

NONTORIC HAMILTONIAN CIRCLE ACTIONS ON FOUR-DIMENSIONAL SYMPLECTIC ORBIFOLDS

S. F. SINGER, J. TALVACCHIA, AND N. WATSON

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ABSTRACT. We construct four-dimensional symplectic orbifolds admitting Hamiltonian circle actions with isolated fixed points, but not admitting any Hamiltonian action of a two-torus. One example is linear, and one example is compact.

In this note we construct four-dimensional symplectic orbifolds admitting Hamiltonian circle actions with isolated fixed points, but not admitting any Hamiltonian action of a two-torus. This work is part of our ongoing attempt to generalize to orbifolds the classification of four-dimensional compact symplectic manifolds with Hamiltonian circle actions due to Ahara and Hattori, Audin and Karshon ([AH], [Au], [K]) and is part of a larger program to understand torus actions of complexity one on orbifolds. The complexity of a Hamiltonian torus action on an orbifold is defined to be the difference between half of the dimension of the orbifold and the dimension of the torus. The complexity zero case (i.e., n -torus actions on $2n$ -dimensional orbifolds) has been analyzed by Lerman and Tolman [LT].

One of Karshon's theorems states that if a Hamiltonian circle action on a compact four-dimensional manifold has isolated fixed points, then the action can be extended to the action of a two-torus. It follows that if a compact symplectic four-dimensional manifold has a Hamiltonian circle action with isolated fixed points, then it is a toric variety. One of our orbifolds is compact, proving that Karshon's result does not generalize to orbifolds. Our result is slightly stronger: not only do the circle actions fail to extend, but the underlying spaces admit no Hamiltonian two-torus action whatsoever.

In the first section we consider a compact orbifold that is the quotient of a non-singular four-dimensional toric variety by a finite group. Specifically, we consider $\mathbf{CP}^1 \times \mathbf{CP}^1$ modulo an action of the group $\mathbf{Z}/4\mathbf{Z}$. The natural Hamiltonian action of the two-torus on $\mathbf{CP}^1 \times \mathbf{CP}^1$ does not descend to the quotient; however, the action of one circle subgroup does descend, giving the required circle action.

In the second section we consider the vector orbifold \mathbf{C}^2/Γ where Γ is a finite subgroup of $\mathbf{U}(2)$ that is not abelian. Here the diagonal action of \mathbf{S}^1 on \mathbf{C}^2 descends to the orbifold but, due to the nature of the group Γ , there is no possible two-torus

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action on the orbifold. Unlike symplectic manifolds, where the only obstructions to large torus actions are global, symplectic orbifolds can have local obstructions.

1. A COMPACT EXAMPLE: $\mathbf{CP}^1 \times \mathbf{CP}^1/\Gamma$

Consider the real four-dimensional manifold $\mathbf{CP}^1 \times \mathbf{CP}^1$. We denote an arbitrary point on this manifold by (x, y) , with $x, y \in \mathbf{CP}^1$. This manifold has a standard symplectic form given by

$$\omega = \frac{i}{2} \left(\frac{dx \wedge d\bar{x}}{(1 + |x|^2)^2} + \frac{dy \wedge d\bar{y}}{(1 + |y|^2)^2} \right).$$

Let Γ denote the group of symplectomorphisms generated by $(x, y) \mapsto (-y, x)$. Note that $\Gamma = \mathbf{Z}/4\mathbf{Z}$. Then $\mathbf{CP}^1 \times \mathbf{CP}^1/\Gamma$ is a symplectic orbifold. We denote a point of this orbifold by $[x, y]$, the Γ -equivalence class of (x, y) . The only singular points of this orbifold are $[0, 0]$, $[\infty, \infty]$ and $[0, \infty]$, and these have cyclic orbifold structure groups of orders 4, 4 and 2, respectively.

We can define a Hamiltonian circle action on $\mathbf{CP}^1 \times \mathbf{CP}^1/\Gamma$. The function

$$\phi([x, y]) = \frac{1}{4} \left(\frac{|x|^2}{(1 + |x|^2)} + \frac{|y|^2}{(1 + |y|^2)} \right)$$

is well-defined on $\mathbf{CP}^1 \times \mathbf{CP}^1/\Gamma$, and its Hamiltonian flow is given by $[e^{it/2}x, e^{it/2}y]$. Notice that the minimum period of this flow is 2π . The fixed points are $[0, 0]$, $[\infty, \infty]$ and $[0, \infty]$. The image of ϕ is $[0, \frac{1}{2}]$. The function ϕ is a Morse function. One can apply the results of [LT], section 4.1, to see that the associated Morse polynomial is $1 + \chi^2 + \chi^4$, that this polynomial must equal the Poincaré polynomial, and hence that $\dim H^2 = 1$, i.e., the second cohomology is one-dimensional.

It remains to show that this orbifold admits no effective Hamiltonian action of a two-torus. We argue by contradiction. Suppose

$$\Phi : \mathbf{CP}^1 \times \mathbf{CP}^1/\Gamma \rightarrow \mathbf{R}^2$$

is the Hamiltonian for an effective two-torus action. Let Δ be the image of Φ . Then Δ is a convex polytope. By applying Bott–Morse theory to a generic one-dimensional projection of the moment map Φ and comparing with the Poincaré polynomial, we know that the number of vertices of Δ is equal to $\dim H^2 + 2 = 3$. Hence Δ is a triangle. Because $\mathbf{CP}^1 \times \mathbf{CP}^1/\Gamma$ has only three singular points, of orders 4, 4 and 2, it follows from the normal form theorem for group actions on orbifolds, in section 3.5 of [LT], that these points must correspond to the vertices of the triangle. Without loss of generality (by choosing an automorphism of the two-torus) we may assume that the edge between the two vertices of order 4 is parallel to the vector $(1, 0)$. Let (a, b) and (c, d) be primitive integer vectors parallel to the other two sides.

Because the order of an isolated singular point in a toric variety is given by the determinant of the matrix of primitive integer vectors parallel to edges emanating from the corresponding vertex of the associated polytope, we find that

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = \pm 2, \quad b = \pm 4 \quad \text{and} \quad d = \pm 4.$$

But these three equations cannot be simultaneously satisfied by integers, as $\begin{vmatrix} a & c \\ 4 & 4 \end{vmatrix}$ must be divisible by 4 if a and c are integers. Contradiction! Hence $\mathbf{CP}^1 \times \mathbf{CP}^1/\Gamma$ admits no effective action of a two-torus.

We remark that this orbifold has a Kaehler structure inherited from $\mathbf{CP}^1 \times \mathbf{CP}^1$. While Karshon has shown that any compact symplectic four-dimensional manifold with a Hamiltonian circle action (with no assumptions on the fixed-point sets) must be Kaehler [K] and Tolman has constructed a six-dimensional, non-Kaehler compact symplectic manifold with a Hamiltonian action of a two-torus [T], it is unknown whether every compact symplectic four-dimensional orbifold with a Hamiltonian circle action must be Kaehler.

2. A VECTOR ORBI-SPACE EXAMPLE: \mathbf{C}^2/Γ

Let Γ be a finite subgroup of $\mathbf{U}(2)$ that is not abelian. (For example, let Γ be a dihedral group.) Consider \mathbf{C}^2 with coordinates (z_1, z_2) and symplectic form

$$\omega = \frac{i}{2}(dz_1 \wedge d\bar{z}_1 + dz_2 \wedge d\bar{z}_2).$$

The diagonal Hamiltonian action of \mathbf{S}^1 on (\mathbf{C}^2, ω) given by

$$(e^{i\theta}, (z_1, z_2)) \rightarrow (e^{i\theta} z_1, e^{i\theta} z_2)$$

with moment map

$$\phi(z_1, z_2) = \frac{1}{4}(|z_1|^2 + |z_2|^2)$$

descends to \mathbf{C}^2/Γ . There is one singular point of the orbifold at $[(0, 0)]$ which is also an isolated fixed point of the \mathbf{S}^1 action. The orbifold structure group of $[(0, 0)]$ is Γ . By Lemma 6.2 in [LT], however, there can be no faithful Hamiltonian two-torus action on \mathbf{C}^2/Γ . The image of the moment map Φ for any such two-torus action would have to be a simplicial cone containing $\Phi([(0, 0)])$ as a vertex, and the orbifold structure group at $[(0, 0)]$ would have to be a quotient of lattices, and hence abelian. But Γ is not abelian; hence the vector orbi-space admits no Hamiltonian two-torus action.

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DEPARTMENT OF MATHEMATICS, HAVERFORD COLLEGE, HAVERFORD, PENNSYLVANIA 19041
E-mail address: ssinger@haverford.edu

DEPARTMENT OF MATHEMATICS, SWARTHMORE COLLEGE, SWARTHMORE, PENNSYLVANIA 19081
E-mail address: jtalvac1@swarthmore.edu

DEPARTMENT OF MATHEMATICS, THE WHARTON SCHOOL, UNIVERSITY OF PENNSYLVANIA,
PHILADELPHIA, PENNSYLVANIA 19104