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# NORMALITY CRITERIA FOR FAMILIES OF HOLOMORPHIC MAPPINGS OF SEVERAL COMPLEX VARIABLES INTO $P^N(C)$

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Abstract. By applying the heuristic principle in several complex variables obtained by Aladro and Krantz, we shall prove some normality criteria for families of holomorphic mappings of several complex variables into $P^N(C)$, the complex N-dimensional projective space, related to Green's and Nochka's Picard type theorems. The equivalence of normality to being uniformly Montel at a point will be obtained. Some examples will be given to complement our theory in this paper.

## 1. Introduction

Let $f(z)$ be a meromorphic function on the complex plane. By the second main theorem of value distribution theory [11, Th. 2.1] (note $m(r, f'/f) = 0$ as $r$ becomes sufficiently large if $f$ is a rational function), we have the following Picard type theorems.

**Theorem A.** *If there exist three mutually distinct points $w_1, w_2$ and $w_3$ on the Riemann sphere such that $f(z) - w_i$ $(i = 1, 2, 3)$ has no zero on the complex plane, then $f$ is a constant.*

**Theorem B.** *If there exist mutually distinct points $w_1, w_2, ..., w_q$ $(q \geq 3)$ on the Riemann sphere such that $f(z)-w_i$ has no zero of multiplicities $< m_i (i = 1, 2, ..., q)$ on the complex plane for $q$ positive integers $m_i (i = 1, 2, ..., q)$ with $1/m_1 + 1/m_2 + ... + 1/m_q < q - 2$, then $f$ is a constant.*

Let $F$ be a family of meromorphic functions defined on a domain $D$ of the complex plane. $F$ is said to be normal on $D$ if every sequence of functions of $F$ has a subsequence which converges uniformly on every compact subset of $D$ with respect to the spherical metric to a meromorphic function or identically $\infty$ on $D$. Montel [15] first realized the scope and coherence of these families, and used them to give a particularly unified treatment of Picard's theorems, Schottky's and Landau's theorems. Perhaps the most celebrated criteria for normality in one complex variable are the following Montel type theorems related to Theorem A and Theorem B.

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**Theorem C** ([6]). *Let $F$ be a family of meromorphic functions on a domain $D$ of the complex plane. Suppose that there exist three mutually distinct points $w_1, w_2$ and $w_3$ on the Riemann sphere such that $f(z) - w_i$ $(i = 1, 2, 3)$ has no zero on $D$ for each $f \in F$. Then $F$ is a normal family on $D$.*

**Theorem D** ([6], [7]). *Let $F$ be a family of meromorphic functions on a domain $D$ of the complex plane. Suppose that there exist mutually distinct points $w_1, w_2, ..., w_q$ $(q \geq 3)$ on the Riemann sphere such that $f(z) - w_i$ has no zero of multiplicities $< m_i$ $(i = 1, 2, ..., q)$ on $D$ for each $f$ in $F$ and for $q$ fixed positive integers $m_i (i = 1, 2, ..., q)$ with $1/m_1 + 1/m_2 + ... + 1/m_q < q - 2$. Then $F$ is a normal family on $D$.*

The fact that Picard type theorems and normality criteria were so intimately related led Bloch to the hypothesis that a family of meromorphic functions which have a property P in common on a domain $D$ is normal on $D$ if the property P forces a meromorphic function on the complex plane to be a constant. This hypothesis is called Bloch's heuristic principle in complex function theory (see [1], [6], [19] and [23]). Rubel [19] gave some counterexamples to Bloch's heuristic principle. Although the principle is false in general, many authors proved normality criteria for families of meromorphic functions by starting from Picard theorems. Hence an interesting topic is to make the principle rigorous and to find its applications. Zalcman [23] gave a well-known heuristic principle in the theory of functions. There are many investigations in this field for one complex variable (see, e.g., [1], [19], [23] and their references for related results).

In the case of higher dimension, the notion of normal family has proved its importance in geometric function theory in several complex variables (see, e.g., [13], [14], [17], [21], and [22]). Bloch [2], Green [8], [9] and Nochka [16] established some Picard type theorems for holomorphic mappings into $P^N(C)$, the complex N-dimensional projective space, which generalized Theorem A and Theorem B respectively. Fujimoto [7] and Nochka [16] gave some normality criteria related to Green's and Nochka's Picard type theorems [8], [16] in several complex variables by using various methods. Recently Aladro and Krantz [1] proved a criterion for normality in several complex variables and for the first time implemented a Zalcman type heuristic principle in this more general content.

In this paper, by modifying the heuristic principle obtained by Aladro and Krantz [1], we shall prove some normality criteria for families of holomorphic mappings of several complex variables into $P^N(C)$ related to Green's and Nochka's Picard type theorems [8], [16]. The equivalence of normality to being uniformly Montel at a point will be given. Some examples will be included to complement our theory.

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## 2. Statement of results

For the general reference of this article, see [12], [14], [17] and [22].

Let $P^N(C)$ be a complex N-dimensional projective space and let $\rho : C^{N+1} - \{0\} \to P^N(C)$ be the standard projective mapping. A subset $H$ of $P^N(C)$ is called a hyperplane if there is an N-dimensional linear subspace $\widetilde{H}$ of $C^{N+1}$ such that $\rho(\widetilde{H} - \{0\}) = H$. If we write $(C^{N+1})^*$ for the dual space of $C^{N+1}$, then there is an $\alpha \in (C^{N+1})^* - \{0\}$ such that $\widetilde{H} = \{\alpha = 0\} = \{x \in C^{N+1} : \alpha(x) = 0\}$. Let $B^*$ be the set of Euclidean unit vectors in $(C^{N+1})^*$. Then $\alpha, \beta \in B^*$ satisfy $\widetilde{H} = \{\alpha = 0\} = \{\beta = 0\}$ if and only if $\alpha = c\beta$ with $c \in C$ and $|c| = 1$. Let $H_1, ..., H_{N+1}$ be hyperplanes in $P^N(C)$. Let $\alpha_i = (\alpha_1^i, ..., \alpha_{N+1}^i) \in B^*$ such that $\widetilde{H_i} = \{\alpha_i = 0\}(i = 1, ..., N+1)$. Define

$$D(H_1, ..., H_{N+1}) := \left| \det \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_{N+1} \end{pmatrix} \right|$$

which only depends on $H_i(i = 1, ..., N+1)$ but does not depend on the choice of $\alpha_i \in B^*$ with $\widetilde{H}_i = \{\alpha_i = 0\}(i = 1, ..., N+1)$. When $N = 1$, $D(a, b)$ is just the spherical distance between $a, b \in P(C)$.

**Definition 1.** Let $H_1, ..., H_q(q \geq N+1)$ be hyperplanes in $P^N(C)$. Define

$$D(H_1, ..., H_q) := \prod D(H_{i_1}, ..., H_{i_{N+1}}),$$

where the product $\prod$ is taken over all $\{i_1, ..., i_{N+1}\}$ with $1 \leq i_1 < i_2 < ... < i_{N+1} \leq q$. We say the hyperplane family $H_1, ..., H_q(q \geq N+1)$ in $P^N(C)$ is in general position if $D(H_1, ..., H_q) > 0$.

Let $D$ be a domain in $C^n$ and $h(z)$ a nonidentically zero holomorphic function on $D$. For a point $a = (a_1, ..., a_n) \in D$ we expand $h(z)$ as a compactly convergent series

$$h(u_1 + a_1, ..., u_n + a_n) = \sum_{k=0}^{\infty} p_k(u_1, ..., u_n)$$

on a neighborhood of $a$, where $p_k$ is either identically zero or a homogeneous polynomial of degree $k$. The number $\min\{k, p_k(u) \not\equiv 0\}$ is said to be the zero multiplicity of $h(z)$ at the point $a$.

Set $\Delta_n = \Delta_n(z_0, r) = \{(z_1, ..., z_n) \in C^n; |z_i - z_i^0| < r, i = 1, ..., n\}$ for $z_0 = (z_1^0, ..., z_n^0) \in C^n$ and $r > 0$. Let $f$ be a holomorphic mapping of $\Delta_n$ into $P^N(C)$. Then there exists a holomorphic mapping $\widetilde{f} = (f_1, ..., f_{N+1})$ of $\Delta_n$ into $C^{N+1}$ such that $\widetilde{f}^{-1}(0) = \emptyset$ and $\rho(\widetilde{f}(z)) = f(z)$ on $\Delta_n$. We call $\widetilde{f}$ a reduced representation of $f$ on $\Delta_n$. Let $H$ be a hyperplane of $P^N(C)$ and $\widetilde{H} = \{\alpha = 0\}$. We say that the holomorphic mapping $f$ intersects the hyperplane $H$ with multiplicity $m < \infty$ on $\Delta_n$ if $f(\Delta_n) \not\subset H$, $f(\Delta_n) \cap H \neq \emptyset$ and the holomorphic function $\alpha(\widetilde{f}(z))$ on $\Delta_n$ has zero multiplicities $\geq m$ at all the zeros of $\alpha(\widetilde{f}(z))$ on $\Delta_n$, while at least one zero has multiplicity $m$. We say that the holomorphic mapping $f$ intersects the hyperplane $H$ with multiplicity $\infty$ on $\Delta_n$ if $f(\Delta_n) \subset H$ or $f(\Delta_n) \cap H = \emptyset$. Hence we always have $m \geq 1$.

Let $f$ be a holomorphic mapping of a domain $D$ in $C^n$ into $P^N(C)$ and let $H$ be a hyperplane of $P^N(C)$. We say that the holomorphic mapping $f$ intersects the hyperplane $H$ with multiplicity at least $m$ on $D$ if $f$ intersects $H$ with multiplicity at least $m$ in any $\Delta_n$ contained in $D$.

Bloch [2] and Green [8], [9] gave the following Picard type theorem:

**Theorem E** ([2], [8])**.** *A holomorphic mapping $f : C \longrightarrow P^N(C)$ that omits $2N+1$ hyperplanes in general position in $P^N(C)$ is a constant.*

In 1983, Nochka [16] improved Theorem E and proved the following Cartan conjecture.

**Theorem F** ([16]). *Suppose that $q \geq 2N+1$ hyperplanes $H_1, ..., H_q$ are given in general position in $P^N(C)$, along with $q$ positive integers $m_1, ..., m_q$ (some of them may be $\infty$). If*

$$\sum_{j=1}^{q}(1 - N/m_j) > N + 1,$$

*then there does not exist a nonconstant holomorphic mapping $f : C \longrightarrow P^N(C)$ such that $f$ intersects $H_j$ with multiplicity at least $m_j (j = 1, ..., q)$.*

In fact, Theorem F has been extended by Ru and Stoll [18].

**Definition 2.** A family $F$ of holomorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$ is said to be normal on $D$ if any sequence in $F$ contains a subsequence which converges uniformly on compact subsets of $D$ to a holomorphic mapping of $D$ into $P^N(C)$ and $F$ is said to be normal at a point $z_0$ in $D$ if $F$ is normal on some neighborhood of $z_0$ in $D$.

Using Cartan's second main theorem for nondegenerate holomorphic curves [4] and the method in [6], Fujimoto [7] proved some normality criteria for a family of nondegenerate meromorphic mappings into $P^N(C)$ related to Theorem F. Nochka [16] gave a normality criterion for a family of holomorphic curves in $P^N(C)$ by Theorem F and a lemma of Brody [3]. Using completely different methods, we shall prove the following results related to Theorem E and Theorem F.

**Theorem 1.** *Let $F$ be a family of holomorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$. Suppose that for each $f \in F$, there exist $q \geq 2N+1$ hyperplanes $H_1(f), ..., H_q(f)$ (which may depend on $f$) in $P^N(C)$ such that $f$ intersects $H_j(f)$ with multiplicity at least $m_j (j = 1, ..., q)$, where $m_j (j = 1, ..., q)$ are fixed positive integers which are independent of $f$ and may be $\infty$, with*

$$\sum_{j=1}^{q}(1 - N/m_j) > N + 1$$

*and*

$$\inf\{D(H_1(f), ..., H_q(f)); f \in F\} > 0.$$

*Then $F$ is a normal family on $D$.*

By Theorem 1 we immediately have the following corollaries.

**Corollary 2.** *Let $F$ be a family of holomorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$. Suppose that for each $f \in F$, there exist $2N+1$ hyperplanes $H_1(f), ..., H_{2N+1}(f)$ in $P^N(C)$ such that*

$$f(D) \cap H_i(f) = \emptyset$$

*for $i = 1, 2, ..., 2N+1$ and*

$$\inf\{D(H_1(f), ..., H_{2N+1}(f)); f \in F\} > 0.$$

*Then $F$ is a normal family on $D$.*

**Corollary 3.** *Let $F$ be a family of holomorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$. Suppose that $H_1, ..., H_q$ are $q \geq 2N+1$ hyperplanes given in general position in $P^N(C)$, along with $q$ positive integers $m_j (j = 1, ..., q)$ such that*

$$\sum_{j=1}^{q}(1 - N/m_j) > N + 1.$$

*If each $f \in F$ intersects $H_j$ with multiplicity at least $m_j (j = 1, ..., q)$, then $F$ is a normal family on $D$.*

*Remark.* Fujimoto [7] proved Corollary 3 in the case $N = 1$ and Nochka [16] gave Corollary 3 in the case $n = 1$. Cf. [7, Th. 8.1].

**Definition 3.** Let $F$ be a family of holomorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$. $F$ is said to be uniformly Montel on $D$ if for any $f \in F$, there exist $2N+1$ hyperplanes $H_1(f), ..., H_{2N+1}(f)$ (which may depend on $f$) located in $P^N(C)$ in general position such that

$$f(D) \cap H_i(f) = \emptyset$$

for $i = 1, 2, ..., 2N+1$ and

$$\inf\{D(H_1(f), ..., H_{2N+1}(f)); f \in F\} > 0$$

and $F$ is said to be uniformly Montel at a point $z_0$ in $D$ if $F$ is uniformly Montel on some neighborhood of $z_0$ in $D$.

For example, let $F$ be a family of holomorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$. Then $F$ is obviously uniformly Montel on $D$ if there exist $2N+1$ hyperplanes $H_1, ..., H_{2N+1}$ located in $P^N(C)$ in general position with $f(D) \cap H_i = \emptyset$ $(i = 1, 2, ..., 2N+1)$ for any $f \in F$.

We shall prove the following necessary and sufficient Montel type criterion for normality in several complex variables:

**Theorem 4.** *Let $F$ be a family of holomorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$. Then $F$ is normal at a point $z_0$ in $D$ if and only if $F$ is uniformly Montel at the point $z_0$.*

By Theorem 4 the following corollary is obvious.

**Corollary 5.** *Let $F$ be a family of holomorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$. Then $F$ is a normal family on $D$ if and only if $F$ is uniformly Montel at every point in $D$.*

## 3. Two examples

Now we give two examples to complement our theory in this paper.

**Example 1.** Here we give an example to explain that it is essential for the hyperplanes $H_i(f)$ in Definition 3 to depend on $f$ in Theorem 4.

Let $z = (z_1, ..., z_{n+1}) \in C^{n+1}$. We call $z$ a rational point in $C^{n+1}$ if all $\mathrm{Re}(z_i)$, $\mathrm{Im}(z_i) (i = 1, 2, ..., n+1)$ are rational numbers. Let $E$ be a subset in $C^{n+1}$ defined by

$$E := \{(z_1, ..., z_{n+1}) \in C^{n+1}; 1 \leq |z_1|^2 + ... + |z_{n+1}|^2 \leq 2\}.$$

Then all rational points in $E$ are dense in $E$ and countable. Let

$$\{z^{(1)}, z^{(2)}, ..., z^{(i)}, ...\}$$

be the set of all rational points in $E$.

Define

$$z^{(i)} := (z_1^{(i)}, ..., z_{n+1}^{(i)})$$

and

$$B_{\frac{1}{2}} := \{z = (z_1, ..., z_{n+1}) \in C^{n+1} : |z_1|^2 + ... + |z_{n+1}|^2 < \frac{1}{4}\}.$$

We define holomorphic mapping $f_i$ of $B_{\frac{1}{2}}$ into $P^n(C)$ which has a reduced representation

$$\widetilde{f}_i(z) := z + z^{(i)} = (z_1 + z_1^{(i)}, z_2 + z_2^{(i)}, ..., z_{n+1} + z_{n+1}^{(i)})$$

from $z \in B_{\frac{1}{2}}$ into $C^{n+1}(i = 1, 2, ...)$.

It is very easy to see that $\{f_i\}_{i=1}^{\infty}$ is a normal family on $B_{\frac{1}{2}}$ and thus by Corollary 5 $\{f_i\}$ is uniformly Montel at every point in $B_{\frac{1}{2}}$. But for any fixed small domain $G \subset B_{\frac{1}{2}}$, we have

$$(\bigcup_{i=1}^{\infty} \widetilde{f}_i(G)) \cap \widetilde{H} \neq \emptyset$$

for any $n$-dimensional linear subspace $\widetilde{H}$ of $C^{n+1}$, i.e.,

$$(\bigcup_{i=1}^{\infty} f_i(G)) \cap H \neq \emptyset$$

for any hyperplane $H$ of $P^n(C)$.

Hence for any given hyperplane $H$ in $P^n(C)$, not all of $f_i(z)(i = 1, 2, ...)$, restricted on any fixed neighborhood in $B_{\frac{1}{2}}$, can omit $H$.

**Example 2.** If $F$ is uniformly Montel on $D$, then $F$ is a normal family on $D$. Here we give an example to explain that even if $F$ is normal on $D$, then $F$ is possibly not uniformly Montel on $D$.

Let $F = \{nz\}_{n=1}^{\infty}$ be defined on $D := \{z \in C; 0 < |z| < 1\}$. Then $F$ is normal on $D$. Now we shall verify that $F$ is not uniformly Montel on $D$.

In fact, for any $a_n, b_n \in P(C) - \{nz; z \in D\} - \{0\}$ we have $a_n \to \infty$, $b_n \to \infty$ as $n$ tends to $+\infty$ and thus $D(a_n, b_n) \to 0$ as $n$ tends to $+\infty$, where $D(a, b)$ is the spherical distance between $a$ and $b$ in $P(C)$. Hence for any three points $a_n, b_n, c_n \in P(C) - \{nz; z \in D\}$ we have

$$D(a_n, b_n, c_n) \longrightarrow 0$$

as $n$ tends to $+\infty$ (see Definition 1 for $D(a_n, b_n, c_n)$). So $F$ is not uniformly Montel on $D$.

## 4. Proof of Theorem 1

**Lemma 1.** *Let $F$ be a family of holomorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$. The family $F$ is not normal on $D$ if and only if there exist a compact set $K_0 \subset D$ and sequences $\{f_i\} \subset F$, $\{p_i\} \subset K_0$, $\{r_i\}$ with $r_i > 0$ and $r_i \to 0^+$ and $\{u_i\} \subset C^n$ Euclidean unit vectors such that*

$$g_i(z) := f_i(p_i + r_i u_i z),$$

*where $z \in C$ satisfies $p_i + r_i u_i z \in D$, converges uniformly on compact subsets of $C$ to a nonconstant holomorphic mapping $g$ of $C$ into $P^N(C)$.*

For the proof of Lemma 1, see [1, Th. 3.1], [10, Th. 6.5]. Cf. [23].

We use the standard notation for a holomorphic mapping element into $P^N(C)$: $(f, D)$ denotes the holomorphic mapping $f$ of a domain $D$ in $C^n$ into $P^N(C)$. We distinguish between $(f, D)$ and $(f, D_1)$ if $D \neq D_1$. Write $\Delta_r = \{z \in C; |z| < r\}$. Let $\mathrm{H}(\Delta_r, D)$ denote the set of holomorphic mappings of $\Delta_r$ into $D$. We shall give a Zalcman type heuristic principle which plays a key role in proving Theorem 1.

**Lemma 2.** *Let $P$ be a property (i.e., a set) of holomorphic mappings of some domains $D$ in $C^n$ into $P^N(C)$ which satisfies the following conditions:*

*(i) If two domains $D_1, D_2$ in $C^n$ satisfy $(f, D_2) \in P$ and $D_1 \subset D_2$, then $(f, D_1) \in P$.*

*(ii) Define $P_1 := \{(f \circ \varphi, \Delta_r); (f, D) \in P$ and $\varphi(z) = az + b$, $\varphi \in H(\Delta_r, D)$ where $a, b \in C^n$, for some $r > 0\}$. Let $\{s_i\}$ be a sequence satisfying $0 < s_1 < s_2 < \ldots$, $s_i \to +\infty$ and $(f_i \circ \varphi_i, \Delta_{s_i}) \in P_1$. If $f_i \circ \varphi_i \to g$ uniformly on compact subsets of $C$, then $(g, C) \in P_1$.*

*(iii) If $(g, C) \in P_1$, then $g$ is a constant mapping of $C$ into $P^N(C)$.*

*Then for any domain $D$ in $C^n$ the family of holomorphic mappings of $D$ into $P^N(C)$ satisfying $(f, D) \in P$ is normal on $D$.*

Lemma 2 slightly improves Principle 4.1 in [1]. Cf. [23].

*Proof of Lemma 2.* Let $F$ be the family of all holomorphic mappings on $D$ into $P^N(C)$ which have property $P$. If $F$ is not normal on $D$, then by Lemma 1 there exist balls $B(p, r) \subset B(p, r_0) \subset D$ with $0 < r < r_0$ and $\{p_i\} \subset B(p, r)$, $\{f_i\} \subset F$, $\{r_i\}$ with $r_i > 0$ and $r_i \to 0^+$ and $\{u_i\} \subset C^n$ Euclidean unit vectors such that

$$g_i(z) := f_i(p_i + r_i u_i z)$$

with $z \in \Delta_{s_i} := \{z \in C; |z| < s_i\}$ ($s_i = \frac{r_0 - r}{r_i} \to +\infty$) converges uniformly on compact subsets of $C$ to a nonconstant holomorphic mapping $g$ of $C$ into $P^N(C)$. Since $(g_i, \Delta_{s_i}) \in P_1$, the limit $(g, C) \in P_1$ by (ii). But $P_1$ contains no nonconstant holomorphic mapping of $C$ into $P^N(C)$. So we have a contradiction. Thus $F$ must be normal on $D$. We get Lemma 2.

From the proof of Lemma 2, we get the following modification of Lemma 2.

**Lemma 2′.** *Let $P$ be a property (i.e., a set) of holomorphic mappings of some domains $D$ in $C^n$ into $P^N(C)$ which satisfies the following conditions:*

*(i) If two domains $D_1, D_2$ in $C^n$ satisfy $(f, D_2) \in P$ and $D_1 \subset D_2$, then $(f, D_1) \in P$.*

*(ii) Define $P_1 = \{(f \circ \varphi, \Delta_r\}; (f, D) \in P$ and $\varphi(z) = az + b$, $\varphi \in H(\Delta_r, D)$ where $a, b \in C^n$, for some $r > 0\}$. Let $\{s_i\}$ be a sequence satisfying $0 < s_1 < s_2 < \ldots$, $s_i \to +\infty$ and $(f_i \circ \varphi_i, \Delta_{s_i}) \in P_1$. If $f_i \circ \varphi_i \to g$ uniformly on compact subsets of $C$, then $g(z)$ $(z \in C)$ is a constant mapping of $C$ into $P^N(C)$.*

*Then, for any domain $D$ in $C^n$ the family of holomorphic mappings of $D$ into $P^N(C)$ satisfying $(f, D) \in P$ is normal on $D$.*

*Proof of Theorem 1.* Let $P$ be the property: a holomorphic mapping $f$ of a domain $D$ in $C^n$ into $P^N(C)$ intersects $q \geq 2N + 1$ hyperplanes $H_i(f)$ with multiplicity at least $m_i (i = 1, 2, \ldots, q)$ which may be $\infty$ with

$$\sum_{i=1}^{q} (1 - N/m_i) > N + 1$$

and

$$D(H_1(f), \ldots, H_q(f)) \geq \delta_0,$$

where $m_i (i = 1, 2, \ldots, q)$ are fixed positive integers or $\infty$, and $\delta_0$ is a fixed positive number which is independent of $f$ and $D$.

Thus (i) of Lemma 2′ is trivially satisfied. We shall verify (ii) of Lemma 2′.

Define $P_1 = \{(f\circ\varphi, \Delta_r); (f, D) \in P$ and $\varphi(z) = az + b, \varphi \in \mathrm{H}(\Delta_r, D)$ where $a, b \in C^n$, for some $r > 0\}$.

By the definition of multiplicity we have the following result: If $(f\circ\varphi, \Delta_r) \in P_1$, then the holomorphic mapping $f\circ\varphi$ of $\Delta_r$ into $P^N(C)$ intersects hyperplane $H_i(f)$ with multiplicity at least $m_i (i = 1, 2, ..., q)$.

Let $\{s_i\}$ be a sequence satisfying $0 < s_1 < s_2 < ..., s_i \to +\infty$ and $(f_i\circ\varphi_i, \Delta_{s_i}) \in P_1$. Suppose that $f_i\circ\varphi_i \to g_0$ uniformly on compact subsets of $C$. Consider hyperplane sequences $\{H_k(f_i)\}_{i=1}^{\infty} (k = 1, 2, ..., q)$ and take $\{\alpha_k^i\}_{i=1}^{\infty} \subset B^* (k = 1, 2, ..., q)$ satisfying $\widetilde{H}_k(f_i) = \{\alpha_k^i = 0\}$. Since $B^*$ is a compact subset of $(C^{N+1})^*$, we can find points $\alpha_k^0 \in B^* (k = 1, 2, ..., q)$ and subsequences (again denoted by themselves) such that $\alpha_k^i \to \alpha_k^0$ as $i \to +\infty (k = 1, 2, ..., q)$. Let $H_k^0 = \rho(\{\alpha_k^0 = 0\} - \{0\}) (k = 1, 2, ..., q)$ be hyperplanes of $P^N(C)$. We have

$$D(H_1^0, ..., H_q^0) \geq \liminf_{i\to+\infty} D(H_1(f_i), ..., H_q(f_i)) \geq \delta_0.$$

Thus the hyperplanes $H_1^0, ..., H_q^0$ are located in $P^N(C)$ in general position. We shall prove that the holomorphic mapping $g_0$ of $C$ into $P^N(C)$ intersects the hyperplane $H_k^0$ with multiplicity at least $m_k (k = 1, 2, ..., q)$.

Let $\widetilde{g}_0(z)$ be a reduced representation of $g_0(z)$ on $C$. Consider the entire function $\alpha_k^0(\widetilde{g}_0(z))(z \in C)$ for a fixed $k (k = 1, 2, ..., q)$. If $\alpha_k^0(\widetilde{g}_0(z)) \equiv 0$ on $C$ or $\alpha_k^0(\widetilde{g}_0(z)) \neq 0$ everywhere in $C$, then $g_0$ intersects $H_k^0$ with multiplicity $\infty$. Suppose that $\alpha_k^0(\widetilde{g}_0(z)) \not\equiv 0$ on $C$ and $\alpha_k^0(\widetilde{g}_0(z_0)) = 0$ for $z_0 \in C$. Choose $r > 0$ such that $z_0$ is the only zero point of $\alpha_k^0(\widetilde{g}_0(z))$ on $E := \{z \in C; |z - z_0| \leq r\}$. Then $g_i := f_i\circ\varphi_i (i = 1, 2, ...)$ has a reduced representation $\widetilde{g}_i$ of $\Delta_{s_i}$ into $C^{N+1}$ such that $\alpha_k^i(\widetilde{g}_i(z)) \to \alpha_k^0(\widetilde{g}_0(z))$ $(i \to +\infty)$ uniformly on $E$. By Hurwitz's theorem (see [5]), there exists a positive integer $N_0$ such that $\alpha_k^i(\widetilde{g}_i(z))$ and $\alpha_k^0(\widetilde{g}_0(z))$ have the same number of zeros with counting multiplicities on $E$ for $i \geq N_0$. Since $g_i = f_i\circ\varphi_i$ of $E$ into $P^N(C)$ intersects $H_k(f_i)$ with multiplicity at least $m_k$, $z_0$ is zero of $\alpha_k^0(\widetilde{g}_0(z))$ with multiplicity at least $m_k$. Hence $g_0$ intersects $H_k^0$ with multiplicity at least $m_k (k = 1, 2, ..., q)$. By Theorem F $g_0$ is a constant mapping of $C$ into $P^N(C)$. Thus (ii) of Lemma 2′ is satisfied.

By Lemma 2′ we get Theorem 1. The proof of Theorem 1 is completed.

## 5. Proof of Theorem 4

**Lemma 3.** *Let $\Omega \subset C^n$ be a hyperbolic domain and let $M$ be a compact complex Hermitian manifold with metric $ds_M^2$. A family $F$ of holomorphic mappings from $\Omega$ into $M$ is a normal family on $\Omega$ if and only if for each compact $E \subset \Omega$, there exists a constant $C(E) > 0$ such that for all $z \in E$ and $\xi \in T_z(\Omega)$,*

$$\sup\{ds_M^2(f(z), df(z)(\xi)); f \in F\} \leq C(E) K_\Omega(z, \xi),$$

*where $df(z)$ is the mapping from $T_z(\Omega)$ into $T_{f(z)}(M)$ induced by $f$ and $K_\Omega$ denotes the infinitesimal Kobayashi metric.*

In fact, Lemma 3 is only a generalization of the classical Marty normality criterion (e.g., see Th. 3.8 in p.158 of [5]). See [10, Lemma 2.7] or [1, Th. 1.1] for the proof of Lemma 3.

**Lemma 4.** *Let $F$ be a family of holomorphic mappings of a domain $D$ in $C^n$ into $P^N(C)$. If $F$ is normal at a point $z_0 \in D$, then $F$ is uniformly Montel at $z_0$.*

*Proof of Lemma 4.* First note that the group of unitary transformations of the space $C^{N+1}$ induces a unitary group $U(N+1)$ of $P^N(C)$ which acts transitively on $P^N(C)$ and leaves the Fubini-Study form invariant. Hence the unitary group $U(N+1)$ acts transitively on $P^N(C)$ and all $T \in U(N+1)$ are isometric mappings with respect to the Fubini-Study metric on $P^N(C)$.

Let $H_1^0, ..., H_{2N+1}^0$ be $2N+1$ hyperplanes in $P^N(C)$ in general position and let $\omega_0$ be a point in $P^N(C) - \bigcup_{i=1}^{2N+1} H_i^0$. Hence there exists a neighborhood

$$G_{\varepsilon_0}(\omega_0) := \{\omega \in P^N(C); d_{P^N}(\omega, \omega_0) < \varepsilon_0\}$$

with

$$G_{\varepsilon_0}(\omega_0) \subset P^N(C) - \bigcup_{i=1}^{2N+1} H_i^0,$$

where $d_{P^N}(\omega_1, \omega_2)$ is the Fubini-Study distance between $\omega_1$ and $\omega_2$ in $P^N(C)$.

Since $F$ is normal at $z_0$ in $D$, there exists a small bounded neighborhood $V$ (so $V$ is hyperbolic) of $z_0$ such that $F$ is normal on $V$. Hence for a smaller bounded neighborhood $U$ of $z_0$ with $\overline{U} \subset V$, by Lemma 3 there exists a constant $C(\overline{U}) > 0$ such that for all $z \in U$ and $\xi \in T_z(V)$,

$$\sup\{ds^2_{P^N}(f(z), df(z)(\xi)); f \in F\} \le C(\overline{U}) K_V(z, \xi),$$

where $df(z)$ is the mapping from $T_z(V)$ into $T_{f(z)}(P^N(C))$ induced by $f$, $ds^2_{P^N}$ and $K_V$ denote the Fubini-Study metric on $P^N(C)$ and the infinitesimal Kobayashi metric on $V$ respectively.

Hence from the definition of the integrated distance there exists some neighborhood $U_{\varepsilon_0}(z_0)$ (which may depend on $\varepsilon_0$) of $z_0$ in $U$ such that

$$f(z) \in \{\omega \in P^N(C); d_{P^N}(\omega, f(z_0)) < \varepsilon_0\}$$

for all $z \in U_{\varepsilon_0}(z_0)$ and all $f \in F$.

Let $\Gamma_\omega \in U(N+1)$ denote some unitary transformation on $P^N(C)$ with $\Gamma_\omega(\omega_0) = \omega$ where $\omega$ is a point in $P^N(C)$ and define $\Gamma_\omega(A) := \{\Gamma_\omega(p) \in P^N(C); p \in A\}$ for $A \subset P^N(C)$. Then we have

$$\begin{aligned} \{\omega \in P^N(C); d_{P^N}(\omega, f(z_0)) < \varepsilon_0\} &= \Gamma_{f(z_0)}(\{\omega \in P^N(C); d_{P^N}(\omega, \omega_0) < \varepsilon_0\}) \\ &\subset \Gamma_{f(z_0)}(P^N(C) - \bigcup_{i=1}^{2N+1} H_i^0) \\ &\subset P^N(C) - \bigcup_{i=1}^{2N+1} \Gamma_{f(z_0)}(H_i^0). \end{aligned}$$

Hence for any $f$ in $F$, $f$ restricted on $U_{\varepsilon_0}(z_0)$ omits these $2N+1$ hyperplanes $\Gamma_{f(z_0)}(H_i^0)(i = 1, ..., 2N+1)$ in $P^N(C)$ in general position. It is easy to see

$$D(\Gamma_{f(z_0)}(H_1^0), ..., \Gamma_{f(z_0)}(H_{2N+1}^0)) = D(H_1^0, ..., H_{2N+1}^0)$$

for all $f$ in $F$.

Then $F$ is uniformly Montel on $U_{\varepsilon_0}(z_0)$. The proof of Lemma 4 is completed.

*Proof of Theorem 4.* Combining Corollary 2 and Lemma 4, we immediately have Theorem 4. This proves Theorem 4.

*Remark.* It seems clear that the idea in this paper suggests some insights into the family of meromorphic mappings of several complex variables into $P^N(C)$. This line of thought will be pursued in another paper.

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