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A CHARACTERIZATION OF THE CLIFFORD TORUS

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Abstract. In this paper, we prove that an n-dimensional closed minimal hypersurface M with Ricci curvature $Ric(M) \geq \dfrac{n}{2}$ of a unit sphere $S^{n+1}(1)$ is isometric to a Clifford torus if $n \leq S \leq n + \frac{14(n+4)}{9n+30}$, where S is the squared norm of the second fundamental form of M.

1. Introduction

Let M be an n-dimensional closed minimal hypersurface in a unit sphere $S^{n+1}(1)$ of dimension $n+1$. Let S denote the squared norm of the second fundamental form of M. From the Gauss equation (see section 2), we know that S, which is extrinsic by definition, is actually an intrinsic quantity. It is well-known that Chern, do Carmo and Kobayashi [3] and Lawson [4] obtained independently that Clifford tori are the only closed minimal hypersurfaces of the unit sphere with $S = n$. When the scalar curvature of M is constant, Yang and the first named author proved in [6] and [7] that if $n \leq S \leq n + \frac{n}{3}$, then M is isometric to a Clifford torus $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$. A natural problem is that, for a closed minimal hypersurface M of a unit sphere, whether there exists a constant $\epsilon(n) > 0$ such that if $n \leq S \leq n + \epsilon(n)$, then $S = n$ and M is isometric to a Clifford torus $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$. The first named author [2] gave a positive answer under the additional condition that M has only two distinct principal curvatures. In general, it still remains open and it is a very hard problem. On the other hand, the Clifford torus $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$ is a closed minimal hypersurface in $S^{n+1}(1)$ with $S = n$ and its Ricci curvature varies between $\dfrac{n(m-1)}{m}$ and $\dfrac{n(n-m-1)}{n-m}$. If $2 \leq m \leq n-2$, then $Ric(M) \geq \dfrac{n}{2}$. Hence it is natural to ask

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whether there exists a constant $\epsilon(n) > 0$ such that if M is a closed minimal hypersurface with $Ric(M) \geq \frac{n}{2}$ and $n \leq S \leq n + \epsilon(n)$, then $S = n$ and M is isometric to a Clifford torus $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$ $(1 < m < n-1)$. In this paper, we give an affirmative answer for the above problem.

Theorem 1. *Let M be an n-dimensional closed minimal hypersurface of a unit sphere $S^{n+1}(1)$ with Ricci curvature $Ric(M) \geq \frac{n}{2}$. If*

$$n \leq S \leq n + \frac{14(n+4)}{9n+30},$$

then $S = n$ and M is isometric to a Clifford torus $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$ $(1 < m < n-1)$.

In particular, if $n \leq 5$, we obtain the following

Theorem 2. *Let M be an n-dimensional $(n \leq 5)$ closed minimal hypersurface of a unit sphere $S^{n+1}(1)$. If*

$$n \leq S \leq n + \epsilon(n),$$

then $S = n$ and M is isometric to a Clifford torus $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$, where $\epsilon(3) = \frac{42}{85}$, $\epsilon(4) = \frac{8}{31}$ and $\epsilon(5) = \frac{3(21-5\sqrt{17})}{28+3\sqrt{17}}$.

Remark. For $n \leq 5$, Peng-Terng [5] proved the following: Let M be an n-dimensional $(n \leq 5)$ closed minimal hypersurface of a unit sphere $S^{n+1}(1)$. If

$$n \leq S \leq n + \epsilon_1(n),$$

then $S = n$, where $\epsilon_1(n) = \dfrac{6 - 1.13n}{5 + \sqrt{17}}$. It is obvious that our pinching constant in Theorem 2 is larger than theirs.

2. Local formulae

Let M be an n-dimensional hypersurface in a unit sphere $S^{n+1}(1)$. We choose a local orthonormal frame field $\{e_1, \ldots, e_{n+1}\}$ in $S^{n+1}(1)$, restricted to M, so that $e_1, \ldots, e_n$ are tangent to M. Let $\omega_1, \ldots, \omega_{n+1}$ denote the dual coframe field in $S^{n+1}(1)$. Then, in M,

$$\omega_{n+1} = 0.$$

It follows from Cartan's Lemma that

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{2.0}$$

The second fundamental form α and the mean curvature of M are defined by

$$\alpha = \sum_{i,j} h_{ij}\omega_i\omega_j e_{n+1} \quad \text{and} \quad nH = \sum_i h_{ii}, \tag{2.1}$$

respectively. We recall that M is by definition a minimal hypersurface if the mean curvature of M is identically zero. The connection form ω_{ij} is characterized by the structure equations

$$\begin{cases} d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, \quad \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\ \Omega_{ij} = \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \end{cases} \tag{2.2}$$

where Ω_{ij} (resp. R_{ijkl}) denotes the curvature form (resp. the components of the curvature tensor) of M. The Gauss equation is given by

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}). \tag{2.3}$$

The covariant derivative $\nabla\alpha$ of the second fundamental form α of M with components h_{ijk} is given by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{jk}\omega_{ik} + \sum_k h_{ik}\omega_{jk}.$$

Then the exterior derivative of (2.0) together with the structure equation yields the following Codazzi equation:

$$h_{ijk} = h_{ikj} = h_{jik}. \tag{2.4}$$

From the Codazzi equation, we know that h_{ijk} is symmetric in the indices i, j and k. Similarly, we have the covariant derivative $\nabla^2\alpha$ of $\nabla\alpha$ with components h_{ijkl} as follows:

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_l h_{ljk}\omega_{il} + \sum_l h_{ilk}\omega_{jl} + \sum_l h_{ijl}\omega_{kl},$$

and it is easy to get the following Ricci formula:

$$h_{ijkl} - h_{ijlk} = \sum_m h_{im}R_{mjkl} + \sum_m h_{mj}R_{mikl}. \tag{2.5}$$

Similarly, we also have

$$h_{ijklm} - h_{ijkml} = \sum_r h_{rjk}R_{rilm} + \sum_r h_{irk}R_{rjlm} + \sum_r h_{ijr}R_{rklm}, \tag{2.6}$$

where the h_{ijklm}'s are the components of the covariant derivative $\nabla^3\alpha$ of $\nabla^2\alpha$. We should remark that h_{ijkl} and h_{ijklm} are symmetric in the first three indices i, j and k and generally not symmetric in the other ones. The components of the Ricci curvature and the scalar curvature are given by

$$R_{ij} = (n-1)\delta_{ij} - \sum_k h_{ik}h_{jk}, \tag{2.7}$$

$$R = n(n-1) - \sum_{i,j} h_{ij}^2. \tag{2.8}$$

Now we compute certain local formulae. For any fixed point p in M, we can choose a local orthonormal frame field $e_1, \ldots, e_n$ such that

$$h_{ij} = \begin{cases} 0 & \text{if} \quad i \neq j, \\ \lambda_i & \text{if} \quad i = j. \end{cases} \tag{2.9}$$

The following formulas can be obtained by a direct computation (cf. [1]). Let

$$S := \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2,$$

$$\frac{1}{2}\Delta S = \sum_{i,j,k} h_{ijk}^2 - S(S-n), \tag{2.10}$$

$$\begin{aligned}\frac{1}{2}\Delta \sum_{i,j,k} h_{ijk}^2 = \sum_{i,j,k,l} h_{ijkl}^2 + (2n+3-S)\sum_{i,j,k} h_{ijk}^2 \\ + 3(2B-A) - \frac{3}{2}|\nabla S|^2,\end{aligned} \tag{2.11}$$

where $A = \sum_{i,j,k} \lambda_i^2 h_{ijk}^2$ and $B = \sum_{i,j,k} \lambda_i \lambda_j h_{ijk}^2$.

3. Proofs of the theorems

At first we give two algebraic lemmas which will play a crucial role in the proofs of our theorems.

Lemma 1. *Let a_i $(i = 1, 2, 3, 4)$ be real numbers satisfying $\sum_i a_i = 0$ and $\sum_i a_i^2 = a$. Then $\sum_i a_i^4 \le \frac{7}{12}a^2$.*

Proof. We maximize the function $\sum_i a_i^4$ with the constraints $\sum_i a_i = 0$ and $\sum_i a_i^2 = a$. By means of the method of the Lagrange multiplier, we solve the following problem:

$$f = \sum_i a_i^4 + \lambda \sum_i a_i + \mu(\sum_i a_i^2 - a),$$

where λ and μ are the Lagrange multipliers. The maximum point of $\sum_i a_i^4$ is a critical point of f. Taking the derivative of f with respect to a_i, we have

$$f_{a_i} = 4a_i^3 + \lambda + 2\mu a_i = 0.$$

Hence, at most three of the a_i's are distinct with each other at a critical point of f. We consider the following three cases.

(1) Three of the a_i's are distinct with each other. Without loss of generality, we denote them by a_1, a_2, a_3 and assume $a_1 = a_4$; then

$$2a_1 + a_2 + a_3 = 0, \quad 2a_1^2 + a_2^2 + a_3^2 = a.$$

Hence,

$$\begin{aligned}\sum_i a_i^4 &= 2a_1^4 + a_2^4 + a_3^4 = \frac{(a+2a_1^2)^2}{2} - 14a_1^4 \\ &= \frac{a^2}{2} + 2aa_1^2 - 12a_1^4 \le \frac{7}{12}a^2,\end{aligned}$$

i.e.,

$$\sum_i a_i^4 \le \frac{7}{12}a^2.$$

(2) Two of the a_i's are distinct with each other. Without loss of generality, we denote them by a_1, a_2 and assume $a_1 = a_4$ and $a_2 = a_3$ or $a_1 = a_3 = a_4$; then $\sum_i a_i^4 \le \frac{7}{12}a^2$.

(3) If all of the a_i's are the same, then $\sum_i a_i^4 = 0$.

Therefore, we conclude

$$\sum_i a_i^4 \leq \frac{7}{12} a^2.$$

This completes the proof of Lemma 1. $\square$

Lemma 2. *Let a_{ij} and b_i $(i, j = 1, \dots, n)$ be real numbers satisfying $\sum_i b_i = 0$, $\sum_i b_i^2 = b > 0$, $\sum_{i,j} b_i a_{ij} = \frac{1}{2} b(n-b)$ and $\sum_{i,j} b_j a_{ij} = \frac{1}{2} b(n-b)$. Then*

$$\sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2 \geq \frac{3b(n-b)^2}{2(n+4)}.$$

Proof. We consider $F = \sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2$ as a function of a_{ij} with constraints $\sum_{i,j} b_i a_{ij} = \frac{1}{2} b(n-b)$ and $\sum_{i,j} b_j a_{ij} = \frac{1}{2} b(n-b)$. Let

$$f := \sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2 + \lambda [\sum_{i,j} b_i a_{ij} - \frac{1}{2} b(n-b)] + \mu [\sum_{i,j} b_j a_{ij} - \frac{1}{2} b(n-b)],$$

where λ and μ are the Lagrange multipliers. It is obvious that the minimum point of F is a critical point of f. Taking the derivative of f with respect to a_{ij}, we get

$$f_{a_{ii}} = 2a_{ii} + \lambda b_i + \mu b_i = 0, \quad \text{for} \quad i, \tag{3.1}$$

$$f_{a_{ij}} = 6a_{ij} + \lambda b_i + \mu b_j = 0, \quad \text{for} \quad i \neq j. \tag{3.2}$$

Hence

$$\sum_i a_{ii} f_{a_{ii}} = 2 \sum_i a_{ii}^2 + \lambda \sum_i a_{ii} b_i + \mu \sum_i a_{ii} b_i = 0$$

and

$$\sum_{i \neq j} a_{ij} f_{a_{ij}} = 6 \sum_{i \neq j} a_{ij}^2 + \lambda \sum_{i \neq j} a_{ij} b_i + \mu \sum_{i \neq j} a_{ij} b_j = 0.$$

Therefore,

$$2[\sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2] = \lambda \frac{1}{2} b(b-n) + \mu \frac{1}{2} b(b-n). \tag{3.3}$$

From (3.1) and (3.2), we have

$$2 \sum_i b_i a_{ii} + (\lambda + \mu) \sum_i b_i^2 = 0, \tag{3.4}$$

$$6 \sum_{i \neq j} b_i a_{ij} + \lambda \sum_{i \neq j} b_i^2 + \mu \sum_{i \neq j} b_i b_j = 0$$

and

$$6 \sum_{i \neq j} b_j a_{ij} + \lambda \sum_{i \neq j} b_i b_j + \mu \sum_{i \neq j} b_j^2 = 0.$$

From (3.4) and the two equalities above, we get

$$-4 \sum_i b_i a_{ii} + 3b(n-b) + \lambda n b = 0$$

and

$$-4\sum_i b_i a_{ii} + 3b(n-b) + \mu nb = 0,$$

$$\lambda + \mu = \frac{6(b-n)}{(n+4)}.$$

According to (3.3), we obtain

$$f_{min} = \frac{3b(n-b)^2}{2(n+4)}.$$

Thus we have finished the proof of Lemma 2. □

For any fixed point p in M, we can choose a local frame field $e_1, \dots, e_n$ such that

$$h_{ij} = \lambda_i \delta_{ij}. \tag{3.5}$$

Defining $f_3 = \sum_i \lambda_i^3$ and $f_4 = \sum_i \lambda_i^4$, then f_3 and f_4 are functions defined globally on M.

Proposition 1. *Let M be a minimal hypersurface in $S^{n+1}(1)$. Then*

$$\sum_{i,j,k,l} h_{ijkl}^2 \geq \frac{3}{2}(Sf_4 - f_3^2 - 2S^2 + nS) + \frac{3S(S-n)^2}{2(n+4)}$$

holds.

Proof. From the Ricci formula (2.5) and the Gauss equation (2.3), we have

$$\begin{aligned} h_{iijj} - h_{jjii} = h_{ijij} - h_{ijji} &= \sum_m h_{im} R_{mjij} + \sum_m h_{mj} R_{miij} \\ &= \lambda_i R_{ijij} + \lambda_j R_{jiij} = (\lambda_i - \lambda_j) R_{ijij} \\ &= (\lambda_i - \lambda_j)(1 + \lambda_i \lambda_j). \end{aligned} \tag{3.6}$$

We define

$$u_{ijkl} = \frac{1}{4}(h_{ijkl} + h_{lijk} + h_{klij} + h_{jkli}). \tag{3.7}$$

Since h_{ijkl} is symmetric in the indices i, j, k, from formula (3.6), we obtain

$$\sum_{i,j,k,l} h_{ijkl}^2 \geq \sum_{i,j,k,l} u_{ijkl}^2 + \frac{3}{2}[Sf_4 - f_3^2 - 2S^2 + nS]. \tag{3.8}$$

Since $\Delta h_{ij} = (n-S)h_{ij}$ and $\sum_i h_{iikl} = 0$, we have

$$\sum_{i,j} u_{iijj} \lambda_i = \sum_{i,j} u_{iijj} \lambda_j = \frac{1}{2} S(n-S).$$

From $\sum_i \lambda_i = 0$ and $\sum_i \lambda_i^2 = S$ and defining $a_{ij} := u_{iijj}$ and $b_i := \lambda_i$, then a_{ij} and b_i satisfy the conditions in Lemma 2. From the definition of u_{ijkl}, we know that u_{ijkl} is symmetric in the indices i, j, k, l. From Lemma 2, we infer

$$\sum_{i,j,k,l} u_{ijkl}^2 \geq \sum_i u_{iiii}^2 + 3\sum_{i \neq j} u_{iijj}^2 \geq \frac{3S(S-n)^2}{2(n+4)}. \tag{3.9}$$

Hence, from (3.8) and (3.9), we obtain

$$\sum_{i,j,k,l} h_{ijkl}^2 \geq \frac{3}{2}(Sf_4 - f_3^2 - 2S^2 + nS) + \frac{3S(S-n)^2}{2(n+4)}.$$

This completes the proof of Proposition 1. □

Proposition 2. *Let M be a closed minimal hypersurface in $S^{n+1}(1)$. Then*

$$\int_M [(S - 2n - \frac{3}{2}) \sum_{i,j,k} h_{ijk}^2 + 2(S-n)f_4 - \frac{3S(S-n)^2}{2(n+4)} + \frac{9}{8}|\nabla S|^2] dM \geq 0$$

holds.

Proof. The following integral formula (3.10) can be found in [2]:

$$\int_M (A - 2B) dM = \int_M [Sf_4 - S^2 - f_3^2 - \frac{1}{4}|\nabla S|^2] dM. \tag{3.10}$$

From the Ricci formula (2.5), by a direct computation, we obtain

$$\frac{1}{4}\Delta f_4 = (n - S)f_4 + 2A + B.$$

Integrating both sides of the above equality, we have

$$\int_M (S-n) f_4 dM = \int_M (2A + B) dM. \tag{3.11}$$

Formulas (3.10) and (3.11) yield

$$\int_M [(S - 4n)f_4 + 3f_3^2 + 3S^2 + \frac{3}{4}|\nabla S|^2] dM \geq 0. \tag{3.12}$$

According to Stokes' formula, we integrate the formula (2.11) and obtain

$$\begin{aligned} \int_M \sum_{i,j,k,l} h_{ijkl}^2 dM \\ = \int_M [-(2n + 3 - S) \sum_{i,j,k} h_{ijk}^2 - 3(2B - A) + \frac{3}{2}|\nabla S|^2] dM. \end{aligned} \tag{3.13}$$

From Proposition 1, (3.10) and (3.13), we infer

(3.14)

$$\int_M \{(S - 2n - \frac{3}{2}) \sum_{i,j,k} h_{ijk}^2 + \frac{3}{4}|\nabla S|^2 + \frac{3}{2}[Sf_4 - f_3^2 - S^2] - \frac{3S(n-S)^2}{2(n+4)}\} dM \geq 0.$$

(3.12)+2 × (3.14) yields

$$\int_M [(S - 2n - \frac{3}{2}) \sum_{i,j,k} h_{ijk}^2 + 2(S-n)f_4 - \frac{3S(S-n)^2}{2(n+4)} + \frac{9}{8}|\nabla S|^2] dM \geq 0.$$

Thus Proposition 2 is valid. □

Proof of Theorem 1. According to (2.10) and Stokes' Theorem, we obtain

$$\int_M \sum_{i,j,k} h_{ijk}^2 dM = \int_M [S(S-n)] dM \tag{3.15}$$

and

$$-\int_M \frac{1}{2}|\nabla S|^2 = \int_M [S\sum_{i,j,k} h_{ijk}^2 + (n-S)S^2]dM. \tag{3.16}$$

From formula (2.7) and the assumption in Theorem 1, we have

$$R_{ii} = n-1-\lambda_i^2 \geq \frac{n}{2}.$$

Therefore,

$$\lambda_i^2 \leq \frac{n-2}{2},$$

$$\sum_i \lambda_i^4 \leq \frac{n-2}{2}\sum_i \lambda_i^2,$$

that is,

$$f_4 \leq \frac{n-2}{2}S. \tag{3.17}$$

From Proposition 2 and (3.17), we have

$$\int_M \{(S-2n-\frac{3}{2})\sum_{i,j,k} h_{ijk}^2 + (n-2)S(S-n) - \frac{3S(S-n)^2}{2(n+4)} + \frac{9}{8}|\nabla S|^2\}dM \geq 0.$$

From (3.15), (3.16) and the above inequality, we infer

$$\int_M \{(-\frac{5}{4}S-n-\frac{7}{2})\sum_{i,j,k} h_{ijk}^2 + [\frac{9}{4}S - \frac{3(S-n)}{2(n+4)}]S(S-n)\}dM \geq 0.$$

Since

$$n \leq S \leq n + \frac{14(n+4)}{9n+30},$$

we have

$$\int_M \{(-\frac{5}{4}S-n-\frac{7}{2})\sum_{i,j,k} h_{ijk}^2 + (\frac{9n}{4}+\frac{7}{2})S(S-n)\}dM \geq 0.$$

Hence

$$\int_M \frac{5}{4}(S-n)\sum_{i,j,k} h_{ijk}^2 dM = 0.$$

Since S and $\sum_{i,j,k} h_{ijk}^2$ are continuous functions, we have $S = n$. Thus, M is isometric to a Clifford torus $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$ $(1 < m < n-1)$ from the result due to Chern, do Carmo and Kobayashi [3] or Lawson [4]. This completes the proof of Theorem 1. □

Proof of Theorem 2. In the case $n = 3$, because $\sum_i \lambda_i = 0$, we have $f_4 = \sum_i \lambda_i^4 = \frac{S^2}{2}$. From Proposition 2, we have

$$\int_M \{(S-2n-\frac{3}{2})\sum_{i,j,k} h_{ijk}^2 + S^2(S-n) - \frac{3S(S-n)^2}{2(n+4)} + \frac{9}{8}|\nabla S|^2\}dM \geq 0.$$

From (3.16) and the above inequality, we have

$$\int_M \{(-\frac{5}{4}S - 2n - \frac{3}{2})\sum_{i,j,k} h_{ijk}^2 + \frac{13}{4}S^2(S-n) - \frac{3S(S-n)^2}{2(n+4)}\}dM \geq 0.$$

Since

$$n \leq S \leq n + \frac{42}{85},$$

we have

$$\int_M \{-\frac{5}{4}(S-n)\sum_{i,j,k} h_{ijk}^2\}dM \geq 0.$$

Hence

$$\int_M \frac{5}{4}(S-n)\sum_{i,j,k} h_{ijk}^2 dM = 0.$$

By making use of the same proof as in Theorem 1, we know that Theorem 2 is true in the case $n = 3$.

In the case $n = 4$, from Lemma 1, we have $f_4 \leq \frac{7}{12}S^2$. By using this inequality, we obtain, from Proposition 2,

$$\int_M \{(S - 2n - \frac{3}{2})\sum_{i,j,k} h_{ijk}^2 + \frac{7}{6}S^2(S-n) - \frac{3S(S-n)^2}{2(n+4)} + \frac{9}{8}|\nabla S|^2\}dM \geq 0.$$

By the same proof as in the case $n = 3$, we know that Theorem 2 is also valid in the case $n = 4$.

In the case $n = 5$, from Proposition 1, (3.10) and (3.13), we have

$$\int_M \{(S - 2n - \frac{3}{2})\sum_{i,j,k} h_{ijk}^2 + \frac{3}{2}(A - 2B) - \frac{3S(S-n)^2}{2(n+4)} + \frac{9}{8}|\nabla S|^2\}dM \geq 0. \tag{3.18}$$

Since

$$\begin{aligned}
3(A-2B) &= \sum_{i,j,k}(\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2\lambda_i\lambda_j - 2\lambda_j\lambda_k - 2\lambda_k\lambda_i)h_{ijk}^2 \\
&= \sum_{i\neq j\neq k\neq i}[2(\lambda_i^2 + \lambda_j^2 + \lambda_k^2) - (\lambda_i + \lambda_j + \lambda_k)^2]h_{ijk}^2 \\
&\quad + 3\sum_{i\neq k}(\lambda_k^2 - 4\lambda_i\lambda_k)h_{iik}^2 - 3\sum_i \lambda_i^2 h_{iii}^2 \\
&\leq \frac{\sqrt{17}+1}{2}S\sum_{i,j,k} h_{ijk}^2,
\end{aligned}$$

by making use of this inequality and (3.18), a similar proof as in the case $n = 3$ yields that Theorem 2 is also valid in this case. We have finished the proof of Theorem 2. □

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