

## A CHARACTERIZATION OF THE CLIFFORD TORUS

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ABSTRACT. In this paper, we prove that an  $n$ -dimensional closed minimal hypersurface  $M$  with Ricci curvature  $Ric(M) \geq \frac{n}{2}$  of a unit sphere  $S^{n+1}(1)$  is isometric to a Clifford torus if  $n \leq S \leq n + \frac{14(n+4)}{9n+30}$ , where  $S$  is the squared norm of the second fundamental form of  $M$ .

### 1. INTRODUCTION

Let  $M$  be an  $n$ -dimensional closed minimal hypersurface in a unit sphere  $S^{n+1}(1)$  of dimension  $n + 1$ . Let  $S$  denote the squared norm of the second fundamental form of  $M$ . From the Gauss equation (see section 2), we know that  $S$ , which is extrinsic by definition, is actually an intrinsic quantity. It is well-known that Chern, do Carmo and Kobayashi [3] and Lawson [4] obtained independently that Clifford tori are the only closed minimal hypersurfaces of the unit sphere with  $S = n$ . When the scalar curvature of  $M$  is constant, Yang and the first named author proved in [6] and [7] that if  $n \leq S \leq n + \frac{n}{3}$ , then  $M$  is isometric to a Clifford torus  $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$ . A natural problem is that, for a closed minimal hypersurface  $M$  of a unit sphere, whether there exists a constant  $\epsilon(n) > 0$  such that if  $n \leq S \leq n + \epsilon(n)$ , then  $S = n$  and  $M$  is isometric to a Clifford torus  $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$ . The first named author [2] gave a positive answer under the additional condition that  $M$  has only two distinct principal curvatures. In general, it still remains open and it is a very hard problem. On the other hand, the Clifford torus  $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$  is a closed minimal hypersurface in  $S^{n+1}(1)$  with  $S = n$  and its Ricci curvature varies between  $\frac{n(m-1)}{m}$  and  $\frac{n(n-m-1)}{n-m}$ . If  $2 \leq m \leq n-2$ , then  $Ric(M) \geq \frac{n}{2}$ . Hence it is natural to ask

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whether there exists a constant  $\epsilon(n) > 0$  such that if  $M$  is a closed minimal hypersurface with  $\text{Ric}(M) \geq \frac{n}{2}$  and  $n \leq S \leq n + \epsilon(n)$ , then  $S = n$  and  $M$  is isometric to a Clifford torus  $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$  ( $1 < m < n - 1$ ). In this paper, we give an affirmative answer for the above problem.

**Theorem 1.** *Let  $M$  be an  $n$ -dimensional closed minimal hypersurface of a unit sphere  $S^{n+1}(1)$  with Ricci curvature  $\text{Ric}(M) \geq \frac{n}{2}$ . If*

$$n \leq S \leq n + \frac{14(n+4)}{9n+30},$$

*then  $S = n$  and  $M$  is isometric to a Clifford torus  $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$  ( $1 < m < n - 1$ ).*

In particular, if  $n \leq 5$ , we obtain the following

**Theorem 2.** *Let  $M$  be an  $n$ -dimensional ( $n \leq 5$ ) closed minimal hypersurface of a unit sphere  $S^{n+1}(1)$ . If*

$$n \leq S \leq n + \epsilon(n),$$

*then  $S = n$  and  $M$  is isometric to a Clifford torus  $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$ , where  $\epsilon(3) = \frac{42}{85}$ ,  $\epsilon(4) = \frac{8}{31}$  and  $\epsilon(5) = \frac{3(21-5\sqrt{17})}{28+3\sqrt{17}}$ .*

*Remark.* For  $n \leq 5$ , Peng-Terng [5] proved the following: Let  $M$  be an  $n$ -dimensional ( $n \leq 5$ ) closed minimal hypersurface of a unit sphere  $S^{n+1}(1)$ . If

$$n \leq S \leq n + \epsilon_1(n),$$

then  $S = n$ , where  $\epsilon_1(n) = \frac{6 - 1.13n}{5 + \sqrt{17}}$ . It is obvious that our pinching constant in Theorem 2 is larger than theirs.

## 2. LOCAL FORMULAE

Let  $M$  be an  $n$ -dimensional hypersurface in a unit sphere  $S^{n+1}(1)$ . We choose a local orthonormal frame field  $\{e_1, \dots, e_{n+1}\}$  in  $S^{n+1}(1)$ , restricted to  $M$ , so that  $e_1, \dots, e_n$  are tangent to  $M$ . Let  $\omega_1, \dots, \omega_{n+1}$  denote the dual coframe field in  $S^{n+1}(1)$ . Then, in  $M$ ,

$$\omega_{n+1} = 0.$$

It follows from Cartan's Lemma that

$$(2.0) \quad \omega_{n+1i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}.$$

The second fundamental form  $\alpha$  and the mean curvature of  $M$  are defined by

$$(2.1) \quad \alpha = \sum_{i,j} h_{ij} \omega_i \omega_j e_{n+1} \quad \text{and} \quad nH = \sum_i h_{ii},$$

respectively. We recall that  $M$  is by definition a minimal hypersurface if the mean curvature of  $M$  is identically zero. The connection form  $\omega_{ij}$  is characterized by the

structure equations

$$(2.2) \quad \begin{cases} d\omega_i + \sum_j \omega_{ij} \wedge \omega_j = 0, & \omega_{ij} + \omega_{ji} = 0, \\ d\omega_{ij} + \sum_k \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\ \Omega_{ij} = \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \end{cases}$$

where  $\Omega_{ij}$  (resp.  $R_{ijkl}$ ) denotes the curvature form (resp. the components of the curvature tensor) of  $M$ . The Gauss equation is given by

$$(2.3) \quad R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}).$$

The covariant derivative  $\nabla\alpha$  of the second fundamental form  $\alpha$  of  $M$  with components  $h_{ijk}$  is given by

$$\sum_k h_{ijk} \omega_k = dh_{ij} + \sum_k h_{jk} \omega_{ik} + \sum_k h_{ik} \omega_{jk}.$$

Then the exterior derivative of (2.0) together with the structure equation yields the following Codazzi equation:

$$(2.4) \quad h_{ijk} = h_{ikj} = h_{jik}.$$

From the Codazzi equation, we know that  $h_{ijk}$  is symmetric in the indices  $i, j$  and  $k$ . Similarly, we have the covariant derivative  $\nabla^2\alpha$  of  $\nabla\alpha$  with components  $h_{ijkl}$  as follows:

$$\sum_l h_{ijkl} \omega_l = dh_{ijk} + \sum_l h_{ljk} \omega_{il} + \sum_l h_{ilk} \omega_{jl} + \sum_l h_{ijl} \omega_{kl},$$

and it is easy to get the following Ricci formula:

$$(2.5) \quad h_{ijkl} - h_{ijlk} = \sum_m h_{im} R_{mjkl} + \sum_m h_{mj} R_{mikl}.$$

Similarly, we also have

$$(2.6) \quad h_{ijklm} - h_{ijkml} = \sum_r h_{rjk} R_{ril m} + \sum_r h_{irk} R_{rjlm} + \sum_r h_{ijr} R_{rklm},$$

where the  $h_{ijklm}$ 's are the components of the covariant derivative  $\nabla^3\alpha$  of  $\nabla^2\alpha$ . We should remark that  $h_{ijkl}$  and  $h_{ijklm}$  are symmetric in the first three indices  $i, j$  and  $k$  and generally not symmetric in the other ones. The components of the Ricci curvature and the scalar curvature are given by

$$(2.7) \quad R_{ij} = (n - 1)\delta_{ij} - \sum_k h_{ik}h_{jk},$$

$$(2.8) \quad R = n(n - 1) - \sum_{i,j} h_{ij}^2.$$

Now we compute certain local formulae. For any fixed point  $p$  in  $M$ , we can choose a local orthonormal frame field  $e_1, \dots, e_n$  such that

$$(2.9) \quad h_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ \lambda_i & \text{if } i = j. \end{cases}$$

The following formulas can be obtained by a direct computation (cf. [1]). Let

$$S := \sum_{i,j} h_{ij}^2 = \sum_i \lambda_i^2,$$

$$(2.10) \quad \frac{1}{2}\Delta S = \sum_{i,j,k} h_{ijk}^2 - S(S-n),$$

$$(2.11) \quad \begin{aligned} \frac{1}{2}\Delta \sum_{i,j,k} h_{ijk}^2 &= \sum_{i,j,k,l} h_{ijkl}^2 + (2n+3-S) \sum_{i,j,k} h_{ijk}^2 \\ &\quad + 3(2B-A) - \frac{3}{2}|\nabla S|^2, \end{aligned}$$

where  $A = \sum_{i,j,k} \lambda_i^2 h_{ijk}^2$  and  $B = \sum_{i,j,k} \lambda_i \lambda_j h_{ijk}^2$ .

### 3. PROOFS OF THE THEOREMS

At first we give two algebraic lemmas which will play a crucial role in the proofs of our theorems.

**Lemma 1.** *Let  $a_i$  ( $i = 1, 2, 3, 4$ ) be real numbers satisfying  $\sum_i a_i = 0$  and  $\sum_i a_i^2 = a$ . Then  $\sum_i a_i^4 \leq \frac{7}{12}a^2$ .*

*Proof.* We maximize the function  $\sum_i a_i^4$  with the constraints  $\sum_i a_i = 0$  and  $\sum_i a_i^2 = a$ . By means of the method of the Lagrange multiplier, we solve the following problem:

$$f = \sum_i a_i^4 + \lambda \sum_i a_i + \mu \left( \sum_i a_i^2 - a \right),$$

where  $\lambda$  and  $\mu$  are the Lagrange multipliers. The maximum point of  $\sum_i a_i^4$  is a critical point of  $f$ . Taking the derivative of  $f$  with respect to  $a_i$ , we have

$$f_{a_i} = 4a_i^3 + \lambda + 2\mu a_i = 0.$$

Hence, at most three of the  $a_i$ 's are distinct with each other at a critical point of  $f$ . We consider the following three cases.

- (1) Three of the  $a_i$ 's are distinct with each other. Without loss of generality, we denote them by  $a_1, a_2, a_3$  and assume  $a_1 = a_4$ ; then

$$2a_1 + a_2 + a_3 = 0, \quad 2a_1^2 + a_2^2 + a_3^2 = a.$$

Hence,

$$\begin{aligned} \sum_i a_i^4 &= 2a_1^4 + a_2^4 + a_3^4 = \frac{(a + 2a_1^2)^2}{2} - 14a_1^4 \\ &= \frac{a^2}{2} + 2aa_1^2 - 12a_1^4 \leq \frac{7}{12}a^2, \end{aligned}$$

i.e.,

$$\sum_i a_i^4 \leq \frac{7}{12}a^2.$$

- (2) Two of the  $a_i$ 's are distinct with each other. Without loss of generality, we denote them by  $a_1, a_2$  and assume  $a_1 = a_4$  and  $a_2 = a_3$  or  $a_1 = a_3 = a_4$ ; then  $\sum_i a_i^4 \leq \frac{7}{12}a^2$ .
- (3) If all of the  $a_i$ 's are the same, then  $\sum_i a_i^4 = 0$ .

Therefore, we conclude

$$\sum_i a_i^4 \leq \frac{7}{12} a^2.$$

This completes the proof of Lemma 1. □

**Lemma 2.** *Let  $a_{ij}$  and  $b_i$  ( $i, j = 1, \dots, n$ ) be real numbers satisfying  $\sum_i b_i = 0$ ,  $\sum_i b_i^2 = b > 0$ ,  $\sum_{i,j} b_i a_{ij} = \frac{1}{2}b(n - b)$  and  $\sum_{i,j} b_j a_{ij} = \frac{1}{2}b(n - b)$ . Then*

$$\sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2 \geq \frac{3b(n - b)^2}{2(n + 4)}.$$

*Proof.* We consider  $F = \sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2$  as a function of  $a_{ij}$  with constraints  $\sum_{i,j} b_i a_{ij} = \frac{1}{2}b(n - b)$  and  $\sum_{i,j} b_j a_{ij} = \frac{1}{2}b(n - b)$ . Let

$$f := \sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2 + \lambda \left[ \sum_{i,j} b_i a_{ij} - \frac{1}{2}b(n - b) \right] + \mu \left[ \sum_{i,j} b_j a_{ij} - \frac{1}{2}b(n - b) \right],$$

where  $\lambda$  and  $\mu$  are the Lagrange multipliers. It is obvious that the minimum point of  $F$  is a critical point of  $f$ . Taking the derivative of  $f$  with respect to  $a_{ij}$ , we get

$$(3.1) \quad f_{a_{ii}} = 2a_{ii} + \lambda b_i + \mu b_i = 0, \quad \text{for } i,$$

$$(3.2) \quad f_{a_{ij}} = 6a_{ij} + \lambda b_i + \mu b_j = 0, \quad \text{for } i \neq j.$$

Hence

$$\sum_i a_{ii} f_{a_{ii}} = 2 \sum_i a_{ii}^2 + \lambda \sum_i a_{ii} b_i + \mu \sum_i a_{ii} b_i = 0$$

and

$$\sum_{i \neq j} a_{ij} f_{a_{ij}} = 6 \sum_{i \neq j} a_{ij}^2 + \lambda \sum_{i \neq j} a_{ij} b_i + \mu \sum_{i \neq j} a_{ij} b_j = 0.$$

Therefore,

$$(3.3) \quad 2 \left[ \sum_i a_{ii}^2 + 3 \sum_{i \neq j} a_{ij}^2 \right] = \lambda \frac{1}{2} b(b - n) + \mu \frac{1}{2} b(b - n).$$

From (3.1) and (3.2), we have

$$(3.4) \quad 2 \sum_i b_i a_{ii} + (\lambda + \mu) \sum_i b_i^2 = 0,$$

$$6 \sum_{i \neq j} b_i a_{ij} + \lambda \sum_{i \neq j} b_i^2 + \mu \sum_{i \neq j} b_i b_j = 0$$

and

$$6 \sum_{i \neq j} b_j a_{ij} + \lambda \sum_{i \neq j} b_i b_j + \mu \sum_{i \neq j} b_j^2 = 0.$$

From (3.4) and the two equalities above, we get

$$-4 \sum_i b_i a_{ii} + 3b(n - b) + \lambda n b = 0$$

and

$$-4 \sum_i b_i a_{ii} + 3b(n-b) + \mu nb = 0,$$

$$\lambda + \mu = \frac{6(b-n)}{(n+4)}.$$

According to (3.3), we obtain

$$f_{min} = \frac{3b(n-b)^2}{2(n+4)}.$$

Thus we have finished the proof of Lemma 2.  $\square$

For any fixed point  $p$  in  $M$ , we can choose a local frame field  $e_1, \dots, e_n$  such that

$$(3.5) \quad h_{ij} = \lambda_i \delta_{ij}.$$

Defining  $f_3 = \sum_i \lambda_i^3$  and  $f_4 = \sum_i \lambda_i^4$ , then  $f_3$  and  $f_4$  are functions defined globally on  $M$ .

**Proposition 1.** *Let  $M$  be a minimal hypersurface in  $S^{n+1}(1)$ . Then*

$$\sum_{i,j,k,l} h_{ijkl}^2 \geq \frac{3}{2}(Sf_4 - f_3^2 - 2S^2 + nS) + \frac{3S(S-n)^2}{2(n+4)}$$

holds.

*Proof.* From the Ricci formula (2.5) and the Gauss equation (2.3), we have

$$(3.6) \quad \begin{aligned} h_{iijj} - h_{jjii} &= h_{ijij} - h_{ijji} = \sum_m h_{im} R_{mji} + \sum_m h_{mj} R_{mii} \\ &= \lambda_i R_{iijj} + \lambda_j R_{jii} = (\lambda_i - \lambda_j) R_{iijj} \\ &= (\lambda_i - \lambda_j)(1 + \lambda_i \lambda_j). \end{aligned}$$

We define

$$(3.7) \quad u_{ijkl} = \frac{1}{4}(h_{ijkl} + h_{lijk} + h_{klji} + h_{jkli}).$$

Since  $h_{ijkl}$  is symmetric in the indices  $i, j, k$ , from formula (3.6), we obtain

$$(3.8) \quad \sum_{i,j,k,l} h_{ijkl}^2 \geq \sum_{i,j,k,l} u_{ijkl}^2 + \frac{3}{2}[Sf_4 - f_3^2 - 2S^2 + nS].$$

Since  $\Delta h_{ij} = (n-S)h_{ij}$  and  $\sum_i h_{iikl} = 0$ , we have

$$\sum_{i,j} u_{iijj} \lambda_i = \sum_{i,j} u_{iijj} \lambda_j = \frac{1}{2}S(n-S).$$

From  $\sum_i \lambda_i = 0$  and  $\sum_i \lambda_i^2 = S$  and defining  $a_{ij} := u_{iijj}$  and  $b_i := \lambda_i$ , then  $a_{ij}$  and  $b_i$  satisfy the conditions in Lemma 2. From the definition of  $u_{ijkl}$ , we know that  $u_{ijkl}$  is symmetric in the indices  $i, j, k, l$ . From Lemma 2, we infer

$$(3.9) \quad \sum_{i,j,k,l} u_{ijkl}^2 \geq \sum_i u_{iiii}^2 + 3 \sum_{i \neq j} u_{iijj}^2 \geq \frac{3S(S-n)^2}{2(n+4)}.$$

Hence, from (3.8) and (3.9), we obtain

$$\sum_{i,j,k,l} h_{ijkl}^2 \geq \frac{3}{2}(Sf_4 - f_3^2 - 2S^2 + nS) + \frac{3S(S-n)^2}{2(n+4)}.$$

This completes the proof of Proposition 1. □

**Proposition 2.** *Let  $M$  be a closed minimal hypersurface in  $S^{n+1}(1)$ . Then*

$$\int_M [(S - 2n - \frac{3}{2}) \sum_{i,j,k} h_{ijk}^2 + 2(S - n)f_4 - \frac{3S(S - n)^2}{2(n + 4)} + \frac{9}{8}|\nabla S|^2]dM \geq 0$$

holds.

*Proof.* The following integral formula (3.10) can be found in [2]:

$$(3.10) \quad \int_M (A - 2B)dM = \int_M [Sf_4 - S^2 - f_3^2 - \frac{1}{4}|\nabla S|^2]dM.$$

From the Ricci formula (2.5), by a direct computation, we obtain

$$\frac{1}{4}\Delta f_4 = (n - S)f_4 + 2A + B.$$

Integrating both sides of the above equality, we have

$$(3.11) \quad \int_M (S - n)f_4dM = \int_M (2A + B)dM.$$

Formulas (3.10) and (3.11) yield

$$(3.12) \quad \int_M [(S - 4n)f_4 + 3f_3^2 + 3S^2 + \frac{3}{4}|\nabla S|^2]dM \geq 0.$$

According to Stokes' formula, we integrate the formula (2.11) and obtain

$$(3.13) \quad \begin{aligned} &\int_M \sum_{i,j,k,l} h_{ijkl}^2dM \\ &= \int_M [-(2n + 3 - S) \sum_{i,j,k} h_{ijk}^2 - 3(2B - A) + \frac{3}{2}|\nabla S|^2]dM. \end{aligned}$$

From Proposition 1, (3.10) and (3.13), we infer

$$(3.14) \quad \int_M \{(S - 2n - \frac{3}{2}) \sum_{i,j,k} h_{ijk}^2 + \frac{3}{4}|\nabla S|^2 + \frac{3}{2}[Sf_4 - f_3^2 - S^2] - \frac{3S(n - S)^2}{2(n + 4)}\}dM \geq 0.$$

(3.12)+2 × (3.14) yields

$$\int_M [(S - 2n - \frac{3}{2}) \sum_{i,j,k} h_{ijk}^2 + 2(S - n)f_4 - \frac{3S(S - n)^2}{2(n + 4)} + \frac{9}{8}|\nabla S|^2]dM \geq 0.$$

Thus Proposition 2 is valid. □

*Proof of Theorem 1.* According to (2.10) and Stokes' Theorem, we obtain

$$(3.15) \quad \int_M \sum_{i,j,k} h_{ijk}^2dM = \int_M [S(S - n)]dM$$

and

$$(3.16) \quad - \int_M \frac{1}{2} |\nabla S|^2 = \int_M [S \sum_{i,j,k} h_{ijk}^2 + (n-S)S^2] dM.$$

From formula (2.7) and the assumption in Theorem 1, we have

$$R_{ii} = n - 1 - \lambda_i^2 \geq \frac{n}{2}.$$

Therefore,

$$\lambda_i^2 \leq \frac{n-2}{2},$$

$$\sum_i \lambda_i^4 \leq \frac{n-2}{2} \sum_i \lambda_i^2,$$

that is,

$$(3.17) \quad f_4 \leq \frac{n-2}{2} S.$$

From Proposition 2 and (3.17), we have

$$\int_M \left\{ (S - 2n - \frac{3}{2}) \sum_{i,j,k} h_{ijk}^2 + (n-2)S(S-n) - \frac{3S(S-n)^2}{2(n+4)} + \frac{9}{8} |\nabla S|^2 \right\} dM \geq 0.$$

From (3.15), (3.16) and the above inequality, we infer

$$\int_M \left\{ (-\frac{5}{4}S - n - \frac{7}{2}) \sum_{i,j,k} h_{ijk}^2 + [\frac{9}{4}S - \frac{3(S-n)}{2(n+4)}] S(S-n) \right\} dM \geq 0.$$

Since

$$n \leq S \leq n + \frac{14(n+4)}{9n+30},$$

we have

$$\int_M \left\{ (-\frac{5}{4}S - n - \frac{7}{2}) \sum_{i,j,k} h_{ijk}^2 + (\frac{9n}{4} + \frac{7}{2}) S(S-n) \right\} dM \geq 0.$$

Hence

$$\int_M \frac{5}{4} (S-n) \sum_{i,j,k} h_{ijk}^2 dM = 0.$$

Since  $S$  and  $\sum_{i,j,k} h_{ijk}^2$  are continuous functions, we have  $S = n$ . Thus,  $M$  is isometric to a Clifford torus  $S^m(\sqrt{\frac{m}{n}}) \times S^{n-m}(\sqrt{\frac{n-m}{n}})$  ( $1 < m < n-1$ ) from the result due to Chern, do Carmo and Kobayashi [3] or Lawson [4]. This completes the proof of Theorem 1.  $\square$

*Proof of Theorem 2.* In the case  $n = 3$ , because  $\sum_i \lambda_i = 0$ , we have  $f_4 = \sum_i \lambda_i^4 = \frac{S^2}{2}$ . From Proposition 2, we have

$$\int_M \left\{ (S - 2n - \frac{3}{2}) \sum_{i,j,k} h_{ijk}^2 + S^2(S-n) - \frac{3S(S-n)^2}{2(n+4)} + \frac{9}{8} |\nabla S|^2 \right\} dM \geq 0.$$



From (3.16) and the above inequality, we have

$$\int_M \left\{ \left(-\frac{5}{4}S - 2n - \frac{3}{2}\right) \sum_{i,j,k} h_{ijk}^2 + \frac{13}{4}S^2(S - n) - \frac{3S(S - n)^2}{2(n + 4)} \right\} dM \geq 0.$$

Since

$$n \leq S \leq n + \frac{42}{85},$$

we have

$$\int_M \left\{ -\frac{5}{4}(S - n) \sum_{i,j,k} h_{ijk}^2 \right\} dM \geq 0.$$

Hence

$$\int_M \frac{5}{4}(S - n) \sum_{i,j,k} h_{ijk}^2 dM = 0.$$

By making use of the same proof as in Theorem 1, we know that Theorem 2 is true in the case  $n = 3$ .

In the case  $n = 4$ , from Lemma 1, we have  $f_4 \leq \frac{7}{12}S^2$ . By using this inequality, we obtain, from Proposition 2,

$$\int_M \left\{ (S - 2n - \frac{3}{2}) \sum_{i,j,k} h_{ijk}^2 + \frac{7}{6}S^2(S - n) - \frac{3S(S - n)^2}{2(n + 4)} + \frac{9}{8}|\nabla S|^2 \right\} dM \geq 0.$$

By the same proof as in the case  $n = 3$ , we know that Theorem 2 is also valid in the case  $n = 4$ .

In the case  $n = 5$ , from Proposition 1, (3.10) and (3.13), we have

$$(3.18) \quad \int_M \left\{ (S - 2n - \frac{3}{2}) \sum_{i,j,k} h_{ijk}^2 + \frac{3}{2}(A - 2B) - \frac{3S(S - n)^2}{2(n + 4)} + \frac{9}{8}|\nabla S|^2 \right\} dM \geq 0.$$

Since

$$\begin{aligned} 3(A - 2B) &= \sum_{i,j,k} (\lambda_i^2 + \lambda_j^2 + \lambda_k^2 - 2\lambda_i\lambda_j - 2\lambda_j\lambda_k - 2\lambda_k\lambda_i) h_{ijk}^2 \\ &= \sum_{i \neq j \neq k \neq i} [2(\lambda_i^2 + \lambda_j^2 + \lambda_k^2) - (\lambda_i + \lambda_j + \lambda_k)^2] h_{ijk}^2 \\ &\quad + 3 \sum_{i \neq k} (\lambda_k^2 - 4\lambda_i\lambda_k) h_{iik}^2 - 3 \sum_i \lambda_i^2 h_{iii}^2 \\ &\leq \frac{\sqrt{17} + 1}{2} S \sum_{i,j,k} h_{ijk}^2, \end{aligned}$$

by making use of this inequality and (3.18), a similar proof as in the case  $n = 3$  yields that Theorem 2 is also valid in this case. We have finished the proof of Theorem 2.  $\square$

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