RELATIVE BRAUER GROUPS
OF DISCRETE VALUED FIELDS

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Abstract. Let $E$ be a non-trivial finite Galois extension of a field $K$. In this paper we investigate the role that valuation-theoretic properties of $E/K$ play in determining the non-triviality of the relative Brauer group, $\text{Br}(E/K)$, of $E$ over $K$. In particular, we show that when $K$ is finitely generated of transcendence degree 1 over a $p$-adic field $k$ and $q$ is a prime dividing $[E : K]$, then the following conditions are equivalent: (i) the $q$-primary component, $\text{Br}(E/K)_q$, is non-trivial, (ii) $\text{Br}(E/K)_q$ is infinite, and (iii) there exists a valuation $\pi$ of $E$ trivial on $k$ such that $q$ divides the order of the decomposition group of $E/K$ at $\pi$.

1. Introduction and Preliminaries

Let $K$ be a field finitely generated of transcendence degree $r \geq 1$ over a field $k$, let $E$ be a non-trivial finite Galois extension of $K$, and let $q$ be a prime dividing $[E : K]$. If $k$ is Hilbertian or if $k$ is arbitrary and $r \geq 2$, the proof of [FS1, Corollary 5] shows that the $q$-primary component of the relative Brauer group of $E$ over $K$, $\text{Br}(E/K)_q$, must be infinite. On the other hand, Roquette [R, Corollary XVIa] has given examples where $\text{Br}(E/K) = \{0\}$ when $k$ is a $p$-adic field, $r = 1$, $K$ has genus 1 over $k$, and $E$ is a non-trivial cyclic extension of $K$. In Roquette’s example, every valuation of $K$ trivial on $k$ splits completely in $E$. In contrast, the proofs of the positive results cited above make use of the fact that there are infinitely many valuations $\pi$ of $E$ trivial on $k$ such that $q$ divides the order of the decomposition group of $E/K$ at $\pi$. In this paper we investigate, for an arbitrary field $K$ and an arbitrary non-trivial finite Galois extension $E$ of $K$, the role that valuation-theoretic properties of $E/K$ play in determining the non-triviality of $\text{Br}(E/K)$. Among other results, we show that when $K$ is finitely generated of transcendence degree 1 over a $p$-adic field $k$ and $E$ is a non-trivial finite Galois extension of $K$, then $\text{Br}(E/K)_q$ is non-trivial if and only if it is infinite and this occurs if and only if there exists a valuation $\pi$ of $E$ trivial on $k$ such that $q$ divides the order of the decomposition group of $E/K$ at $\pi$.
We now establish some of the terminology and notation that we will maintain throughout. We say that $L/F$ is an $\mathcal{H}$-Galois extension of fields if $L$ is a Galois extension of a field $F$ with Galois group $\text{Gal}(L/F) = \mathcal{H}$. In what follows, $K$ will always be a field with one or more discrete (rank one) valuations, $E$ will be a non-trivial $G$-Galois extension of $K$, and $q$ will be a prime dividing $[E : K]$. Let $\pi$ be a discrete valuation of $E$. The completion of $E$ at $\pi$ will be denoted by $E_\pi$ and the residue field by $\overline{E}_\pi$; when $\pi$ is understood we sometimes denote $\overline{E}_\pi$ by $\overline{E}$. An element $b$ of $E_\pi$ generating the maximal ideal of the valuation ring of $E_\pi$ will be referred to as a uniformizing element for $\pi$. We denote the restriction of $\pi$ to $K$ by $\pi_K$; if no misunderstanding is possible, we sometimes write $\pi$ instead of $\pi_K$. We denote the decomposition group of $\pi$ in $\text{Gal}(E/K)$ by $\text{Gal}(E/K)_\pi$; with the above notation and conventions, $E_\pi/K_\pi$ is $G_\pi$-Galois. We denote the ramification degree of $E_\pi$ over $K_\pi$ by $e(E_\pi/K_\pi)$. By a $p$-adic field we mean a finite extension of the field $\mathbb{Q}_p$ of $p$-adic numbers.

We say that $A/F$ is central simple if $A$ is a simple algebra with center $F$ which is finite dimensional over $F$; we denote the class of such an $A$ in $\text{Br}(F)$ by $[A]$. The exponent of $[A]$, $\exp(A)$, is the order of $[A]$ in $\text{Br}(K)$. If $L$ is a field extension of $F$, we refer to the map from $\text{Br}(F)$ to $\text{Br}(L)$ defined by $[A] \rightarrow [A \otimes_F L]$ as restriction and denote it by $\text{res}_{L/F}$. If $F/K$ is a finite separable extension of fields, we denote the corestriction map from $\text{Br}(F)$ to $\text{Br}(K)$ by $\text{cor}_{F/K}$.

Suppose that $K \subseteq F \subseteq E$ and $\text{Gal}(E/F) = \mathcal{H} \subseteq G = \text{Gal}(E/K)$. Then $\text{Br}(E/F) \cong H^2(\mathcal{H}, E^*)$, $\text{Br}(E/K) \cong H^2(G, E^*)$, and $\text{cor}_{K}^F$ restricted to $\text{Br}(E/F)$ corresponds to the cohomological corestriction map from $H^2(\mathcal{H}, E^*)$ to $H^2(G, E^*)$. In particular, if $[A]$ is in the $q$-primary component, $\text{Br}(E/F)_q$, of $\text{Br}(E/F)$, then $\text{cor}_{K}^F([A]) \in \text{Br}(E/K)_q$. All of our non-triviality results for relative Brauer groups are obtained by showing that there exists suitable $F$ and $A$ as above such that $\text{cor}_{K}^F([A]) \neq 0$. The algebras $A$ that we will need will be cyclic algebras; we next review their construction.

Suppose $E/F$ is a possibly infinite $\mathcal{H}$-Galois extension of fields. We let the profinite group $\mathcal{H}$ act trivially on the discrete group $\mathbb{Q}/\mathbb{Z}$ and define the character group of $E/F$, $\chi(E/F)$, to be $H^1(\mathcal{H}, \mathbb{Q}/\mathbb{Z})$, the group of continuous homomorphisms of $\mathcal{H}$ into $\mathbb{Q}/\mathbb{Z}$. If $f \in \chi(E/F)$, then $f(\mathcal{H})$ is a finite subgroup of $\mathbb{Q}/\mathbb{Z}$ and hence is cyclic. Let $L \subseteq E$ be the fixed field of the kernel of $f$. Then $L/F$ is cyclic and we say that the character $f$ defines $L/F$. The short exact sequence of trivial $\mathcal{H}$-modules $0 \rightarrow \mathbb{Z} \rightarrow \mathbb{Q} \rightarrow \mathbb{Q}/\mathbb{Z} \rightarrow 0$ gives rise to a long exact sequence of cohomology groups. Since $H^1(\mathcal{H}, \mathbb{Q}) = H^2(\mathcal{H}, \mathbb{Q}) = 0$, we have an isomorphism $d : H^2(\mathcal{H}, \mathbb{Z}) \rightarrow H^1(\mathcal{H}, \mathbb{Q}/\mathbb{Z}) = \chi(E/F)$. The cup product (e.g. [B, p. 112]) defines a map: $H^2(\mathcal{H}, \mathbb{Z}) \times H^0(\mathcal{H}, E^*) \rightarrow H^2(\mathcal{H}, E^*) \cong \text{Br}(E/F)$. Since $H^0(\mathcal{H}, E^*) \cong F^*$ and $H^2(\mathcal{H}, \mathbb{Z}) \cong \chi(E/F)$, we have a map $\chi(E/F) \times F^* \rightarrow \text{Br}(E/F)$. We denote the image of $(f, b)$ (where $f \in \chi(E/F)$ and $b \in F^*$) under this map by $\Delta(f, b)$. In algebra terms, $\Delta(f, b)$ is represented by the cyclic crossed product $\Delta(L/F, \sigma, b)$ where $f$ defines $L/F$ and where $\sigma \in \text{Gal}(E/F)$ is the generator satisfying $f(\sigma) = (1/[L : F]) + \mathbb{Z}$.

We shall show that $\text{cor}_{K}^F([A]) \neq 0$ for suitable $A$ and $F$ as above by showing that $\text{res}_{K}^F \text{cor}_{K}^F([A]) \neq 0$ for a suitably chosen discrete valuation $\pi$ of $K$. In what follows, we shall freely use standard results about the Brauer group of a complete discrete valued field. We refer the reader to [Se, Chapter XII, Section 3] for a complete discussion of the Brauer group of such a field; we briefly summarize the results we will need.
Let $K$ be a field complete with respect to a discrete valuation, let $T$ denote the maximal unramified extension of $K$, and let $b$ be a fixed uniformizing element for $K$. Then $\chi(T/K) \cong \chi(T/K)$; we denote the image of $f \in \chi(T/K)$ under this isomorphism by $\bar{f}$. Now suppose that $L/K$ is a cyclic unramified extension of $K$ of degree $n$ and let $f \in \chi(T/K)$ define $L/K$. Since the value group of $L$ equals the value group of $K$ and the values of conjugates of an element are equal, $n$ is the smallest positive power of $b$ which is a norm from $L$ to $K$. In particular, $\Delta(f,b)$ has exponent $n$ in $Br(K)$. Let $q$ be a prime not dividing the characteristic of $K$. Then $Br(K)_q$ is the direct sum of the inertial lift (in the sense of [JW, Theorem 2.8]) of $Br(K_q)$ and a subgroup isomorphic to $\chi(T/K)_q$ where the isomorphism is given by $\bar{f} \to \Delta(f,b)$ for $f \in \chi(T/K)_q$ [Se, Theorem 2, p. 194]. We let $ram$ denote the projection of $Br(K)_q$ onto $\chi(T/K)_q$ defined by this direct sum decomposition of $Br(K)_q$. In particular, $ram(\Delta(f,b)) = \bar{f}$.

The following approximation result is basic to our approach; a slightly weaker version appears implicitly in [FSS2, Lemma 4.7] (see also [FS2, Lemma 4]).

**Proposition 1.** Let $K \subseteq F \subseteq L \subseteq E$ where $L/F$ is cyclic of degree $q$, defined by $f \in \chi(E/F)$. Assume that $\pi$ is a discrete valuation of $E$ such that $[L_\pi : F_\pi] = [L : F]$. Identifying $Gal(E_\pi/F_\pi)$ as a subgroup of $Gal(E/F)$, we may view $f$ as an element of $\chi(E_\pi/F_\pi)$ defining $L_\pi/F_\pi$.

1. Let $\bar{b} \in F^*_\pi$. Then there exists $b \in F^*$ such that $\text{res}^{K_\pi}_K \text{cor}^{F_\pi}_K(\Delta(f,b)) = \text{cor}^{F_\pi}_K(\Delta(f,\bar{b}))$.

2. Assume that there exists $\bar{b} \in F^*_\pi$ such that $\text{cor}^{F_\pi}_K(\Delta(f,\bar{b}))$ has exponent $q$ in $Br(K_\pi)$. Then $Br(E/K)_q \neq \{0\}$.

3. If $F_\pi = K_\pi$ and $L_\pi/K_\pi$ is unramified, then $Br(E/K)_q \neq \{0\}$.

4. Assume that there exist infinitely many discrete valuations $\pi$ of $E$ such that the hypotheses of (2) or (3) above hold. Then $Br(E/K)_q$ is infinite.

**Proof.** (1) We argue exactly as in the proof of [FSS2, Lemma 4.7] (see also [FS2, Lemma 4]) but with one minor modification. We need to choose $b$ sufficiently close to $\bar{b}$ $\pi$-adically so that $b\bar{b}^{-1}$ is a norm from $L_\pi$ to $F_\pi$; since $b\bar{b}^{-1} = 1 + b^{-1}(b - \bar{b})$, this is possible by [Se, Chapter 5, Section 2, Proposition 3, and Section 6, Corollary 4]. For $b$ so chosen, $\Delta(f,bb^{-1}) = 0$ in $Br(F_\pi)$ and so $\text{cor}^{F_\pi}_K(\Delta(f,b)) = \text{cor}^{F_\pi}_K(\Delta(f,\bar{b}))$. The proof of [FSS2, Lemma 4.7] now continues without incident.

(2) Assume that there exists $\bar{b} \in F^*_\pi$ such that $\text{cor}^{F_\pi}_K(\Delta(f,\bar{b}))$ has exponent $q$ in $Br(K_\pi)$. Let $b$ be as in (1). Then $\text{res}^{K_\pi}_K \text{cor}^{F_\pi}_K(\Delta(f,b))$ has exponent $q$. Since restriction is a homomorphism and $\Delta(f,b)$ has exponent dividing $q$, $\text{cor}^{F_\pi}_K(\Delta(f,b))$ has exponent $q$. As noted above, $\text{cor}^{F_\pi}_K(\Delta(f,b))$ is also an element of $Br(E/K)$. Thus $Br(E/K)_q \neq \{0\}$.

(3) Assume that $F_\pi = K_\pi$ and $L_\pi/K_\pi$ is unramified. Choose $\bar{b}$ a uniformizing element for $K_\pi$. Then $\Delta(f,\bar{b})$ has exponent $q$. Since $F_\pi = K_\pi$, (3) follows from (2).

(4) Since the hypotheses of (3) imply the hypotheses for (2), we may assume that there exist infinitely many discrete valuations $\pi_i$ of $E$ and $\bar{b}_i \in F^*_\pi$ such that $\text{cor}^{F_{\pi_i}}_{K_{\pi_i}}(\Delta(f,\bar{b}_i))$ has exponent $q$ in $Br(K_{\pi_i})$. Suppose that $|Br(E/K)_q| < n$ for some positive integer $n$. For $1 \leq i \leq n$, we choose $b_i \in F^*$ such that $\text{res}^{K_\pi}_K \text{cor}^{F_\pi}_K(\Delta(f,b_i)) = \text{cor}^{F_{\pi_i}}_{K_{\pi_i}} b_i(\Delta(f,b_i))$ as in the proof of (1) above but with the additional restriction that $b_i$ is a norm from $L_\pi$ to $F_\pi$ for all valuations $\gamma$ of $L$ such that $\gamma$ restricted to $K$ equals $\pi_j$ restricted to $K$ for some $1 \leq j \leq n, j \neq i$. 

Let \( A_{ij} = \text{res}_{E/K}^{K_v} \text{cor}_E^f(\Delta(f, b_i)) \). Then \( A_{ij} = 0 \) for \( i \neq j \) and \( A_{ii} \) has exponent \( q \). It follows that \( \{ \text{cor}_E^f(\Delta(f, b_1)), \ldots, \text{cor}_E^f(\Delta(f, b_n)) \} \) is a set of \( n \) distinct elements in \( \text{Br}(E/K)_q \), contradicting the assumption that \( |\text{Br}(E/K)_q| < n \). This proves (4).

\[ \square \]

Remark. Suppose that \( K \) is either a global field or a regular extension of transcendence degree \( \geq 1 \) of a Hilbertian field (e.g. a regular extension of transcendence degree \( \geq 2 \) of an arbitrary field) and \( E \) is a finite Galois extension of \( K, E \neq K \). Let \( q \) be a prime dividing \( [E : K] \) and \( \sigma \in \text{Gal}(E/K) \) have order \( q \). By the Tchebotarev Density Theorem if \( K \) is a global field [T, p. 163] or by the proof of [FSS1, Theorem 2.6] (see also [FS2, Proposition 3]) if \( K \) is a regular extension of transcendence degree at least 1 of a Hilbertian field, there exist infinitely many discrete valuations \( \pi \) of \( E \) such that \( \langle \pi \rangle \) is the decomposition group of \( E/K \) at \( \pi \).

It follows from Proposition 1, (4) that \( \text{Br}(E/K)_q \) is infinite. Thus Proposition 1 provides another approach to the results of [FKS] and [FS1], at least for Galois extensions \( E \) over \( K \).

2. Relative Brauer groups of function fields over \( p \)-adic fields

Let \( k \) be a \( p \)-adic field, let \( K \) be a finitely generated extension of \( k \) of transcendence degree \( \geq 1 \), \( L \) be a non-trivial finite Galois extension of \( K \), and \( q \) be a prime dividing \( [L : K] \). The proof of [FS1, Corollary 5] shows that if \( r \geq 2 \), then \( \text{Br}(L/K)_q \) is infinite. In this section we consider the case when \( r = 1 \). In view of Proposition 1, it is natural to begin by considering complete fields having \( p \)-adic residue fields.

Lemma 2. Let \( K \) be a field complete with respect to a discrete valuation having a \( p \)-adic residue field, let \( E \) be a non-trivial finite Galois extension of \( K \), and let \( q \) be a prime dividing \( [E : K] \). Assume that there exists an unramified cyclic extension \( L/F \) of degree \( q \) with \( K \subseteq F \subseteq L \subseteq E \) such that \( q \mid e(F/K) \). Let \( f \in \chi(E/F) \) define \( L/F \). Then there exists \( b \in F^* \) such that \( \text{cor}_E^f(\Delta(f, b)) \) has exponent \( q \) in \( \text{Br}(E/K) \).

Proof. Let \( \bar{f} \in \chi(E/F) \) be the character defining \( \bar{L}/\bar{F} \) corresponding to \( f \). Let \( v \) be a uniformizing element for \( \bar{F} \). Since \( \bar{F} \) is complete, \( \Delta(f, v) \) has exponent \( q \). Let \( \Delta(f, b) \) be the inertial lift of \( \Delta(f, v) \) in the sense of [JW, Theorem 2.8].

\[ \text{cor}_E^f(\Delta, b) \] is similar to the inertial lift of the division algebra component of \( e(F/K) \).

\[ \text{cor}_E(\Delta(f, v)) \] [Hw, Theorem 17]. Since the corestriction map from \( \text{Br}(\bar{F}) \) to \( \text{Br}(\bar{K}) \) is injective because \( \bar{F} \) is a \( p \)-adic field [Se, p. 175] and since \( q \mid e(F/K) \),

\[ \text{cor}_E^f(\Delta(f, v)) \] has exponent \( q \). Thus \( \text{cor}_E^f(\Delta, b) \) has exponent \( q \) in \( \text{Br}(E/K) \).

\[ \square \]

Theorem 3. Let \( K \) be a finitely generated field extension of transcendence degree 1 of a \( p \)-adic field \( k \), let \( E/K \) be a finite Galois extension of \( K \), and let \( q \) be a prime dividing \( [E : K] \). Then the following are equivalent:

1. \( \text{Br}(E/K)_q \neq \{0\} \).
2. \( \text{Br}(E/K)_q \) is infinite.
3. There exists a valuation \( \pi \) of \( E \) trivial on \( k \) such that \( q \) divides the order of the decomposition group of \( E/K \) at \( \pi \).

Proof. We will prove the circle of implications: (2) \( \Rightarrow \) (1), (1) \( \Rightarrow \) (3), and (3) \( \Rightarrow \) (2). The implication (2) \( \Rightarrow \) (1) is clear. Assume next that (1) holds so there exists
\(\alpha \in \text{Br}(E/K)\) of order \(q\). Let \(\mathbb{P}(K)\) denote the set of inequivalent valuations of \(K\) trivial on \(k\). There is a natural map \(\text{Br}(K) \rightarrow \prod_{\gamma \in \mathbb{P}(K)} \text{Br}(K_\gamma)\) where \([D] \in \text{Br}(K)\) maps to \((\ldots, [D \otimes K_\gamma], \ldots) \in \prod_{\gamma \in \mathbb{P}(K)} \text{Br}(K_\gamma)\) and a corresponding map \(\text{Br}(E) \rightarrow \prod_{\delta \in \mathbb{P}(E)} \text{Br}(E_\delta)\). Similarly, there is a natural map \(\prod_{\gamma \in \mathbb{P}(K)} \text{Br}(K_\gamma) \rightarrow \prod_{\delta \in \mathbb{P}(E)} \text{Br}(E_\delta)\) induced by the restriction maps from \(\text{Br}(K_\gamma)\) to \(\text{Br}(E_\delta)\) for each \(\gamma \in \mathbb{P}(K)\) and each \(\delta \in \mathbb{P}(E)\) extending \(\gamma\). Combining these natural maps with the restriction map from \(\text{Br}(K)\) to \(\text{Br}(E)\) leads to a commutative diagram:

\[
\begin{array}{ccc}
\text{Br}(E) & \longrightarrow & \prod_{\delta \in \mathbb{P}(E)} \text{Br}(E_\delta) \\
\uparrow & & \uparrow \\
\text{Br}(K) & \longrightarrow & \prod_{\gamma \in \mathbb{P}(K)} \text{Br}(K_\gamma)
\end{array}
\]

Both horizontal maps are monomorphisms by a result due implicitly to Lichtenbaum (see the proof of Theorem 5 of [Li]) and explicitly stated in [Po, Theorem 4.1]. Let \(D\) be the underlying division algebra of \(\alpha\). Since \(\alpha\) has order \(q\) in \(\text{Br}(K)\), there exists a valuation \(\pi\) of \(K\) trivial on \(k\) such that the Brauer class of \(D \otimes K K_\pi\) has order \(q\) in \(\text{Br}(K_\pi)\). Since \(\alpha \in \text{Br}(E/K)\), the image of \(\alpha\) in \(\prod_{\delta \in \mathbb{P}(E)} \text{Br}(E_\delta)\) is trivial.

Let \(\delta\) be any valuation of \(E\) extending \(\pi\). Since \(E_\delta\) splits \(D \otimes K K_\pi\), \(q\) divides \([E_\delta : K_\pi]\) [Pi, Proposition 14.4b(ii) and Lemma 13.4]. Thus (3) holds.

It remains to show that (3) implies (2). Assume that there exists a valuation \(\pi\) of \(E\) trivial on \(k\) such that \(q\) divides the order of the decomposition group of \(E/K\) at \(\pi\). Let \(G_\pi \subseteq G\) be the decomposition group of \(\pi\) over \(K\) and let \(H_\pi\) denote a Sylow \(q\)-subgroup of \(G_\pi\). Since \(q \mid |G_\pi|\), \(H_\pi \neq \{1\}\). Let \(J_\pi\) denote a subgroup of \(H_\pi\) of index \(q\), let \(F\) be the fixed field of \(H_\pi\) and let \(L\) be the fixed field of \(J_\pi\). Then \(L\) is a cyclic extension of \(F\) of degree \(q\) and \(\pi|L\) is a valuation of \(L\) whose decomposition group over \(F\) is non-trivial. In particular, there are valuations of \(F\) trivial on \(k\) not splitting completely in \(L\). By [Sa, Theorem 7.1] there are infinitely many valuations of \(F\) trivial on \(k\) which are unramified in \(L\) of residue class degree \(q\). Since only finitely many valuations of \(K\) ramify in \(E\), there are infinitely many valuations \(\delta\) of \(E\) trivial on \(k\) which are unramified over \(K\) and with \([L_\delta : F_\delta] = q\). By Lemma 2 and Proposition 1, (4), \(\text{Br}(E/K)_q\) is infinite.

As an immediate consequence of Theorem 3, we have:

**Corollary 4.** Let \(K\) be a finitely generated extension of transcendence degree one of a \(p\)-adic field \(k\) and let \(E\) be a non-trivial finite Galois extension of \(K\). Then the following statements are equivalent:

1. \(\text{Br}(E/K) \neq \{0\}\).
2. \(\text{Br}(E/K)\) is infinite.
3. There exists a valuation of \(K\) trivial on \(k\) not splitting completely in \(E\).

Corollary 4 explains the example of Roquette [R, Corollary XVIa] mentioned in the Introduction since in Roquette’s example every valuation of \(K\) trivial on \(k\) splits completely in \(E\). In general, for \(K\) as in Corollary 4, there will exist extensions \(E\) of \(K\) such that every valuation of \(K\) trivial on \(k\) splits completely in \(E\) when the smooth projective variety over \(k\) with function field \(K\) has bad reduction to the residue field of \(k\). Such examples seem to exist for any genus \(\geq 1\) [Sa, Examples 2.7.
and 7.2] but not if $K/k$ has genus 0 [Sa, page 73 and Theorem 7.1]. We refer the reader to [Sa] for a thorough discussion of the existence of these completely split covers.

**Corollary 5.** Let $k$, $K$, and $E$ be as in Corollary 4 and suppose that $K/k$ has genus 0. Then $\text{Br}(E/K)$ is infinite.

**Proof.** Let $k_1$ denote the field of constants of $K$ and let $K_1 = Kk_1$. Since $K_1/k_1$ has genus 0, there are no completely split covers of $K_1$ (see the references to [Sa] above) and so there exists a valuation $\pi_1$ of $K_1$ trivial on $k_1$ not splitting completely in $E$. Let $\pi$ denote the restriction of $\pi_1$ to $K$. Then $\pi$ does not split completely in $E$ and so $\text{Br}(E/K)$ is infinite by Corollary 4.

3. The ramified case

Let $K$ be a field with a discrete valuation $\pi$ and let $E$ be a finite Galois extension of $K$ in which $\pi$ does not split completely. Then there exist $L$ and $F$ with $K \subseteq F \subseteq L \subseteq E$ where $L/F$ is cyclic and some extension $\delta$ of $\pi$ to $F$ is undecomposed in $L$. If $L/F$ is unramified at $\delta$, Proposition 1, (3), can sometimes be used to conclude that $\text{Br}(E/K)$ is non-trivial. In this section we consider what can be said if $L/F$ is ramified at $\delta$. As in the preceding section, we begin with a result about complete fields.

**Lemma 6.** Let $K$ be a field complete with respect to a discrete valuation, let $E$ be a non-trivial finite Galois extension of $K$ of degree not divisible by the characteristic of $K$, and let $q$ be a prime dividing $[E : K]$. Let $G = \text{Gal}(E/K)$, let $H$ be a Sylow $q$-subgroup of $G$, let $F = E^H$, let $J$ be a subgroup of $H$ of index $q$, and let $L = E^J$.

Let $f \in \chi(E/F)$ define $L/F$. Assume that $L/F$ is ramified and that there exists a cyclic extension $\bar{K}$ of degree $q$. Then there exists $b \in F^*$ such that $\text{cor}_{\bar{K}}^E(\Delta(f, b))$ has exponent $q$ in $\text{Br}(E/K)$.

**Proof.** By assumption, $L/F$ is tamely ramified so there exists $\beta \in L$ such that $\beta^q = \alpha$, where $\alpha$ is a uniformizing element for $F$ [W, Proposition 3-4-3]. Let $r = e(F/K)$ and let $\pi$ be a uniformizing element for $K$. Then $\alpha^r = \pi$ for some unit $u \in F$. By assumption, there exists a cyclic unramified extension $M$ of $K$ of degree $q$. Let $g \in \chi(M/K)$ define $M/K$ so $g$ has order $q$.

Consider $A = \Delta_{\text{res}}^E_K(g, u\pi) \in \text{Br}(F)$. Since $g$ has order $q$, $\text{exp}(A)$ is either 1 or $q$. Moreover, $\text{res}_{\bar{K}}^E(A) = \Delta(\text{res}^L_{\bar{K}}(g), u\pi) = \Delta(\text{res}^K_{\bar{K}}(g), \alpha^r) = \Delta(\text{res}_K^L(g), (\beta^r)^q) = q \cdot \Delta^1(\text{res}_K^L(g), \beta^r)$. Since $\text{res}_K^L(g)$ has order 1 or $q$, $\text{res}_K^L(A) = 0$, so $A \in \text{Br}(L/F)$. It follows that there exists $b \in F^*$ such that $A = \Delta(f, b)$ [P1, Chapter 15]. Since $A \in \text{Br}(E/F)$, $\text{cor}_{\bar{K}}^E(A) \in \text{Br}(E/K)$. It thus suffices to show that $\text{cor}_{\bar{K}}^E(A)$ has exponent $q$ in $\text{Br}(K)$.

Let $\bar{T}$ be the separable closure of $\bar{K}$. Recall that we denote the projection map $\text{Br}(F)_q \rightarrow \chi(T/F)$ by $\text{ram}$. By [FSS1, Theorem 1.4], there is a commutative diagram:

$$
\begin{array}{ccc}
\text{Br}(F) & \xrightarrow{\text{ram}} & \chi(T/F) \\
\text{cor} & \downarrow & \text{cor} \\
\text{Br}(K) & \xrightarrow{\text{ram}} & \chi(T/K)
\end{array}
$$

Since $A = \Delta(\text{res}_K^E(g), u\pi) = \Delta(\text{res}_K^E(g), \alpha^r)$ and $\alpha$ is a uniformizing element for $F$, $\text{ram}(A) = r \cdot \text{res}_K^E(g)$ and so $\text{ram}(\text{cor}_{\bar{K}}^E(A)) = \text{cor}(\text{ram}(A)) = r[F : K][\bar{g}] \in \chi(T/K)$.
Since $q$ does not divide $r[F : K]$, $\text{ram}(\text{cor}^F_K(A))$ has order $q$. Thus $\text{cor}^F_K(A)$ has exponent $q$ in $\text{Br}(K)$, as was to be shown. \qed

We next provide an example to show that Lemma 6 does not hold, in general, with $L/F$ unramified.

Example. Let $s, t$ be independent transcendentals over the complex numbers, $\mathbb{C}$, and let $K = \mathbb{C}(t)((s))$ be the Laurent series field in $s$ over $\mathbb{C}(t)$. By the Riemann Existence Theorem, there exists an $A_5$-Galois extension $E_0$ of $\mathbb{C}(t)$, where $A_5$ denotes the alternating group on five letters [M, Folgerung 2, p. 21]. Let $E = KE_0$. Then $K = \mathbb{C}(t)$ and $E = E_0$. Since $E/K$ is unramified, $\text{Br}(E/K) \cong \text{Br}(E/K) \oplus \chi(E/K)$ [Se, Exercise 2, p. 195]. Since $K = \mathbb{C}(t)$, $\text{Br}(K) = \{0\}$ by Tsen's Theorem [P, Section 19.4]. Since $A_5$ is simple, $\chi(E/K) = \{0\}$. Thus $\text{Br}(E/K) = \{0\}$. Since there exist cyclic extensions of $K$ of all possible degrees, this shows that Lemma 6 does not hold, in general, if $L/F$ is unramified.

Theorem 7. Let $E/K$ be a finite Galois extension of fields. Assume that $E$ has a discrete valuation $\pi$ satisfying the following conditions:

1. the decomposition group of $E/K$ at $\pi$ has a normal subgroup of prime index $q$, and
2. there exists a cyclic extension of degree $q$ of the residue field of $K$ at $\pi$.

Then $\text{Br}(E/K)_q \neq \{0\}$. If there exist infinitely many inequivalent discrete valuations $\pi$ satisfying the above conditions, then $\text{Br}(E/K)_q$ is infinite.

Proof. Let the hypotheses be as in Theorem 7, let $Z$ be the decomposition group of $E/K$ at $\pi$, and let $T = E^Z$. Then $\text{Gal}(E/T) = \text{Gal}(E_0/K_\pi)$. By assumption, there exists a field $M \subseteq E$ with $M/T$ cyclic of degree $q$. Let $f \in \chi(E/T)$ define $M/T$ (and so also $M_3/\delta_3$). If $M/T$ is unramified at $\delta T$, then $\text{Br}(E/K)_q \neq \{0\}$ by Proposition 1, (2). Suppose then that $M/T$ is ramified. Let $F$ be the fixed field of a Sylow $q$-subgroup of $Z$ and let $L = MF$. Then $L/F$ is cyclic of degree $q$ and is ramified at $\delta$. By Lemma 6, $\text{Br}(E/K)_q \neq \{0\}$. If there exist infinitely many such $\pi$, then $\text{Br}(E/K)_q$ is infinite by Proposition 1, (4). \qed

Example. This example shows that we cannot drop the requirement in Theorem 7 that there exists a cyclic extension of degree $q$ of the residue field of $K$ at $\pi$. Let $F$ be an algebraically closed field and let $K = F(t)$ where $t$ is transcendental over $F$. Then $\text{Br}(K) = \{0\}$ by Tsen's Theorem [P, Section 19.7]. Let $E = F(\sqrt{t})$ and let $\pi$ be the valuation of $K$ trivial on $F$ having $t$ as uniformizing element. Then the decomposition group of $E/K$ at $\pi$ is cyclic of order $2$ and so the first hypothesis of Theorem 7 is satisfied. This shows that the second hypothesis is needed.

As an almost immediate consequence of Lemmas 2 and 6 we have:

Corollary 8. Let $K$ be a field complete with respect to a discrete valuation with residue field a $p$-adic field and let $E$ be a finite Galois extension of $K$. Then $\text{Br}(E/K)$ is finite and $\text{Br}(E/K)_q$ is non-trivial for every prime $q$ dividing $[E : K]$.

Proof. We show first that $\text{Br}(E/K)$ is finite. Let $\beta = e(E/K) \cdot \text{res}_{K/F} : \chi(T/K) \to \chi(T/E)$ where $T$ is the separable closure of $K$. It follows from [Se, Chapter XII, Section 3, Exercise 2] that there is an exact sequence $0 \to \text{Br}(E/K) \to \text{Br}(E/K) \to \ker(\beta)$. To show that $\text{Br}(E/K)$ is finite, it is enough to show that $\text{Br}(E/K)$ and $\ker(\beta)$ are both finite. Since $K$ is a $p$-adic field, $\text{Br}(K) \cong \mathbb{Q}/\mathbb{Z}$ [P,
Theorem 17.10] and $\text{Br}(E/K)$ corresponds to the kernel of the multiplication map $[E : K] : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z}$. In particular, $\text{Br}(E/K)$ is finite. Any element of $\ker(\beta)$ is a character of order dividing $[E : K]$ and there are only finitely many such characters since $\overline{K}$ has only finitely many extensions of any given degree [La, p. 54]. Thus $\ker(\beta)$ is also finite, proving that $\text{Br}(E/K)$ is finite.

Now let $q$ be a prime dividing $[E : K]$, let $F$ be the fixed field of a Sylow $q$-subgroup of $\text{Gal}(E/K)$, and let $L/F$ be a cyclic extension of degree $q$ where $K \subset F \subset L \subset E$. We note that $\overline{K}$ has a cyclic extension of degree $q$ since $K$ is a $p$-adic field. If $L/F$ is ramified, then $\text{Br}(E/K) \neq \{0\}$ by Lemma 6, while if $L/F$ is unramified, $\text{Br}(E/K) \neq \{0\}$ by Lemma 2. Thus $\text{Br}(E/K)_q \neq \{0\}$, completing the proof of Corollary 8.

References


