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RELATIVE BRAUER GROUPS OF DISCRETE VALUED FIELDS

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ABSTRACT. Let E be a non-trivial finite Galois extension of a field K. In this paper we investigate the role that valuation-theoretic properties of E/K play in determining the non-triviality of the relative Brauer group, $\mathrm{Br}(E/K)$, of E over K. In particular, we show that when K is finitely generated of transcendence degree 1 over a p-adic field k and q is a prime dividing $[E : K]$, then the following conditions are equivalent: (i) the q-primary component, $\mathrm{Br}(E/K)_q$, is non-trivial, (ii) $\mathrm{Br}(E/K)_q$ is infinite, and (iii) there exists a valuation π of E trivial on k such that q divides the order of the decomposition group of E/K at π.

1. Introduction and preliminaries

Let K be a field finitely generated of transcendence degree $r \geq 1$ over a field k, let E be a non-trivial finite Galois extension of K, and let q be a prime dividing $[E : K]$. If k is Hilbertian or if k is arbitrary and $r \geq 2$, the proof of [FS1, Corollary 5] shows that the q-primary component of the relative Brauer group of E over K, $\mathrm{Br}(E/K)_q$, must be infinite. On the other hand, Roquette [R, Corollary XVIa] has given examples where $\mathrm{Br}(E/K) = \{0\}$ when k is a p-adic field, $r = 1$, K has genus 1 over k, and E is a non-trivial cyclic extension of K. In Roquette's example, every valuation of K trivial on k splits completely in E. In contrast, the proofs of the positive results cited above make use of the fact that there are infinitely many valuations π of E trivial on k such that q divides the order of the decomposition group of E/K at π. In this paper we investigate, for an arbitrary field K and an arbitrary non-trivial finite Galois extension E of K, the role that valuation-theoretic properties of E/K play in determining the non-triviality of $\mathrm{Br}(E/K)$. Among other results, we show that when K is finitely generated of transcendence degree 1 over a p-adic field k and E is a non-trivial finite Galois extension of K, then $\mathrm{Br}(E/K)_q$ is non-trivial if and only if it is infinite and this occurs if and only if there exists a valuation π of E trivial on k such that q divides the order of the decomposition group of E/K at π.

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We now establish some of the terminology and notation that we will maintain throughout. We say that L/F is an $\mathcal{H}$-Galois extension of fields if L is a Galois extension of a field F with Galois group $\mathrm{Gal}(L/F) = \mathcal{H}$. In what follows, K will always be a field with one or more discrete (rank one) valuations, E will be a non-trivial $\mathcal{G}$-Galois extension of K, and q will be a prime dividing $[E : K]$. Let π be a discrete valuation of E. The completion of E at π will be denoted by E_π and the residue field by $\overline{E}_\pi$; when π is understood we sometimes denote $\overline{E}_\pi$ by $\overline{E}$. An element $\hat{b}$ of E_π generating the maximal ideal of the valuation ring of E_π will be referred to as a uniformizing element for π. We denote the restriction of π to K by π_K; if no misunderstanding is possible, we sometimes write π instead of π_K. We denote the decomposition group of E/K at π by $\mathcal{G}_\pi$; with the above notation and conventions, E_π/K_π is $\mathcal{G}_\pi$-Galois. We denote the ramification degree of E_π over K_π by $e(E_\pi/K_\pi)$. By a p-adic field we mean a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers.

We say that A/F is central simple if A is a simple algebra with center F which is finite dimensional over F; we denote the class of such an A in $\mathrm{Br}(F)$ by $[A]$. The exponent of $[A]$, $\exp(A)$, is the order of $[A]$ in $\mathrm{Br}(K)$. If L is a field extension of F, we refer to the map from $\mathrm{Br}(F)$ to $\mathrm{Br}(L)$ defined by $[A] \to [A \otimes_F L]$ as restriction and denote it by res_F^L. If F/K is a finite separable extension of fields, we denote the corestriction map from $\mathrm{Br}(F)$ to $\mathrm{Br}(K)$ by cor_K^F.

Suppose that $K \subseteq F \subseteq E$ and $\mathrm{Gal}(E/F) = \mathcal{H} \subseteq \mathcal{G} = \mathrm{Gal}(E/K)$. Then $\mathrm{Br}(E/F) \cong H^2(\mathcal{H}, E^*)$, $\mathrm{Br}(E/K) \cong H^2(\mathcal{G}, E^*)$, and cor_K^F restricted to $\mathrm{Br}(E/F)$ corresponds to the cohomological corestriction map from $H^2(\mathcal{H}, E^*)$ to $H^2(\mathcal{G}, E^*)$. In particular, if $[A]$ is in the q-primary component, $\mathrm{Br}(E/F)_q$, of $\mathrm{Br}(E/F)$, then $\mathrm{cor}_K^F([A]) \in \mathrm{Br}(E/K)_q$. All of our non-triviality results for relative Brauer groups are obtained by showing that there exists suitable F and A as above such that $\mathrm{cor}_K^F([A]) \neq 0$. The algebras A that we will need will be cyclic algebras; we next review their construction.

Suppose E/F is a possibly infinite $\mathcal{H}$-Galois extension of fields. We let the profinite group $\mathcal{H}$ act trivially on the discrete group $\mathbb{Q}/\mathbb{Z}$ and define the character group of E/F, $\chi(E/F)$, to be $H^1(\mathcal{H}, \mathbb{Q}/\mathbb{Z})$, the group of continuous homomorphisms of $\mathcal{H}$ into $\mathbb{Q}/\mathbb{Z}$. If $f \in \chi(E/F)$, then $f(\mathcal{H})$ is a finite subgroup of $\mathbb{Q}/\mathbb{Z}$ and hence is cyclic. Let $L \subseteq E$ be the fixed field of the kernel of f. Then L/F is cyclic and we say that the character f defines L/F. The short exact sequence of trivial $\mathcal{H}$-modules $0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$ gives rise to a long exact sequence of cohomology groups. Since $H^1(\mathcal{H}, \mathbb{Q}) = H^2(\mathcal{H}, \mathbb{Q}) = 0$, we have an isomorphism $d : H^2(\mathcal{H}, \mathbb{Z}) \longrightarrow H^1(\mathcal{H}, \mathbb{Q}/\mathbb{Z}) = \chi(E/F)$. The cup product (e.g. [B, p. 112]) defines a map: $H^2(\mathcal{H}, \mathbb{Z}) \times H^0(\mathcal{H}, E^*) \longrightarrow H^2(\mathcal{H}, E^*) \cong \mathrm{Br}(E/F)$. Since $H^0(\mathcal{H}, E^*) \cong F^*$ and $H^2(\mathcal{H}, \mathbb{Z}) \cong \chi(E/F)$, we have a map $\chi(E/F) \times F^* \longrightarrow \mathrm{Br}(E/F)$. We denote the image of (f, b) (where $f \in \chi(E/F)$ and $b \in F^*$) under this map by $\Delta(f, b)$. In algebra terms, $\Delta(f, b)$ is represented by the cyclic crossed product $\Delta(L/F, \sigma, b)$ where f defines L/F and where $\sigma \in \mathrm{Gal}(E/F)$ is the generator satisfying $f(\sigma) = (1/[L : F]) + \mathbb{Z}$.

We shall show that $\mathrm{cor}_K^F([A]) \neq 0$ for suitable A and F as above by showing that $\mathrm{res}_K^{K_\pi} \mathrm{cor}_K^F([A]) \neq 0$ for a suitably chosen discrete valuation π of K. In what follows, we shall freely use standard results about the Brauer group of a complete discrete valued field. We refer the reader to [Se, Chapter XII, Section 3] for a complete discussion of the Brauer group of such a field; we briefly summarize the results we will need.

Let K be a field complete with respect to a discrete valuation, let T denote the maximal unramified extension of K, and let b be a fixed uniformizing element for K. Then $\chi(T/K) \cong \chi(\overline{T}/\overline{K})$; we denote the image of $f \in \chi(T/K)$ under this isomorphism by $\bar{f}$. Now suppose that L/K is a cyclic unramified extension of K of degree n and let $f \in \chi(T/K)$ define L/K. Since the value group of L equals the value group of K and the values of conjugates of an element are equal, n is the smallest positive power of b which is a norm from L to K. In particular, $\Delta(f,b)$ has exponent n in $\mathrm{Br}(K)$. Let q be a prime not dividing the characteristic of $\overline{K}$. Then $\mathrm{Br}(K)_q$ is the direct sum of the inertial lift (in the sense of [JW, Theorem 2.8]) of $\mathrm{Br}(\overline{K})_q$ and a subgroup isomorphic to $\chi(\overline{T}/\overline{K})_q$ where the isomorphism is given by $\bar{f} \to \Delta(f,b)$ for $\bar{f} \in \chi(\overline{T}/\overline{K})_q$ [Se, Theorem 2, p. 194]. We let *ram* denote the projection of $\mathrm{Br}(K)_q$ onto $\chi(\overline{T}/\overline{K})_q$ defined by this direct sum decomposition of $\mathrm{Br}(K)_q$. In particular, $ram(\Delta(f,b)) = \bar{f}$.

The following approximation result is basic to our approach; a slightly weaker version appears implicitly in [FSS2, Lemma 4.7] (see also [FS2, Lemma 4]).

Proposition 1. *Let $K \subseteq F \subset L \subseteq E$ where L/F is cyclic of degree q, defined by $f \in \chi(E/F)$. Assume that π is a discrete valuation of E such that $[L_\pi : F_\pi] = [L : F]$. Identifying $\mathrm{Gal}(E_\pi/F_\pi)$ as a subgroup of $\mathrm{Gal}(E/F)$, we may view f as an element of $\chi(E_\pi/F_\pi)$ defining L_π/F_π.*

(1) *Let $\hat{b} \in F_\pi^*$. Then there exists $b \in F^*$ such that $\mathrm{res}_K^{K_\pi} \mathrm{cor}_K^F(\Delta(f,b)) = \mathrm{cor}_{K_\pi}^{F_\pi}(\Delta(f,\hat{b}))$.*

(2) *Assume that there exists $\hat{b} \in F_\pi^*$ such that $\mathrm{cor}_{K_\pi}^{F_\pi}(\Delta(f,\hat{b}))$ has exponent q in $\mathrm{Br}(K_\pi)$. Then $\mathrm{Br}(E/K)_q \neq \{0\}$.*

(3) *If $F_\pi = K_\pi$ and L_π/K_π is unramified, then $\mathrm{Br}(E/K)_q \neq \{0\}$.*

(4) *Assume that there exist infinitely many discrete valuations π of E such that the hypotheses of (2) or (3) above hold. Then $\mathrm{Br}(E/K)_q$ is infinite.*

Proof. (1) We argue exactly as in the proof of [FSS2, Lemma 4.7] (see also [FS2, Lemma 4]) but with one minor modification. We need to choose b sufficiently close to $\hat{b}$ π-adically so that $b\hat{b}^{-1}$ is a norm from L_π to F_π; since $b\hat{b}^{-1} = 1+\hat{b}^{-1}(b-\hat{b})$, this is possible by [Se, Chapter 5, Section 2, Proposition 3, and Section 6, Corollary 4]. For b so chosen, $\Delta(f, b\hat{b}^{-1}) = 0$ in $\mathrm{Br}(F_\pi)$ and so $\mathrm{cor}_{K_\pi}^{F_\pi}(\Delta(f,\hat{b})) = \mathrm{cor}_{K_\pi}^{F_\pi}(\Delta(f,b))$. The proof of [FSS2, Lemma 4.7] now continues without incident.

(2) Assume that there exists $\hat{b} \in F_\pi^*$ such that $\mathrm{cor}_{K_\pi}^{F_\pi}(\Delta(f,\hat{b}))$ has exponent q in $\mathrm{Br}(K_\pi)$. Let b be as in (1). Then $\mathrm{res}_K^{K_\pi} \mathrm{cor}_K^F(\Delta(f,b))$ has exponent q. Since restriction is a homomorphism and $\Delta(f,b)$ has exponent dividing q, $\mathrm{cor}_K^F(\Delta(f,b))$ has exponent q. As noted above, $\mathrm{cor}_K^F(\Delta(f,b))$ is also an element of $\mathrm{Br}(E/K)$. Thus $\mathrm{Br}(E/K)_q \neq \{0\}$.

(3) Assume that $F_\pi = K_\pi$ and L_π/K_π is unramified. Choose $\hat{b}$ a uniformizing element for K_π. Then $\Delta(f,\hat{b})$ has exponent q. Since $F_\pi = K_\pi$, (3) follows from (2).

(4) Since the hypotheses of (3) imply the hypotheses for (2), we may assume that there exist infinitely many discrete valuations π_i of E and $\hat{b}_i \in F_{\pi_i}^*$ such that $\mathrm{cor}_{K_{\pi_i}}^{F_{\pi_i}}(\Delta(f,\hat{b}_i))$ has exponent q in $\mathrm{Br}(K_{\pi_i})$. Suppose that $|\mathrm{Br}(E/K)_q| < n$ for some positive integer n. For $1 \leq i \leq n$, we choose $b_i \in F^*$ such that $\mathrm{res}_K^{K_{\pi_i}} \mathrm{cor}_K^F(\Delta(f,b_i)) = \mathrm{cor}_{K_{\pi_i}}^{F_{\pi_i}}(\Delta(f,\hat{b}_i))$ as in the proof of (1) above but with the additional restriction that b_i is a norm from L_γ to F_γ for all valuations γ of L such that γ restricted to K equals π_j restricted to K for some $1 \leq j \leq n$, $j \neq i$.

Let $A_{ij} = \operatorname{res}_K^{K_{\pi_j}} \operatorname{cor}_K^F(\Delta(f, b_i))$. Then $A_{ij} = 0$ for $i \neq j$ and A_{ii} has exponent q. It follows that $\{\operatorname{cor}_K^F(\Delta(f, b_1)), \dots, \operatorname{cor}_K^F(\Delta(f, b_n))\}$ is a set of n distinct elements in $\operatorname{Br}(E/K)_q$, contradicting the assumption that $|\operatorname{Br}(E/K)_q| < n$. This proves (4). □

Remark. Suppose that K is either a global field or a regular extension of transcendence degree ≥ 1 of a Hilbertian field (e.g. a regular extension of transcendence degree ≥ 2 of an arbitrary field) and E is a finite Galois extension of K, $E \neq K$. Let q be a prime dividing $[E : K]$ and let $\sigma \in \operatorname{Gal}(E/K)$ have order q. By the Tchebotarev Density Theorem if K is a global field [T, p. 163] or by the proof of [FSS1, Theorem 2.6] (see also [FS2, Proposition 3]) if K is a regular extension of transcendence degree at least 1 of a Hilbertian field, there exist infinitely many discrete valuations π of E such that $\langle\sigma\rangle$ is the decomposition group of E/K at π. It follows from Proposition 1, (4) that $\operatorname{Br}(E/K)_q$ is infinite. Thus Proposition 1 provides another approach to the results of [FKS] and [FS1], at least for Galois extensions E over K.

2. Relative Brauer groups of function fields over p-adic fields

Let k be a p-adic field, let K be a finitely generated extension of k of transcendence degree $r \geq 1$, let E be a non-trivial finite Galois extension of K, and let q be a prime dividing $[E : K]$. The proof of [FS1, Corollary 5] shows that if $r \geq 2$, then $\operatorname{Br}(E/K)_q$ is infinite. In this section we consider the case when $r = 1$. In view of Proposition 1, it is natural to begin by considering complete fields having p-adic residue fields.

Lemma 2. *Let K be a field complete with respect to a discrete valuation having a p-adic residue field, let E be a non-trivial finite Galois extension of K, and let q be a prime dividing $[E : K]$. Assume that there exists an unramified cyclic extension L/F of degree q with $K \subseteq F \subset L \subseteq E$ such that $q \nmid e(F/K)$. Let $f \in \chi(E/F)$ define L/F. Then there exists $b \in F^*$ such that $\operatorname{cor}_K^F(\Delta(f, b))$ has exponent q in $\operatorname{Br}(E/K)$.*

Proof. Let $\bar{f} \in \chi(\overline{E}/\overline{F})$ be the character defining $\overline{L}/\overline{F}$ corresponding to f. Let v be a uniformizing element for $\overline{F}$. Since $\overline{F}$ is complete, $\Delta(\bar{f}, v)$ has exponent q. Let $\Delta(f, b)$ be the inertial lift of $\Delta(\bar{f}, v)$ in the sense of [JW, Theorem 2.8]. $\operatorname{cor}_K^F(\Delta, b)$ is similar to the inertial lift of the division algebra component of $e(F/K) \cdot \operatorname{cor}_{\overline{K}}^{\overline{F}}(\Delta(\bar{f}, v))$ [Hw, Theorem 17]. Since the corestriction map from $\operatorname{Br}(\overline{F})$ to $\operatorname{Br}(\overline{K})$ is injective because $\overline{K}$ is a p-adic field [Se, p. 175] and since $q \nmid e(F/K)$, $\operatorname{cor}_{\overline{K}}^{\overline{F}}(\Delta(\bar{f}, v))$ has exponent q. Thus $\operatorname{cor}_K^F(\Delta, b)$ has exponent q in $\operatorname{Br}(E/K)$. □

Theorem 3. *Let K be a finitely generated field extension of transcendence degree 1 of a p-adic field k, let E/K be a finite Galois extension of K, and let q be a prime dividing $[E : K]$. Then the following are equivalent:*

(1) $\operatorname{Br}(E/K)_q \neq \{0\}$.
(2) $\operatorname{Br}(E/K)_q$ *is infinite.*
(3) *There exists a valuation π of E trivial on k such that q divides the order of the decomposition group of E/K at π.*

Proof. We will prove the circle of implications: (2) ⇒ (1), (1) ⇒ (3), and (3) ⇒ (2). The implication (2) ⇒ (1) is clear. Assume next that (1) holds so there exists

$\alpha \in \mathrm{Br}(E/K)$ of order q. Let $\mathbb{P}(K)$ denote the set of inequivalent valuations of K trivial on k. There is a natural map $\mathrm{Br}(K) \longrightarrow \prod_{\gamma\in\mathbb{P}(K)} \mathrm{Br}(K_\gamma)$ where $[D] \in \mathrm{Br}(K)$ maps to $(\dots, [D\otimes_K K_\gamma], \dots) \in \prod_{\gamma\in\mathbb{P}(K)} \mathrm{Br}(K_\gamma)$ and a corresponding map $\mathrm{Br}(E) \longrightarrow \prod_{\delta\in\mathbb{P}(E)} \mathrm{Br}(E_\delta)$. Similarly, there is a natural map $\prod_{\gamma\in\mathbb{P}(K)} \mathrm{Br}(K_\gamma) \longrightarrow \prod_{\delta\in\mathbb{P}(E)} \mathrm{Br}(E_\delta)$ induced by the restriction maps from $\mathrm{Br}(K_\gamma)$ to $\mathrm{Br}(E_\delta)$ for each $\gamma \in \mathbb{P}(K)$ and each $\delta \in \mathbb{P}(E)$ extending γ. Combining these natural maps with the restriction map from $\mathrm{Br}(K)$ to $\mathrm{Br}(E)$ leads to a commutative diagram:

$$\begin{array}{ccc} \mathrm{Br}(E) & \longrightarrow & \prod_{\delta\in\mathbb{P}(E)} \mathrm{Br}(E_\delta) \\ \uparrow & & \uparrow \\ \mathrm{Br}(K) & \longrightarrow & \prod_{\gamma\in\mathbb{P}(K)} \mathrm{Br}(K_\gamma) \end{array}$$

Both horizontal maps are monomorphisms by a result due implicitly to Lichtenbaum (see the proof of Theorem 5 of [Li]) and explicitly stated in [Po, Theorem 4.1]. Let D be the underlying division algebra of α. Since α has order q in $\mathrm{Br}(K)$, there exists a valuation π of K trivial on k such that the Brauer class of $D \otimes_K K_\pi$ has order q in $\mathrm{Br}(K_\pi)$. Since $\alpha \in \mathrm{Br}(E/K)$, the image of α in $\prod_{\delta\in\mathbb{P}(E)} \mathrm{Br}(E_\delta)$ is trivial. Let δ be any valuation of E extending π. Since E_δ splits $D \otimes_K K_\pi$, q divides $[E_\delta : K_\pi]$ [Pi, Proposition 14.4b(ii) and Lemma 13.4]. Thus (3) holds.

It remains to show that (3) implies (2). Assume that there exists a valuation π of E trivial on k such that q divides the order of the decomposition group of E/K at π. Let $\mathcal{G}_\pi \subseteq \mathcal{G}$ be the decomposition group of π over K and let $\mathcal{H}_\pi$ denote a Sylow q-subgroup of $\mathcal{G}_\pi$. Since $q|\,|\mathcal{G}_\pi|$, $\mathcal{H}_\pi \neq \{1\}$. Let $\mathcal{J}_\pi$ denote a subgroup of $\mathcal{H}_\pi$ of index q, let F be the fixed field of $\mathcal{H}_\pi$ and let L be the fixed field of $\mathcal{J}_\pi$. Then L is a cyclic extension of F of degree q and $\pi|L$ is a valuation of L whose decomposition group over F is non-trivial. In particular, there are valuations of F trivial on k not splitting completely in L. By [Sa, Theorem 7.1] there are infinitely many valuations of F trivial on k which are unramified in L of residue class degree q. Since only finitely many valuations of K ramify in E, there are infinitely many valuations δ of E trivial on k which are unramified over K and with $[L_\delta : F_\delta] = q$. By Lemma 2 and Proposition 1, (4), $\mathrm{Br}(E/K)_q$ is infinite. □

As an immediate consequence of Theorem 3, we have:

Corollary 4. *Let K be a finitely generated extension of transcendence degree one of a p-adic field k and let E be a non-trivial finite Galois extension of K. Then the following statements are equivalent:*

(1) $\mathrm{Br}(E/K) \neq \{0\}$.
(2) $\mathrm{Br}(E/K)$ *is infinite.*
(3) *There exists a valuation of K trivial on k not splitting completely in E.*

Corollary 4 explains the example of Roquette [R, Corollary XVIa] mentioned in the Introduction since in Roquette's example every valuation of K trivial on k splits completely in E. In general, for K as in Corollary 4, there will exist extensions E of K such that every valuation of K trivial on k splits completely in E when the smooth projective variety over k with function field K has bad reduction to the residue field of k. Such examples seem to exist for any genus ≥ 1 [Sa, Examples 2.7

and 7.2] but not if K/k has genus 0 [Sa, page 73 and Theorem 7.1]. We refer the reader to [Sa] for a thorough discussion of the existence of these completely split covers.

Corollary 5. *Let k, K, and E be as in Corollary 4 and suppose that K/k has genus 0. Then* $\mathrm{Br}(E/K)$ *is infinite.*

Proof. Let k_1 denote the field of constants of K and let $K_1 = Kk_1$. Since K_1/k_1 has genus 0, there are no completely split covers of K_1 (see the references to [Sa] above) and so there exists a valuation π_1 of K_1 trivial on k_1 not splitting completely in E. Let π denote the restriction of π_1 to K. Then π does not split completely in E and so $\mathrm{Br}(E/K)$ is infinite by Corollary 4. □

3. The ramified case

Let K be a field with a discrete valuation π and let E be a finite Galois extension of K in which π does not split completely. Then there exist L and F with $K \subseteq F \subset L \subseteq E$ where L/F is cyclic and some extension δ of π to F is undecomposed in L. If L/F is unramified at δ, Proposition 1, (3), can sometimes be used to conclude that $\mathrm{Br}(E/K)$ is non-trivial. In this section we consider what can be said if L/F is ramified at δ. As in the preceding section, we begin with a result about complete fields.

Lemma 6. *Let K be a field complete with respect to a discrete valuation, let E be a non-trivial finite Galois extension of K of degree not divisible by the characteristic of $\overline{K}$, and let q be a prime dividing $[E : K]$. Let $\mathcal{G} = \mathrm{Gal}(E/K)$, let $\mathcal{H}$ be a Sylow q-subgroup of $\mathcal{G}$, let $F = E^{\mathcal{H}}$, let $\mathcal{J}$ be a subgroup of $\mathcal{H}$ of index q, and let $L = E^{\mathcal{J}}$. Let $f \in \chi(E/F)$ define L/F. Assume that L/F is ramified and that there exists a cyclic extension of $\overline{K}$ of degree q. Then there exists $b \in F^*$ such that $\mathrm{cor}_K^F(\Delta(f,b))$ has exponent q in* $\mathrm{Br}(E/K)$.

Proof. By assumption, L/F is tamely ramified so there exists $\beta \in L$ such that $\beta^q = \alpha$, where α is a uniformizing element for F [W, Proposition 3-4-3]. Let $r = e(F/K)$ and let π be a uniformizing element for K. Then $\alpha^r = u\pi$ for some unit $u \in F$. By assumption, there exists a cyclic unramified extension M of K of degree q. Let $g \in \chi(M/K)$ define M/K so g has order q.

Consider $A = \Delta(\mathrm{res}_K^F(g), u\pi) \in \mathrm{Br}(F)$. Since g has order q, $\exp(A)$ is either 1 or q. Moreover, $\mathrm{res}_F^L(A) = \Delta(\mathrm{res}_K^L(g), u\pi) = \Delta(\mathrm{res}_K^L(g), \alpha^r) = \Delta(\mathrm{res}_K^L(g), (\beta^r)^q) = q \cdot \Delta(\mathrm{res}_K^L(g), \beta^r)$. Since $\mathrm{res}_K^L(g)$ has order 1 or q, $\mathrm{res}_F^L(A) = 0$, so $A \in \mathrm{Br}(L/F)$. It follows that there exists $b \in F^*$ such that $A = \Delta(f,b)$ [Pi, Chapter 15]. Since $A \in \mathrm{Br}(E/F)$, $\mathrm{cor}_K^F(A) \in \mathrm{Br}(E/K)$. It thus suffices to show that $\mathrm{cor}_K^F(A)$ has exponent q in $\mathrm{Br}(K)$.

Let $\overline{T}$ be the separable closure of $\overline{K}$. Recall that we denote the projection map $\mathrm{Br}(F)_q \longrightarrow \chi(\overline{T}/\overline{F})$ by *ram*. By [FSS1, Theorem 1.4], there is a commutative diagram:

$$\begin{array}{ccc} \mathrm{Br}(F) & \xrightarrow{ram} & \chi(\overline{T}/\overline{F}) \\ {\scriptstyle \mathrm{cor}}\big\downarrow & & {\scriptstyle \mathrm{cor}}\big\downarrow \\ \mathrm{Br}(K) & \xrightarrow{ram} & \chi(\overline{T}/\overline{K}) \end{array}$$

Since $A = \Delta(\mathrm{res}_K^F(g), u\pi) = \Delta(\mathrm{res}_K^F(g), \alpha)^r$ and α is a uniformizing element for F, $ram(A) = r \cdot \mathrm{res}_{\overline{K}}^{\overline{F}}(\bar{g})$ and so $ram(\mathrm{cor}_K^F(A)) = \mathrm{cor}(ram(A)) = r[\overline{F} : \overline{K}]\bar{g} \in \chi(\overline{T}/\overline{K})$.

Since q does not divide $r[\overline{F} : \overline{K}]$, $ram(\mathrm{cor}_K^F(A))$ has order q. Thus $\mathrm{cor}_K^F(A)$ has exponent q in $\mathrm{Br}(K)$, as was to be shown. □

We next provide an example to show that Lemma 6 does not hold, in general, with L/F unramified.

Example. Let s, t be independent transcendentals over the complex numbers, $\mathbb{C}$, and let $K = \mathbb{C}(t)((s))$ be the Laurent series field in s over $\mathbb{C}(t)$. By the Riemann Existence Theorem, there exists an A_5-Galois extension E_0 of $\mathbb{C}(t)$, where A_5 denotes the alternating group on five letters [M, Folgerung 2, p. 21]. Let $E = KE_0$. Then $\overline{K} = \mathbb{C}(t)$ and $\overline{E} = E_0$. Since E/K is unramified, $\mathrm{Br}(E/K) \cong \mathrm{Br}(\overline{E}/\overline{K}) \oplus \chi(\overline{E}/\overline{K})$ [Se, Exercise 2, p. 195]. Since $\overline{K} = \mathbb{C}(t)$, $\mathrm{Br}(\overline{K}) = \{0\}$ by Tsen's Theorem [Pi, Section 19.4]. Since A_5 is simple, $\chi(\overline{E}/\overline{K}) = \{0\}$. Thus $\mathrm{Br}(E/K) = \{0\}$. Since there exist cyclic extensions of $\overline{K}$ of all possible degrees, this shows that Lemma 6 does not hold, in general, if L/F is unramified.

Theorem 7. *Let E/K be a finite Galois extension of fields. Assume that E has a discrete valuation π satisfying the following conditions:*

(1) *the decomposition group of E/K at π has a normal subgroup of prime index q, and*
(2) *there exists a cyclic extension of degree q of the residue field of K at π.*

Then $\mathrm{Br}(E/K)_q \neq \{0\}$. *If there exist infinitely many inequivalent discrete valuations π satisfying the above conditions, then* $\mathrm{Br}(E/K)_q$ *is infinite.*

Proof. Let the hypotheses be as in Theorem 7, let $\mathcal{Z}$ be the decomposition group of E/K at π, and let $T = E^{\mathcal{Z}}$. Then $\mathrm{Gal}(E/T) = \mathrm{Gal}(E_\pi/K_\pi)$. By assumption, there exists a field $M \subseteq E$ with M/T cyclic of degree q. Let $f \in \chi(E/T)$ define M/T (and so also M_δ/T_δ). If M/T is unramified at $\delta|T$, then $\mathrm{Br}(E/K)_q \neq \{0\}$ by Proposition 1, (2). Suppose then that M/T is ramified. Let F be the fixed field of a Sylow q-subgroup of $\mathcal{Z}$ and let $L = MF$. Then L/F is cyclic of degree q and is ramified at δ. By Lemma 6, $\mathrm{Br}(E/K)_q \neq \{0\}$. If there exist infinitely many such π, then $\mathrm{Br}(E/K)_q$ is infinite by Proposition 1, (4). □

Example. This example shows that we cannot drop the requirement in Theorem 7 that there exists a cyclic extension of degree q of the residue field of K at π. Let F be an algebraically closed field and let $K = F(t)$ where t is transcendental over F. Then $\mathrm{Br}(K) = \{0\}$ by Tsen's Theorem [Pi, Section 19.7]. Let $E = F(\sqrt{t})$ and let π be the valuation of K trivial on F having t as uniformizing element. Then the decomposition group of E/K at π is cyclic of order 2 and so the first hypothesis of Theorem 7 is satisfied. This shows that the second hypothesis is needed.

As an almost immediate consequence of Lemmas 2 and 6 we have:

Corollary 8. *Let K be a field complete with respect to a discrete valuation with residue field a p-adic field and let E be a finite Galois extension of K. Then* $\mathrm{Br}(E/K)$ *is finite and* $\mathrm{Br}(E/K)_q$ *is non-trivial for every prime q dividing* $[E : K]$.

Proof. We show first that $\mathrm{Br}(E/K)$ is finite. Let $\beta = e(E/K) \cdot \mathrm{res}_{\overline{K}}^{\overline{E}} : \chi(\overline{T}/\overline{K}) \longrightarrow \chi(\overline{T}/\overline{E})$ where $\overline{T}$ is the separable closure of $\overline{K}$. It follows from [Se, Chapter XII, Section 3, Exercise 2] that there is an exact sequence $0 \longrightarrow \mathrm{Br}(\overline{E}/\overline{K}) \longrightarrow \mathrm{Br}(E/K) \longrightarrow \ker(\beta)$. To show that $\mathrm{Br}(E/K)$ is finite, it is enough to show that $\mathrm{Br}(\overline{E}/\overline{K})$ and $\ker(\beta)$ are both finite. Since $\overline{K}$ is a p-adic field, $\mathrm{Br}(\overline{K}) \cong \mathbb{Q}/\mathbb{Z}$ [Pi, Theorem 17.10] and $\mathrm{Br}(\overline{E}/\overline{K})$ corresponds to the kernel of the multiplication map $[\overline{E}:\overline{K}] : \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z}$. In particular, $\mathrm{Br}(\overline{E}/\overline{K})$ is finite. Any element of $\ker(\beta)$ is a character of order dividing $[\overline{E}:\overline{K}]$ and there are only finitely many such characters since $\overline{K}$ has only finitely many extensions of any given degree [La, p. 54]. Thus $\ker(b)$ is also finite, proving that $\mathrm{Br}(E/K)$ is finite.

Now let let q be a prime dividing $[E : K]$, let F be the fixed field of a Sylow q-subgroup of $\mathrm{Gal}(E/K)$, and let L/F be a cyclic extension of degree q where $K \subseteq F \subset L \subseteq E$. We note that $\overline{K}$ has a cyclic extension of degree q since $\overline{K}$ is a p-adic field. If L/F is ramified, then $\mathrm{Br}(E/K) \neq \{0\}$ by Lemma 6, while if L/F is unramified, $\mathrm{Br}(E/K) \neq \{0\}$ by Lemma 2. Thus $\mathrm{Br}(E/K)_q \neq \{0\}$, completing the proof of Corollary 8. □

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