turn follows easily from b). Indeed, every such S is contained in $N_G(T)$, so it will be sufficient to show that $N_G(T)/T$ has only finitely many elements of order p, up to conjugation. Since by b) the last group is virtually free of finite rank, we are done by the case $n = 1$ already considered.

References

[1] J. L. Dyer, G. P. Scott, *Periodic automorphisms of free groups*, Comm. Algebra **3**, 195-201 (1975). MR **51:**5762

[2] D. Gildenhuys, L. Ribes, *Profinite groups and boolean graphs*, J. Pure Appl. Algebra **12**, 21-47 (1978). MR **81g:**20059

[3] W. N. Herfort, L. Ribes, *On automorphisms of free pro-p-groups I*, Proc. Am. Math. Soc. **108**, 287-295 (1990). MR **90d:**20048

[4] W. N. Herfort, L. Ribes, P. A. Zalesskii, *Fixed points of automorphisms of free pro-p groups of rank 2*, Canad. J. Math. **47**, 383-404 (1995). MR **96k:**20053

[5] W. Herfort, L. Ribes, P. A. Zalesskii, *Finite extensions of free pro-p groups of rank at most two*, Preprint 1996, to appear in Israel J. Math.

[6] J. Neukirch, *Freie Produkte pro-endlicher Gruppen und ihre Kohomologie*, Arch. Math. **22**, 337-357 (1971). MR **50:**490

[7] C. Scheiderer, *Real and Étale Cohomology*, Lect. Notes Math. **1588**, Springer, Berlin 1994. MR **96c:**14018

[8] C. Scheiderer, *Farrell cohomology and Brown theorems for profinite groups*, Manuscr. math. **91**, 247-281 (1996). MR **97j:**20050

[9] J.-P. Serre, *Sur la dimension cohomologique des groupes profinis*, Topology **3**, 413-420 (1965). MR **31:**4853

[10] J.-P. Serre, *Cohomologie Galoisienne*, Cinquième édition. Lect. Notes Math. **5**, Springer, Berlin, 1994. MR **96b:**12010

[11] P. A. Zalesskiĭ, O. V. Mel'nikov, *Subgroups of profinite groups acting on trees*, Math. USSR Sbornik **63**, 405-424 (1989) (English translation). MR **90f:**20041

Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany
E-mail address: `claus.scheiderer@mathematik.uni-regensburg.de`