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INTEGRATION AND HOMOGENEOUS FUNCTIONS

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Abstract. We show that integrating a (positively) homogeneous function f on a compact domain $\Omega \subset R^n$ reduces to integrating a related function on the boundary $\partial\Omega$. The formula simplifies when the boundary $\partial\Omega$ is determined by homogeneous functions. Similar results are also presented for integration of exponentials and logarithms of homogeneous functions.

1. Introduction

We consider the integration of a continuous (positively) homogeneous function $f: R^n \to R$ on a compact domain Ω with boundary $\partial\Omega$. Using Euler's identity for homogeneous functions, Green's formula simplifies so that integrating f on Ω reduces to integrating a (simply related) function on the boundary $\partial\Omega$. In the particular case where $\Omega := \{x \in R^n \,|\, g_i(x) \leq a_i,\ i = 1, \ldots, m\}$ and where the functions g_i are each (positively) homogeneous of degree p_i, the formula is even simpler. Actually, an alternative proof that uses only Euler's formula is also outlined.

We thus extend to more general domains a result in [4] for integrating a homogeneous function on a convex polytope.

A potential application is the integration of arbitrary continuous functions on a compact set Ω. Indeed, as the polynomials are sums of homogeneous polynomials, and are dense in $C(\Omega)$, the space of continuous functions on Ω with the sup norm, or even in $L_1(\Omega)$, one could approximate $\int_\Omega f dx$ by $\sum_i \int_\Omega P_i dx$, where the P_i's are homogeneous polynomials, and therefore use the previous result. This result could also be used in finite element methods for the integration of a poynomial on each individual volume element.

Finally, we also provide simple formulas for the integration of exponentials and logarithms of homogeneous functions. For instance, integrating $\log f$ on a convex polytope Ω reduces to integrating $\log f$ on $\partial\Omega$ and to computing the volume of Ω.

2. Integration of a homogeneous function

Let $f : R^n \to R$ be a real (positively) *homogeneous* function of degree p (or in short, f is p-homogeneous), i.e. $f(\lambda x) = \lambda^p f(x)$ for all $\lambda > 0,\ x \in R^n$. For a (positively) p-homogeneous function that is continuously differentiable, Euler's

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formula states that

$$pf(x) = \langle \nabla f(x), x\rangle \quad \text{for all } x. \tag{2.1}$$

2.1. **On the Riemann-Green formula.** We first treat the 2-dimensional case. Let Ω be a compact domain in R^2 with boundary $\partial\Omega$. Let $P(x,y)$ and $Q(x,y)$ be continuously differentiable on Ω. Then, under some regularity conditions, the well-known Riemann-Green formula (cf. [5]) states that

$$\iint_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\, dy = \int_{\partial\Omega} P\, dx + Q\, dy. \tag{2.2}$$

This yields:

Lemma 2.1. *Let $f(x,y)$ be a continuously differentiable p-homogeneous function on Ω. Then*

$$(p+2)\iint_\Omega f\, dx\, dy = \int_{\partial\Omega} f(x,y)(x\, dy - y\, dx). \tag{2.3}$$

Proof. As f is p-homogeneous, from (2.1)

$$pf(x,y) = x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\ \forall (x,y) \in \Omega.$$

Let $P(x,y) := -yf(x,y)$ and $Q(x,y) := xf(x,y)$ for every $(x,y) \in \Omega$.

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2f(x,y) + x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = (p+2)f(x,y)\ \forall (x,y) \in \Omega.$$

Hence, applying the Riemann-Green formula (2.2) to P and Q yields

$$(p+2)\iint_\Omega f\, dx\, dy = \int_{\partial\Omega} f(x,y)(x\, dy - y\, dx),$$

the desired result. □

When $f \equiv 1$ (i.e. $p = 0$), one immediately retrieves the well-known formula

$$\text{area}(\Omega) \ = \ \frac{1}{2}\int_{\partial\Omega} x\, dy - y\, dx.$$

2.2. **The general case.** The previous result generalizes to R^n as follows:

Lemma 2.2. *Let $f : R^n \to R$ be a continuously differentiable p-homogeneous function and Ω a compact domain R^n with boundary $\partial\Omega$. Then*

$$(n+p)\int_\Omega f d\omega = \int_{\partial\Omega} \langle \vec{M}, \vec{n}\rangle f(M)\, d\sigma, \tag{2.4}$$

where $\vec{n}$ is the unit outward-pointing normal to $\partial\Omega$.

Proof. With notation as in [6], let X be the vector field $X := \sum_j x_j \frac{\partial}{\partial x_j}$. From Proposition 2.3 in [6], we have

$$\int_\Omega \text{div}(X) f\, d\omega + \int_\Omega X f\, d\omega = \int_{\partial\Omega} \langle X, \vec{n}\rangle\, f\, d\sigma. \tag{2.5}$$

Now, $\text{div}(X) = n$ and from (2.1), $Xf = pf$ so that the result follows. □

Thus, integrating f on Ω reduces to integrating $\langle \vec{M}, \vec{n}\rangle f$ on the boundary $\partial\Omega$. We next show that Lemma 2.2 further simplifies in the case where the boundary $\partial\Omega$ is determined by homogeneous functions.

3. Boundary determined by homogeneous functions

We now consider the special case where

$$\Omega(\subset R^n) := \{x \in R^n |\ g_i(x) \leq a_i,\ i = 1, \ldots, m\}, \tag{3.1}$$

and the functions g_i are each continuously differentiable (positively) p_i-homogeneous functions, $i = 1, \ldots, m$. We still assume that Ω is compact. Finally, let $\Omega_i := \{x \in \Omega \,|\, g_i(x) = a_i\}$ so that $\partial\Omega = \bigcup_{i=1}^m \Omega_i$.

Theorem 3.1. *Assume that f is a continuously differentiable q-homogeneous function and the functions g_i are each continuously differentiable and p_i-homogeneous, $i = 1, \ldots, m$. Then*

$$(n+q)\int_\Omega f d\omega = \sum_{i=1}^m p_i a_i \int_{\Omega_i} \frac{f}{\|\nabla g_i\|} d\sigma. \tag{3.2}$$

In particular, with $f \equiv 1$, one obtains

$$n \times \mathrm{vol}(\Omega) \;=\; \sum_{i=1}^m p_i a_i \int_{\Omega_i} \|\nabla g_i\|^{-1} d\sigma. \tag{3.3}$$

Proof. Lemma 2.2 applies so that

$$(n+q)\int_\Omega f d\omega \;=\; \sum_{i=1}^m \int_{\Omega_i} \langle X, \vec{n}\rangle f d\sigma. \tag{3.4}$$

Now, $\vec{n} = \|\nabla g_i\|^{-1}\nabla g_i$ on Ω_i. Therefore, using (2.1), we have on Ω_i

$$\langle X, \nabla g_i\rangle = \langle x, \nabla g_i(x)\rangle = p_i g_i(x) = p_i a_i,$$

which yields the desired result. □

Remark. When Ω is a convex polytope, i.e. $g_i(x) = \langle A_i, x\rangle$ and $p_i = 1$, $i = 1, \ldots, m$, then $\|\nabla g_i\| = \|A_i\|$ and thus (3.2) simplifies to

$$(n+q)\int_\Omega f d\omega \;=\; \sum_{i=1}^m \frac{a_i}{\|A_i\|}\int_{\Omega_i} f d\sigma.$$

In addition, one may iterate the process for $\int_{\Omega_i} f d\sigma$ (cf. [4] for more details).

An alternative proof. Interestingly enough, there is a direct proof that does not use Green's formula. It was used in [4] for the case where Ω is a convex polytope.

Write $a_i = b_i^{p_i}$, $i = 1, \ldots, m$, and let

$$h(b) := \int_\Omega f d\omega = \int_{\{g_i(x) \leq b_i^{p_i},\ i=1,\ldots,m.\}} f dx. \tag{3.5}$$

It is immediate that $h(\lambda b) = \lambda^{n+q} h(b)$, i.e. h is $(n+q)$-homogeneous and continuously differentiable. In addition, one may show that

$$\frac{\partial h}{\partial b_i} = p_i b_i^{p_i - 1}\int_{\Omega_i} \frac{f}{\|\nabla g_i\|} d\sigma.$$

Hence, applying Euler's formula to h yields

$$(n+q)h(b) = \langle \nabla h(b), b\rangle = \sum_{i=1}^m p_i a_i \int_{\Omega_i} \frac{f}{\|\nabla g_i\|} d\sigma, \tag{3.6}$$

the desired result.

Example. In R^2, consider the domain

$$\Omega := \{(x,y) \in R^2 \,|\, xy \leq 1;\ x^2+y^2 \leq R^2;\ x,y \geq 0\}$$

and its volume $\int_\Omega d\omega$ (i.e. $f \equiv 1$ is 0-homogeneous).

Note that in (3.3), the coordinate axis boundaries of Ω do not contribute since the boundary values a_i are zero in this case.

Both $g_i(x,y) := xy$ and $g_2(x,y) := x^2+y^2$ are 2-homogeneous. $\Omega_1 := \{(x,y) \in \Omega |\ xy = 1\}$ and $\Omega_2 := \{(x,y) \in \Omega |\ x^2+y^2 = R^2\}$. $||\nabla g_1(M)|| = \sqrt{x^2+y^2}$ so that on Ω_1 we have $||\nabla g_1(M)|| = \sqrt{(x^4+1)/x^2}$ and $d\sigma = \sqrt{(x^4+1)/x^4}dx$, which yields

$$\int_{\Omega_1} \|\nabla g_1\|^{-1} d\sigma \;=\; \int_{a^{-1}}^{a} x^{-1}dx \;=\; 2\log(a), \tag{3.7}$$

where $x = a$ is the solution to $x^2+y^2 = R^2,\ xy = 1$.

Similarly, $\|\nabla g_2\| = 2R$ on Ω_2, so that

$$\int_{\Omega_2} \|\nabla g_2\|^{-1} d\sigma = 2 \times (1/2R) \times (R\alpha) = \alpha,$$

where $\alpha = \arctan(a^{-2})$. Hence, Theorem 3.1 yields

$$2\,\mathrm{vol}(\Omega) = 4\log(a) + R^2 \arctan(a^{-2}),$$

i.e. $\mathrm{vol}(\Omega) = 2\log(a) + R^2\arctan(a^{-2})/2$. This can be retrieved directly.

If we now take $\Omega := \{(x,y)|\ xy \leq 1; b \leq y \leq a; x \geq 0\}$ with $a \geq b > 0$, we get $\Omega_2 := \{(x,y) \in \Omega|\ y = a\}$ and $\Omega_3 := \{(x,y) \in \Omega| -y = -b\}$. Therefore,

$$\int_{\Omega_1} \|\nabla g_1\|^{-1} d\sigma = \log(\frac{a}{b});\ \int_{\Omega_2} \|\nabla g_2\|^{-1} d\sigma = \frac{1}{a};\ \int_{\Omega_3} \|\nabla g_3\|^{-1} d\sigma = \frac{1}{b},$$

so that Theorem 3.1 yields

$$2\mathrm{vol}(\Omega) = 2\log(\frac{a}{b}) + a\frac{1}{a} - b\frac{1}{b} = 2\log(\frac{a}{b}),$$

i.e. $\mathrm{vol}(\Omega) = \log(a/b)$, which is immediate to check.

4. Integrating exponentials and logarithms

In this section, we consider exponentials and logarithms of homogeneous functions. Indeed, we show that integrating such functions on Ω reduces to integrating a related function on the boundary $\partial\Omega$.

4.1. **Exponentials.** We first consider the case of an exponential $e^{h(x)}$ where $h : R^n \to R$ is q-homogeneous. The particular case where $h = \langle c, x\rangle$ was considered in [1] and [2] and was used to compute the number of integral points in a convex polytope. We have the following result.

Theorem 4.1. *Let Ω be the convex polytope $\{x \in R^n \,|\, \langle A_i, x\rangle \leq a_i,\ i = 1,\dots,m\}$.*

(a) Let $h : R^n \to R$ be q-homogeneous and continuously differentiable. Then, for every $k = 0, 1, \dots,$

$$(n+qk)\int_\Omega h^k e^h\, d\omega \;+\; q\int_\Omega h^{k+1}e^h\, d\omega \;=\; \sum_{i=1}^m \frac{a_i}{\|A_i\|}\int_{\Omega_i} h^k e^h\, d\sigma. \tag{4.1}$$

(b) Let $h : R^n \to R$ be q-homogeneous, continuously differentiable and strictly positive on Ω. Then, for every $k = 1, \ldots,$

$$(n - qk)\int_\Omega h^{-k}e^h \, d\omega \, + \, q\int_\Omega h^{-k+1}e^h \, d\omega \, = \, \sum_{i=1}^m \frac{a_i}{\|A_i\|}\int_{\Omega_i} h^{-k}e^h \, d\sigma. \tag{4.2}$$

(c) If $p := n/q$ is an integer, then

$$q\int_\Omega h^{-p+1}e^h \, d\omega \, = \, \sum_{i=1}^m \frac{a_i}{\|A_i\|}\int_{\Omega_i} h^{-p}e^h \, d\sigma \tag{4.3}$$

and $\int_\Omega h^k e^h \, d\omega$, $k = -p+1, \ldots, -1, 0, 1, \ldots,$ can all be expressed as integrals on $\partial\Omega$. In particular,

$$\begin{aligned} q\int_\Omega e^h \, d\omega \quad = \quad & \sum_{i=1}^m \frac{a_i}{\|A_i\|}\int_{\Omega_i} e^h[h^{-1} - (p-1)h^{-2} + (p-1)(p-2)h^{-3} \\ & + \cdots + (-1)^{p+1}(p-1)!h^{-p}]d\sigma. \end{aligned} \tag{4.4}$$

(d) With $h := \langle c, x\rangle$, and for every $\xi \in R^n$, we get

$$\langle c, \xi\rangle \int_\Omega e^{\langle c,x\rangle} \, d\omega \, = \, \sum_{i=1}^m \frac{\langle A_i, \xi\rangle}{\|A_i\|}\int_{\Omega_i} e^{\langle c,x\rangle} \, d\sigma. \tag{4.5}$$

Proof. Again, with $f := h^k e^h$ and with the vector field $X := \sum_i x_i \frac{\partial}{\partial x_i}$, Green's formula yields

$$n\int_\Omega f \, d\omega + \int_\Omega Xf \, d\omega \, = \, \int_{\partial\Omega} \langle X, \vec{n}\rangle \, f \, d\sigma. \tag{4.6}$$

Using

$$Xf = (kh^{k-1} + h^k)e^h \sum_{i=1}^n x_i \frac{\partial h}{\partial x_i} = q(kh^k + h^{k+1})e^h, \text{ and } \langle X, \vec{n}\rangle = \frac{a_i}{\|A_i\|} \text{ on } \Omega_i,$$

we obtain

$$(n + qk)\int_\Omega h^k e^h \, d\omega + q\int_\Omega h^{k+1}e^h \, d\omega = \sum_{i=1}^m \frac{a_i}{\|A_i\|}\int_{\Omega_i} h^k e^h \, d\sigma, \tag{4.7}$$

i.e. (4.1). Similarly, with $f := h^{-k}e^h$ and similar arguments we obtain (4.2).

To get (c), just notice from (4.2) that $\int_\Omega h^{-n/q+1}e^h d\omega$ is expressed directly as an integral over $\partial\Omega$, which yields (4.3). Using (4.2) with $k := n/q - 1$ and (4.3), one obtains $\int h^{-n/q+2}e^h d\omega$ as an integral on $\partial\Omega$. Therefore, iterating and using (4.2) and (4.1), all the $\int_\Omega h^k e^h d\omega$, for $k := -n/q + 1, \ldots, -1, 0, 1, \ldots,$ are expressed as integrals on $\partial\Omega$.

To get (d), apply Green's formula (4.6), but now with $X = \sum_i \xi_i \partial/\partial x_i$, so that $\mathrm{div}(X) = 0$. $\square$

The formula (4.1) remains valid even with $h \equiv 0$ and $k = 0$. With the convention $0^0 = 1$, (4.1) reduces to the volume formula (3.3) of Ω.

Note that (4.5) was obtained in [1] using Stokes' formula, and a similar argument was used to show that it suffices to consider the vertices of the polyhedron. The same can be done using the above argument.

Indeed, let $\mathcal{H}_i$ be the $(n-1)$-dimensional affine variety that contains Ω_i, and $\{u_1, \ldots, u_{n-1}\}$ an orthonormal basis of the associated vector space. $x \in \Omega_i$ can be written $x_0 + \sum_{i=1}^{n-1} y_i u_i$ with $x_0 \in \Omega_i$, arbitrary. On Ω_i, consider the vector field $\sum_1^{n-1} \xi_i \partial/\partial y_i$. Green's formula yields

$$\langle c, \sum_{i=1}^{n-1} \xi_i u_i \rangle \int_{\Omega_i} e^{\langle c,x\rangle} d\sigma \;=\; \sum_{j\neq i} \langle \xi, \vec{n}_i \rangle \int_{\Omega_{ij}} e^{\langle c,x\rangle} d\nu,$$

where $\Omega_{ij} := \Omega_i \cap \Omega_j$ and $\vec{n}_i$ (in the basis $\{u_1, \ldots, u_{n-1}\}$) is the unit outward-pointing normal to Ω_{ij}. Obviously, the process can be repeated up to the 0-dimensional faces, i.e. the vertices of Ω.

4.2. **Logarithms.** Consider now the function $\log f$ where $f : R^n \to R$ is continuously differentiable, q-homogeneous and strictly positive on Ω.

Lemma 4.2. *Let $\Omega \subset R^n$ be a compact domain with boundary $\partial\Omega$, and let $f : R^n \to R$ be continuously differentiable, q-homogeneous and strictly positive on Ω. Then*

$$n \int_{\Omega} \log f \, d\omega \;+\; q \times \mathrm{vol}(\Omega) \;=\; \int_{\partial\Omega} \langle \vec{M}, \vec{n} \rangle \, \log f(M) \, d\sigma. \tag{4.8}$$

In addition, if $\Omega := \{x \in R^n \,|\, \langle A_i, x \rangle \le a_i, \, i = 1, \ldots, m\}$, and $\Omega_i := \{x \in \Omega \,|\, \langle A_i, x \rangle = a_i\}$, then

$$n \int_{\Omega} \log f \, d\omega \;+\; q \times \mathrm{vol}(\Omega) \;=\; \sum_{i=1}^{m} \frac{a_i}{\|A_i\|} \int_{\Omega_i} \log f \, d\sigma. \tag{4.9}$$

Proof. The proof is again the same as for the exponential. Note that with the vector field $X := \sum_j x_i \partial/\partial x_j$, we have $X \log f = q$ and $\mathrm{div}(X) = n$, so that (2.5) reduces to (4.9). $\square$

Hence, integrating $\log f$ on a convex polytope Ω reduces to integrating $\log f$ on the boundary $\partial\Omega$ and to computing the volume of the polyhedron.

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