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THE MULTIDIMENSIONAL p-ADIC GREEN FUNCTION

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Abstract. A proof much simpler than the one given by Bikulov (*Investigation of the p-adic Green function*, Teoret. Mat. Fiz. **87** (1991), 376–390) for properties of the 2-dimensional p-adic Green function is shown. By this method one can treat a multidimensional case, and some sharp estimates are obtained.

1. Introduction

In recent years there has been a growing interest in p-adic analysis (see for instance [1]-[8]). In [8] by V. S. Vladimirov, P. V. Volovich and E. I. Zelenov and in [1] by A. Kh. Bikulov, some properties of 1-dimensional and 2-dimensional p-adic Green functions were studied.

In this note, a proof much simpler than the one in [1] is shown. In this way a multidimensional case with some sharp estimates is proved.

Let us recall here the definition of the p-adic Green function. As for the classical Green function on the field of real numbers (even on a Riemannian manifold), the p-adic Green function can be defined as the solution of the equation

$$(\Delta_p + m^2)G(x) = \delta(x), \tag{*}$$

where m is a positive real, $\delta(x)$ is the Dirac function, Δ_p is an operator defined by

$$(\Delta_p\varphi)(x) = \int_{Q_p^n} |(y,y)|_p \tilde{\varphi}(y)\chi((x,y))dy, \tag{**}$$

where $y = (y_1, \dots, y_n) \in Q_p^n$, $|(y,y)|_p = |y_1^2 + \dots + y_n^2|_p$, $|.|_p$ is the p-adic norm ([7], [8]), $(x,y) = x_1y_1 + \dots + x_ny_n$, $\varphi(x)$ is the test function on $Q_p^n(\varphi(x) \in \mathcal{D}(Q_p^n))$, $\tilde{\varphi}(y)$ denotes the p-adic Fourier transform of $\varphi(x), \chi(x)$ is the additive character on Q_p, and dy denotes the Haar measure on Q_p^n (see [1], [2], [3], [8]).

By applying the p-adic Fourier transform to equality (*), from equalities (*), (**) we get

$$(|(y,y)|_p + m^2)\tilde{G}(-y) = \tilde{\delta}(-y) = 1,$$

and the p-adic inverse Fourier transform gives an equivalent definition of the p-adic

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Green function as follows:

$$G(x) = \int_{Q_p^n} \frac{\chi((x,y))dy}{|(y,y)|_p + m^2},$$

which we use in the present paper.

Note that even though the p-adic Green function can be defined in the usual way, its properties are essentially different from respective properties of the classical Green one. This statement can be easily seen by the theorems in the following sections.

2. 2-Dimensional case

In this section we study properties of the 2-dimensional p-adic Green function

$$G(x) = \int_{Q_p^2} \frac{\chi((x,y))dy}{|(y,y)|_p + m^2},$$

where m is a positive real.

If we set $z = (x,x)$, then $|z|_p = |x_1^2 + x_2^2|_p$ and we obtain

Theorem 1. *The Green function $G(x)$ has the following properties: For $p \equiv 3 \pmod 4$*

i) $G(x) = (1 - \frac{1}{p^2})\sum_{k=0}^{+\infty} \frac{p^{-2k}}{m^2|z|_p + p^{-2k}} - \frac{1}{m^2|z|_p + p^2}$, for $z \neq 0$;

ii) $G(x) = G(|z|_p) > 0$, $\forall z \neq 0$;

iii)

$$\frac{1}{p^2}\ln\frac{m^2|z|_p + 1}{m^2|z|_p} + \frac{p^2-1}{p^2}\frac{1}{m^2|z|_p + 1} - \frac{1}{m^2|z|_p + p^2} \leq G(|z|_p) \leq \ln\frac{m^2|z|_p + 1}{m^2|z|_p} - \frac{1}{m^2|z|_p + p^2}.$$

Proof. i) Since $p \equiv 3 \pmod 4$ we get $|z|_p = |x_1^2 + x_2^2|_p = \max(|x_1|_p^2, |x_2|_p^2)$ (see [8]), so from $x \neq 0$, it implies that $z \neq 0$ and

$$G(|z|_p) = \sum_{\gamma=-\infty}^{+\infty} \int_{|y|_p = p^\gamma} \frac{1}{p^{2\gamma} + m^2}\chi((x,y))dy$$
$$= \sum_{\gamma-\infty}^{+\infty} \frac{1}{p^{2\gamma} + m^2} \cdot \begin{cases} p^{2\gamma}(1 - \frac{1}{p^2}), & \text{for } |x|_p \leq p^{-\gamma}, \\ -p^{2(\gamma-1)}, & \text{for } |x|_p = p^{-\gamma+1}, \\ 0, & \text{for } |x|_p \geq p^{-\gamma+2}. \end{cases}$$

By $x \neq 0$, we set $|x|_p = p^N$, so $|z|_p = p^{2N}$, and

$$G(|z|_p) = \sum_{\gamma=-\infty}^{+\infty} \frac{1}{p^{2\gamma} + m^2} \cdot \begin{cases} p^{2\gamma}(1 - \frac{1}{p^2}), & \text{for } \gamma \leq -N, \\ -p^{2(\gamma-1)}, & \text{for } \gamma = -N+1, \\ 0, & \text{for } \gamma \geq -N+2. \end{cases}$$
$$= \sum_{\gamma=-\infty}^{-N} \frac{1}{p^{2\gamma} + m^2} p^{2\gamma}(1 - \frac{1}{p^2}) - \frac{p^{-2N}}{p^{2(-N+1)} + m^2}.$$

Setting $k = -\gamma - N$ and $p^{2N} = |z|_p$ we get

$$G(|z|_p) = (1 - \frac{1}{p^2}) \sum_{k=0}^{+\infty} \frac{p^{-2k}}{m^2|z|_p + p^{-2k}} - \frac{1}{m^2|z|_p + p^2}.$$

□

ii) It is easy to see

$$G(|z|_p) = \sum_{k=0}^{+\infty} \frac{p^{-2k}}{m^2|z|_p + p^{-2k}} - \sum_{k=0}^{+\infty} \frac{p^{-2k-2}}{m^2|z|_p + p^{-2k}} - \frac{p^0}{m^2|z|_p + p^0.p^2}$$

or

$$G(|z|_p) = \sum_{k=0}^{+\infty} p^{-2k} \left[\frac{1}{m^2|z|_p + p^{-2k}} - \frac{1}{m^2|z|_p + p^{-2k}p^2} \right]$$

$$= \sum_{k=0}^{+\infty} \frac{(p^2-1)p^{-4k}}{(m^2|z|_p + p^{-2k})(m^2|z|_p + p^{-2k}p^2)} > 0, \ \forall z \neq 0.$$

iii) Set $m^2|z|_p = a > 0, p^{-2} = q$. Obviously $0 < q < 1$ and from i) of the theorem

(1) $$G(|z|_p) = \sum_{k=0}^{+\infty} \frac{1}{a + q^k}(q^k - q^{k+1}) - \frac{1}{a + p^2}$$

$$= -\sum_{k=0}^{+\infty} \frac{1}{a + q^k} \int_k^{k+1} d(q^x) - \frac{1}{a + p^2}.$$

By $0 < q < 1$

$$-\int_k^{k+1} d(q^x) = -\int_k^{k+1} q^x \ln q dx > 0, \ \frac{1}{a + q^k} \leq \frac{1}{a + q^x}, \ \frac{1}{a + q^k} \geq \frac{1}{a + q^{x-1}}.$$

So

$$-\frac{1}{a + q^k} \int_k^{k+1} q^x \ln q dx \leq -\int_k^{k+1} \frac{q^x \ln q dx}{a + q^x}.$$

In view of (1) we get

$$G(|z|_p) \leq -\sum_{k=0}^{\infty} \int_k^{k+1} \frac{q^x \ln q dx}{a + q^x} - \frac{1}{a + p^2}$$

$$= \ln \frac{m^2|z|_p + 1}{m^2|z|_p} - \frac{1}{m^2|z|_p + p^2},$$

that is the second inequality in iii) is proved.

For proving the first inequality in iii) we do as follows:

$$G(|z|_p) = (1 - q) \sum_{k=0}^{+\infty} \frac{q^k}{a + q^k} - \frac{1}{a + p^2}$$

$$\geq \frac{1 - q}{a + 1} - \frac{1}{a + p^2} - q \int_0^{\infty} \frac{q^t \ln q dt}{a + q^t}$$

$$= \frac{1 - p^{-2}}{m^2|z|_q + 1} - \frac{1}{m^2|z|_p + p^2} + \frac{1}{p^2} \ln \frac{a + 1}{a}. \quad \square$$

Remark. The estimates in iii) can be made sharper as follows:

$$\begin{aligned} G(|z|_p) &= -\sum_{k=0}^{+\infty} \frac{1}{a+q^k} \int_k^{k+1} q^x \ln q dx - \frac{1}{a+p^2} \\ &= \frac{1-q}{a+1} - \frac{1}{a+p^2} - \sum_{k=1}^{\infty} \frac{1}{a+q^k} \int_k^{k+1} q^x \ln q dx \\ &< \frac{1-q}{a+1} - \frac{1}{a+p^2} - \sum_{k=1}^{\infty} \int_k^{k+1} \frac{q^x \ln q}{a+q^x} dx \\ &= \frac{1-q}{a+1} - \frac{1}{a+p^2} - \int_1^{\infty} \frac{q^x \ln q dx}{a+q^x} \\ &= \frac{1-p^{-2}}{a+1} - \frac{1}{a+p^2} + \ln \frac{a+p^{-2}}{a}. \end{aligned}$$

For the inequality $>$, obviously

$$G(|z|_p) > \frac{1-q}{a+1} + \frac{1-q}{a+q} - \frac{1}{a+p^2} + \frac{1}{p^2} \ln \frac{a+q}{a}.$$

Theorem 2. *For $p \equiv 1 \pmod 4, p$ is a prime number, we have*

$$\begin{aligned} G(x) &= G(|z|_p) \\ &= \sum_{n=0}^{+\infty} (1-\frac{1}{p}) p^{-n} \left[\frac{(n+1)(1-\frac{1}{p})}{p^{-n}+m^2|z|_p} - \frac{2}{p^{-n}p+m^2|z|_p} \right] + \frac{1}{p^2+m^2|z|_p}, \end{aligned}$$

where

$$G(x) = \int\int_{Q_p^2} \frac{\chi((x,y))dy}{|y_1^2+y_2^2|_p+m^2}.$$

Proof. Since $p \equiv 1 \pmod 4$ there exists $\tau \in Q_p$ such that $\tau^2 = -1$. Let us denote $\tau = \sqrt{-1}$. We get

$$y_1^2 + y_2^2 = (y_1 + \tau y_2)(y_1 - \tau y_2).$$

Setting

$$t = y_1 + \tau y_2, \;\; \overline{t} = y_1 - \tau y_2$$

we have $dy_1 dy_2 = dt d\overline{t}$ and $(x,y) = at + \overline{a}\overline{t}$ with $a = \frac{1}{2}(x_1 + \frac{1}{\tau}x_2),\ \overline{a} = \frac{1}{2}(x_1 - \frac{1}{\tau}x_2)$.

So $|a\overline{a}| = |z|_p$, and

$$\begin{aligned} G(x) &= \int_{Q_p^2} \frac{\chi(at)\chi(\overline{a}\overline{t})dtd\overline{t}}{|t|_p|\overline{t}|_p+m^2} \\ &= \lim_{N\to+\infty} \int_{|\overline{t}|\le p^N} \chi(\overline{a}\overline{t})d\overline{t} \int_{|t|_p\le p^N} \frac{1}{|t|_p|\overline{t}|_p+m^2} \chi(at)dt. \end{aligned} \tag{2}$$

Setting $|a|_p = p^k$, with sufficiently large N such that $|a|_p > p^{-N}$, i.e. $-k < N$, we obtain

$$
\begin{aligned}
I_1 &= \int_{|t|_p \le p^N} \frac{1}{|t|_p|\bar{t}|_p + m^2}\chi(at) \\
&= \sum_{\gamma=-\infty}^{N} \frac{1}{p^\gamma|\bar{t}|_p + m^2} \cdot \begin{cases} (1-\frac{1}{p})p^\gamma, & \gamma \le -k, \\ -p^{\gamma-1}, & \gamma = -k+1, \\ 0, & \gamma \ge -k+2, \end{cases} \\
&= \sum_{\gamma=-\infty}^{-k} (1-\frac{1}{p})p^\gamma \frac{1}{p^\gamma|\bar{t}|_p + m^2} - \frac{p^{-k}}{p^{-k+1}|\bar{t}|_p + m^2}.
\end{aligned} \tag{3}
$$

Obviously (2), (3) with $|\bar{t}|_p = p^\beta$, $|\overline{a}|_p = p^h > p^{-N}$ give

$$
\begin{aligned}
G(x) &= \sum_{\gamma=k}^{+\infty}\sum_{\beta=h}^{+\infty}(1-\frac{1}{p})^2 \frac{p^{-(\gamma+\beta)}}{p^{-(\gamma+\beta)} + m^2} \\
&\quad + \sum_{\gamma=k}^{+\infty}(\frac{1}{p}-1)\frac{p^{-\gamma-h}}{p^{-\gamma-h+1}+m^2} \\
&\quad + \sum_{k=h}^{+\infty}(\frac{1}{p}-1)\frac{p^{-\beta-k}}{p^{-\beta-k+1}+m^2} + \frac{1}{p^2+m^2|z|_p}.
\end{aligned} \tag{4}
$$

It is not difficult to calculate the first term I_2 in (4)

$$
I_2 = \sum_{n=0}^{+\infty}(n+1)(1-\frac{1}{p})^2\frac{p^{-n}}{p^{-n}+m^2|z|_p}
$$

and to see that, in (4) the second term I_3 is equal to the third term I_4 and

$$
I_3 = I_4 = \sum_{n=0}^{+\infty}(\frac{1}{p}-1)\frac{p^{-n}}{pp^{-n}+m^2|z|_p}.
$$

Therefore

$$
\begin{aligned}
G(x) &= G(|z|_p) \\
&= \sum_{n=0}^{+\infty}\Big[\frac{(n+1)(1-\frac{1}{p})^2p^{-n}}{p^{-n}+m^2|z|_p} + \frac{2(\frac{1}{p}-1)p^{-n}}{p^{-n}p+m^2|z|_p}\Big] + \frac{1}{p^2+m^2|z|_p}.
\end{aligned}
$$

□

3. n-Dimensional case

We now study the properties of the n-dimensional p-adic Green function.

Theorem 3. *Let n be a positive interger and p be a prime number satisfying*

$$
a_1^2 + a_2^2 + \ldots + a_n^2 \not\equiv 0 \pmod{p},\ \forall a_1, a_2, \ldots, a_n \in \{1, 2, \ldots, p-1\}.
$$

Then the Green function defined by

$$
G(x) = \int_{Q_p^n} \frac{\chi((x,y))dy}{|(y,y)|_p + m^2} \tag{5}
$$

is calculated as follows:

$$G(x) = G(|z|_p) = \Big(\sum_{k=0}^{+\infty} \frac{p^{-nk}(1-\frac{1}{p^n})}{m^2|z|_p + p^{-2k}} - \frac{1}{m^2|z|_p + p^2} \Big) \frac{1}{|z|_p^{\frac{n-2}{2}}},$$

where $z = x_1^2 + \ldots + x_n^2,\ |(y,y)|_p = |y_1^2 + \ldots + y_n^2|_p, z \neq 0.$

Proof. By the assumptions on p, if $|y|_p = p^\gamma$, then $|(y,y)|_p = |y|_p^2 = p^{2\gamma}$. Consequently

$$\begin{aligned} G(x) &= \sum_{\gamma=-\infty}^{+\infty} \int_{|y|_p = p^\gamma} \frac{\chi((x,y))dy}{p^{2\gamma}+m^2} \\ &= \sum_{\gamma=-\infty}^{+\infty} \frac{1}{p^{2\gamma}+m^2} \cdot \begin{cases} p^{n\gamma}(1-\frac{1}{p^n}), & \text{for } \gamma \leq -N (|x|_p = p^N), \\ -p^{n(\gamma-1)}, & \text{for } \gamma = -N+1, \\ 0, & \text{for } \gamma \geq -N+2, \end{cases} \\ &= \sum_{\gamma=-\infty}^{-N} \frac{p^{n\gamma}(1-\frac{1}{p^n})}{p^{2\gamma}+m^2} - \frac{1}{p^{2(-N+1)}+m^2} p^{-nN} \\ &= (1-\frac{1}{p^n}) \frac{1}{|z|_p^{\frac{n-2}{2}}} \sum_{k=0}^{+\infty} \frac{p^{-nk}}{m^2|z|_p + p^{-2k}} - \frac{1}{|z|_p^{\frac{n-2}{2}}(p^2 + m^2|z|_p)}. \end{aligned}$$

□

Theorem 4. *Consider the same assumptions as in Theorem 3. Then the Green function* $G(|z|_p)$ *defined by (5) has the following properties:*

a) $G(|z|_p) > 0,\ \forall z \neq 0;$

b)

$$G(|z|_p) = \frac{p^2-1}{1-p^{-(n+2)}} \cdot \frac{1}{m^4|z|_p^{1+\frac{n}{2}}} + O(\frac{1}{|z|_p^{1+\frac{n}{2}}})$$

as $m^2|z|_p \to +\infty;$

c)

$$\frac{1}{|z|_p^{\frac{n-2}{2}}} \Big(\frac{1-q}{a+1} - \frac{1}{a+p^2} + qI_n \Big) < G(|z|_p) < \frac{1}{|z|_p^{\frac{n-2}{2}}} \Big(-\frac{1}{a+p^2} + I_n \Big),$$

where $a = m^2|z|_p > 0,\ q = \frac{1}{p^n},$

(6a) $$I_n = n\Big[\frac{1}{n-2} - \frac{a}{n-4} + \frac{a^2}{n-6} - \ldots + \frac{(-1)^{k-1}a^{k-1}}{2} \ln \frac{a+1}{a} \Big],$$

if $n = 2k$, k *is an integer* ≥ 1 *and*

(6b) $$I_n = n\Big[\frac{1}{n-2} - \frac{a}{n-4} + \frac{a^2}{n-6} - \ldots + (-1)^{k-1}a^{k-1}\frac{1}{\sqrt{a}} \arctan \frac{1}{\sqrt{a}} \Big],$$

if $n = 2k-1$, k *is an integer* ≥ 1 *(for* $k = 1$ *there are no* $n - 2j$ *terms in* I_n*).*

Proof. a) From Theorem 3 it follows that

$$G(|z|_p) = \frac{1}{|z|_p^{\frac{n-2}{2}}}\left[\sum_{k=0}^{+\infty}\frac{p^{-nk}(1-p^{-n})}{m^2|z|_p+p^{-2k}} - \frac{1}{m^2|z|_p+p^2}\right]$$

$$= \frac{1}{|z|_p^{\frac{n-2}{2}}}\left[\sum_{k=0}^{+\infty}\frac{p^{-nk}}{m^2|z|_p+p^{-2k}} - \sum_{k=1}^{+\infty}\frac{p^{-nk}}{m^2|z|_p+p^2p^{-2k}} - \frac{p^{-n.0}}{m^2|z|_p+p^2p^{-2.0}}\right].$$

Consequently

$$(7)\qquad \begin{aligned} G(|z|_p) &= \frac{1}{|z|_p^{\frac{n-2}{2}}}\sum_{k=0}^{+\infty}\left[\frac{p^{-nk}}{m^2|z|_p+p^{-2k}} - \frac{p^{-nk}}{m^2|z|_p+p^2p^{-2k}}\right] \\ &= \frac{1}{|z|_p^{\frac{n-2}{2}}}\sum_{k=0}^{+\infty}(p^2-1)\frac{p^{-(n+2)k}}{(m^2|z|_p+p^{-2k})(m^2|z|_p+p^2p^{-2k})} > 0. \end{aligned}$$

b) Setting $t = \frac{1}{m^2|z|_p}$ we have

$$(8)\qquad \begin{aligned} \frac{1}{m^2|z|_p+p^{-2k}} &= t(\frac{1}{1+p^{-2k}t}) = t(1-p^{-2k}t+\mathrm{O}(t)) \\ &= \frac{1}{m^2|z|_p} - \frac{p^{-2k}}{m^4|z|_p^2} + \mathrm{O}(\frac{1}{|z|_p^2}), \end{aligned}$$

as $m^2|z|_p \to +\infty$.

Similarly

$$(9)\qquad \frac{1}{m^2|z|_p+p^2p^{-2k}} = \frac{1}{m^2|z|_p} - \frac{p^2p^{-2k}}{m^4|z|_p^2} + \mathrm{O}(\frac{1}{|z|_p^2}).$$

By (8), (9) and (7)

$$\begin{aligned} G(|z|_p) &= \frac{1}{|z|_p^{\frac{n-2}{2}}}\sum_{k=0}^{+\infty}p^{-nk}\left[-\frac{p^{-2k}}{m^4|z|_p^2} + \frac{p^2p^{-2k}}{m^4|z|_p^2} + \mathrm{O}(\frac{1}{|z|_p^2})\right] \\ &= \frac{1}{|z|_p^{\frac{n-2}{2}}}\sum_{k=0}^{+\infty}\frac{(p^2-1)p^{-(n+2)k}}{m^4|z|_p^2} + \mathrm{O}(\frac{1}{|z|_p^{1+\frac{n}{2}}}) \\ &= \frac{1}{|z|_p^{\frac{n-2}{2}}}\cdot\frac{p^2-1}{1-p^{-(n+2)}}\cdot\frac{1}{m^4|z|_p^2} + \mathrm{O}(\frac{1}{|z|_p^{1+\frac{n}{2}}}) \\ &= \frac{p^2-1}{1-p^{-(n+2)}}\cdot\frac{1}{m^4|z|_p^{1+\frac{n}{2}}} + \mathrm{O}(\frac{1}{|z|_p^{1+\frac{n}{2}}}). \end{aligned}$$

c) By Theorem 3

$$(10)\qquad G(|z|_p) = \frac{1}{|z|_p^{\frac{n-2}{2}}}\left[\sum_{k=0}^{+\infty}\frac{p^{-nk}(1-p^{-n})}{m^2|z|_p+p^{-2k}} - \frac{1}{m^2|z|_p+p^2}\right].$$

Setting $a = m^2|z|_p > 0, q = p^{-n}$ we get

$$\begin{aligned} s &= \sum_{k=0}^{+\infty} \frac{p^{-nk}(1-p^{-n})}{m^2|z|_p + p^{-2k}} = \sum_{k=0}^{+\infty}(1-q)\frac{q^k}{a+q^{\frac{2k}{n}}} \\ &= \sum_{k=0}^{+\infty} \frac{1}{a+q^{\frac{2k}{n}}}(q^k - q^{k+1}) \\ &= -\sum_{k=0}^{+\infty} \frac{1}{a+q^{\frac{2k}{n}}} \int_k^{k+1} d(q^x). \end{aligned} \tag{11}$$

By $k \leq x \leq k+1, 0 < q < 1$

$$\frac{1}{a+q^{\frac{2(x-1)}{n}}} \leq \frac{1}{a+q^{\frac{2k}{n}}} \leq \frac{1}{a+q^{\frac{2x}{n}}}.$$

But

$$-\int_k^{k+1} d(q^x) > 0,$$

consequently

$$-\int_k^{k+1} \frac{d(q^x)}{a+q^{\frac{2(x-1)}{n}}} < -\frac{1}{a+q^{\frac{2k}{n}}}\int_k^{k+1} d(q^x) < -\int_k^{k+1} \frac{d(q^x)}{a+q^{\frac{2x}{n}}}. \tag{12}$$

By (11), (12) it is easy to see

$$\frac{1-q}{a+1} - \sum_{k=1}^{+\infty}\int_k^{k+1} \frac{d(q^x)}{a+q^{\frac{2(x-1)}{n}}} < s < -\sum_{k=0}^{+\infty}\int_k^{k+1} \frac{d(q^x)}{a+q^{\frac{2x}{n}}}. \tag{13}$$

From (13) it follows that

$$\frac{1-q}{a+1} - q\int_0^{+\infty} \frac{d(q^x)}{a+q^{\frac{2x}{n}}} < s < -\int_0^{+\infty} \frac{d(q^x)}{a+q^{\frac{2x}{n}}}. \tag{14}$$

Setting

$$I_n = -\int_0^{+\infty} \frac{d(q^x)}{a+q^{\frac{2x}{n}}}$$

we obtain

$$I_n = -\int_1^0 \frac{nt^{n-1}dt}{a+t^2} = n\int_0^1 \frac{t^{n-1}dt}{a+t^2}.$$

Let $n \geq 3$. Then

$$I_n = n\int_0^1 \Big(t^{n-3} - a\frac{t^{n-3}}{a+t^2}\Big)dt = n\Big(\frac{1}{n-2} - aI_{n-2}\Big).$$

By induction it is not difficult to prove

$$\text{(15a)} \qquad I_n = n\left[\frac{1}{n-2} - \frac{a}{n-4} + \frac{a^2}{n-6} - \ldots + \frac{(-1)^{k-1}a^{k-1}}{2}\ln\frac{a+1}{a}\right], \quad \text{if } n = 2k$$

$$\text{(15b)} \qquad I_n = n\left[\frac{1}{n-2} - \frac{a}{n-4} + \frac{a^2}{n-6} - \ldots + \frac{(-1)^{k-1}a^{k-1}}{\sqrt{a}}\arctan\frac{1}{\sqrt{a}}\right], \quad \text{if } n = 2k-1,$$

$k \geq 1$; for $k = 1$ it is assumed there are no $n - 2j$ terms in I_n.

From (15a), (15b) and (10) with

$$G(|z|_p) = \frac{1}{|z|_p^{\frac{n-2}{2}}}\left[s - \frac{1}{a+p^2}\right]$$

we get the estimates

$$\frac{1}{|z|_p^{\frac{n-2}{2}}}\left[\frac{1-q}{a+1} - \frac{1}{a+p^2} + qI_n\right] < G(|z|_p) < \frac{1}{|z|_p^{\frac{n-2}{2}}}\left[-\frac{1}{a+p^2} + I_n\right].$$

□

Corollary 1. *For $n = 1$ we obtain*

$$(1-\frac{1}{p})\frac{\sqrt{|z|_p}}{m^2|z|_p+1} - \frac{\sqrt{|z|_p}}{m^2|z|_p+p^2} + \frac{1}{mp}\arctan\frac{1}{m\sqrt{|z|_p}} < G(|z|_p)$$
$$< -\frac{\sqrt{|z|_p}}{m^2|z|_p+p^2} + \frac{1}{m}\arctan\frac{1}{m\sqrt{|z|_p}}.$$

Corollary 2. *For $n = 2$ we have the estimates iii) in Theorem 1.*

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