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DIMENSION OF A MINIMAL NILPOTENT ORBIT

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ABSTRACT. We show that the dimension of the minimal nilpotent coadjoint orbit for a complex simple Lie algebra is equal to twice the dual Coxeter number minus two.

Let $\mathfrak{g}$ be a finite dimensional complex simple Lie algebra. We fix a Cartan subalgebra $\mathfrak{h}$, a root system $\Delta \subset \mathfrak{h}^*$ and a set of positive roots $\Delta_+ \subset \Delta$. Let ρ be half the sum of all positive roots. Denote by θ the highest root and normalize the Killing form

$$(,) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$$

by the condition $(\theta, \theta) = 2$. The dual Coxeter number $h^\vee$ can be defined as $h^\vee = (\rho, \theta) + 1$ (cf. [K]). This intrinsic number of the Lie algebra $\mathfrak{g}$ plays an important role in representation theory (cf. e.g. [K]).

As is well known there exists a unique nonzero nilpotent (co)adjoint orbit of minimal dimension. A coadjoint orbit can be identified with an adjoint one by means of the Killing form. For more detail on the nilpotent orbits, we refer to the excellent exposition [CM] and the references therein. Our result of this short note is the following theorem.

Theorem 1. *The dimension of the minimal nonzero nilpotent orbit equals* $2h^\vee - 2$.

We start with the following well-known lemma; cf., for example, Lemma 4.3.5, [CM].

Lemma 1. *The dimension of the minimal nonzero nilpotent orbit is equal to one plus the number of positive roots not orthogonal to* θ.

We call a root α in Δ_+ *special* if $\theta - \alpha$ is also a root. The subset of special roots, denoted by $\mathbb{S}$, was singled out in [KW, W] for some other purposes. It is easy to see that we can also define the set $\mathbb{S}$ equivalently as follows.

Lemma 2. *The set* $\mathbb{S}$ *is characterized by the property:* $r_\theta(\alpha) = \alpha - \theta$, *if* $\alpha \in \mathbb{S}$; $r_\theta(\alpha) = \alpha$, *if* $\alpha \in \Delta_+ - (\mathbb{S} \cup \{\theta\})$. *In other words,* $\mathbb{S} \cup \{\theta\}$ *is the set of positive roots not orthogonal to* θ.

The following lemma is taken from [KW, W]. The simple proof given below follows [W].

Lemma 3. *The number of special roots is* $\#\mathbb{S} = 2(h^\vee - 2)$.

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Proof. Since $(\theta,\theta)=2$ and $(\rho,\theta)=h^\vee-1$, we have

$$r_\theta\rho=\rho-\frac{2(\rho,\theta)}{(\theta,\theta)}\theta=\rho-(h^\vee-1)\theta. \tag{1}$$

On the other hand, it follows from Lemma 2 that

$$\begin{aligned} r_\theta\rho &= r_\theta\left(\frac{1}{2}\sum_{\alpha\in\Delta_+}\alpha\right)\\ &= \frac{1}{2}\left(\sum_{\theta\neq\alpha\in\Delta_+} r_\theta(\alpha)-\theta\right)\\ &= \frac{1}{2}\left(\sum_{\theta\neq\alpha\in\Delta_+}\alpha-(\#\mathbb{S})\theta-\theta\right)\\ &= \rho-\frac{1}{2}\left(\#\mathbb{S}+2\right)\theta. \qquad (2)\end{aligned}$$

Thus this lemma follows by comparing the right hand sides of the equations (1) and (2). ◻

By combining Lemmas 1, 2 and 3, we prove our theorem. We have an immediate corollary from Lemmas 2 and 3.

Corollary 1. *The length of the reflection r_θ is $l\left(r_\theta\right)=2h^\vee-3$.*

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