

## THE NORMAL HOMOGENEOUS SPACE $(\mathbf{SU}(3) \times \mathbf{SO}(3))/\mathbf{U}^\bullet(2)$ HAS POSITIVE SECTIONAL CURVATURE

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ABSTRACT. We give a new description of the positively curved seven-dimensional Aloff-Wallach spaces  $M_{kl}^7$ . In particular, this description exhibits  $M_{11}^7$  as a normal homogeneous space, although it does not occur in Berger's classification (1961).

A theorem of Berger [1961] states that a simply connected, normal homogeneous space of positive sectional curvature is either diffeomorphic to a compact rank-one symmetric space,  $\mathbb{S}^n$ ,  $\mathbb{C}\mathbb{P}^n$ ,  $\mathbb{H}\mathbb{P}^n$ ,  $\mathbb{C}\mathbb{a}\mathbb{P}^2$ , or to one of the two following exceptional spaces:

- 1.)  $V_1 := \mathbf{Sp}(2)/\mathbf{SU}(2)$ , where the Lie algebra  $\mathfrak{su}(2) \subset \mathfrak{sp}(2)$  is given by

$$\text{span}_{\mathbb{R}} \left\{ \begin{pmatrix} \frac{3}{2}i & 0 \\ 0 & \frac{1}{2}i \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & j \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{2}\sqrt{3}i \\ \frac{1}{2}\sqrt{3}i & k \end{pmatrix} \right\}.$$

- 2.)  $V_2 := \mathbf{SU}(5)/\mathbf{H}$ , where the group  $\mathbf{H}$  is given by

$$\mathbf{H} := \left\{ \begin{pmatrix} zA & 0 \\ 0 & \bar{z}^4 \end{pmatrix} \mid A \in \mathbf{Sp}(2) \subset \mathbf{SU}(4), z \in \mathbf{S}^1 \subset \mathbb{C} \right\} \subset \mathbf{U}(4) \subset \mathbf{SU}(5).$$

In particular  $\mathbf{H}$  is isomorphic to  $(\mathbf{Sp}(2) \times \mathbf{S}^1)/\{\pm(\text{id}, 1)\}$ .

Here normal homogeneous means that the metrics on  $V_1$  and  $V_2$  are induced from biinvariant metrics on  $\mathbf{Sp}(2)$  and  $\mathbf{SU}(5)$ , respectively. This theorem is not correct. We will show that there is a third exception:

- 3.)  $V_3 := (\mathbf{SU}(3) \times \mathbf{SO}(3))/\mathbf{U}^\bullet(2)$ , where  $\mathbf{U}^\bullet(2)$  is the image under the embedding  $(\iota, \pi): \mathbf{U}(2) \hookrightarrow \mathbf{SU}(3) \times \mathbf{SO}(3)$  given by the natural inclusion

$$\iota(A) := \begin{pmatrix} A & 0 \\ 0 & \det A^{-1} \end{pmatrix} \quad \text{for } A \in \mathbf{U}(2)$$

and the projection  $\pi: \mathbf{U}(2) \rightarrow \mathbf{U}(2)/\mathbf{S}^1 \cong \mathbf{SO}(3)$ ; here  $\mathbf{S}^1 \subset \mathbf{U}(2)$  denotes the center of  $\mathbf{U}(2)$ . On  $\mathbf{SU}(3) \times \mathbf{SO}(3)$  we consider the 1-parameter family of biinvariant metrics  $\tilde{g}_\lambda := -(B_{\mathfrak{su}(3)} \times \lambda B_{\mathfrak{so}(3)})$  for  $\lambda > 0$ , where  $B_{\mathfrak{su}(3)}$  and  $B_{\mathfrak{so}(3)}$  are the Killing forms of  $\mathfrak{su}(3)$  and  $\mathfrak{so}(3)$ , respectively. The induced metric on the quotient  $V_3$  which turns the projection into a Riemannian submersion will also be denoted by  $\tilde{g}_\lambda$ .

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In fact, Aloff and Wallach [AW1972] have introduced  $(V_3, \tilde{g}_\lambda)$  from a different point of view. They studied for positive integers  $k$  and  $l$  the groups

$$T_{kl} := \left\{ \begin{pmatrix} z^k & & \\ & z^l & \\ & & \bar{z}^{k+l} \end{pmatrix} \mid z \in S^1 \subset \mathbb{C} \right\} \subset U(2) \subset SU(3),$$

and the 1-parameter family of left-invariant,  $\text{Ad}(U(2))$ -invariant metrics on  $SU(3)$  given by

$$g_t(u + x, v + y) := -(1 + t) B_{\mathfrak{su}(3)}(u, v) - B_{\mathfrak{su}(3)}(x, y)$$

where  $t \in (-1, \infty)$ ,  $u, v \in \mathfrak{u}(2)$  and  $x, y \in \mathfrak{u}(2)^\perp$ . Aloff and Wallach have shown that the space  $(M_{kl}^7, g_t) := (SU(3), g_t)/T_{kl}$  has positive sectional curvature if and only if  $t \in (-1, 0)$ .

In the special case  $k = l = 1$  we will prove

**Proposition 1.**  $(V_3, \tilde{g}_\lambda)$  is isometric to  $(M_{11}^7, g_t)$  for  $t = -\frac{3}{2\lambda+3}$ .

According to the proof that we give below, the isometry is obtained upon combining the canonical diffeomorphisms between the following spaces

$$\begin{aligned} SU(3)/T_{11} &\cong ((SU(3)/T_{11}) \times U(2))/U^\bullet(2) \\ &= (SU(3) \times (U(2)/S^1))/U^\bullet(2) \\ &\cong (SU(3) \times SO(3))/U^\bullet(2), \end{aligned}$$

where  $U^\bullet(2) \subset SU(3) \times U(2)$  denotes the image of  $U(2)$  under the diagonal embedding  $(\iota, \text{id})$  and  $S^1 = T_{11}$  is the center of  $U(2)$ .

By Proposition 1 the 1-parameter family of normal homogeneous metrics on  $V_3$  coincides with the positively curved Aloff–Wallach metrics on  $M_{11}^7$ . Püttmann [1997] has computed the pinching constants of the metrics  $g_t$  on  $M_{11}^7$ , i.e., the ratios of the minimum and the maximum of their sectional curvatures. He arrived at the conclusion that the optimal pinching constant for these metrics is  $\frac{1}{37}$  and is attained at  $t = -\frac{3}{5}$ . It is a curious fact that the optimal metric  $g_{-3/5}$  is Einstein and, by Proposition 1, also the natural metric induced by the Killing form on the product  $\mathfrak{su}(3) \times \mathfrak{so}(3)$ .

The other positively curved Aloff–Wallach examples are not normal homogeneous spaces. However, they may be considered as “normal biquotients”, since their metrics can be constructed from biinvariant metrics on Lie groups in the following way:

**Proposition 2.** We consider the biinvariant metric  $\hat{g}_\lambda$  on  $SU(3) \times U(2)$  given by

$$\hat{g}_\lambda((x, u), (y, v)) := -B_{\mathfrak{su}(3)}(x, y) - \frac{2}{3}\lambda B_{\mathfrak{su}(3)}(\iota_*(u), \iota_*(v))$$

for  $x, y \in \mathfrak{su}(3)$ ,  $u, v \in \mathfrak{u}(2)$  and assume  $t = -\frac{3}{2\lambda+3}$ . The map

$$\begin{aligned} F: (SU(3), g_t) &\rightarrow (SU(3) \times U(2), \hat{g}_\lambda)/U^\bullet(2), \\ A &\mapsto (A, e) \cdot U^\bullet(2) \end{aligned}$$

itself is an isometry. Furthermore, it induces an isometry between the Aloff–Wallach space  $(M_{kl}^7, g_t)$  and the biquotient

$$(Q_{kl}^7, \hat{g}_\lambda) := (\{e\} \times T_{kl}) \backslash (SU(3) \times U(2), \hat{g}_\lambda)/U^\bullet(2).$$

The factor  $\frac{2}{3}$  in the definition of  $\hat{g}_\lambda$  has been chosen such that the restriction of  $-\hat{g}_1$  to  $\mathfrak{su}(3) \times \mathfrak{su}(2)$  coincides with the Killing form of  $\mathfrak{su}(3) \times \mathfrak{su}(2)$ .

- Remarks.*
1. Wallach [W1972] and Bérard Bergery [1976] have classified the simply connected, positively curved, homogeneous spaces in even and odd dimensions. More precisely, they listed all pairs  $(\mathfrak{g}, \mathfrak{h})$  of Lie algebras with  $\mathfrak{h} \subset \mathfrak{g}$  that correspond to positively curved, homogeneous spaces  $\mathbf{G}/\mathbf{H}$ . Using this classification, it can be shown that a simply connected, normal homogeneous space of positive sectional curvature is either diffeomorphic to a compact rank-one symmetric space or to one of the exceptional spaces  $V_1$ ,  $V_2$  and  $V_3$ .
  2. The pair  $(A_2 \oplus A_1, A_1^\bullet \oplus \mathbb{R})$  corresponding to  $V_3$  appears in the classification of Bérard Bergery [1976]. For the geometric properties of  $V_3$ , however, he refers to an earlier paper [1975] where he claims that  $V_3$  does not admit a normal homogeneous metric of positive sectional curvature. This remark is related to an omission in Berger's classification [1961] of positively curved, normal homogeneous spaces, which can be traced back to the treatment of the non-simple case on page 218 in his paper. There the equation " $\dim(\mathbf{T}'_1) = \dim(\mathbf{T} \cap \mathbf{L}_1) + 1$ " does not hold for the pair  $(A_2 \oplus A_1, A_1^\bullet \oplus \mathbb{R})$ .
  3. The natural operation of  $\mathrm{Sp}(2)$  on  $V_1$  has kernel  $\{\pm \mathrm{id}\}$ . Using the fact that  $\mathrm{Sp}(2)/\{\pm \mathrm{id}\} \cong \mathrm{SO}(5)$  it follows that  $V_1$  can be described as quotient  $\mathrm{SO}(5)/\mathrm{SO}(3)$ . Eschenburg<sup>1</sup> has observed that the corresponding embedding  $\mathrm{SO}(3) \hookrightarrow \mathrm{SO}(5)$  is induced by the canonical action of  $\mathrm{SO}(3)$  on the space of traceless, symmetric, real  $3 \times 3$ -matrices.
  4. The pinching constants of  $V_1$  and  $V_2$  are  $\frac{1}{37}$  and  $\frac{16}{29 \cdot 37}$ , respectively (see Eliasson [1966] and Heintze [1971]). Püttmann [1997] proved that  $V_2$  admits a homogeneous metric with pinching constant  $\frac{1}{37}$ , too. This metric is obtained upon shrinking the natural metric in the direction of the fibers of the canonical projection  $V_2 = \mathrm{SU}(5)/\mathbf{H} \rightarrow \mathrm{SU}(5)/\mathrm{U}(4) \cong \mathbb{C}\mathbb{P}^4$ . Thus  $V_2$  with this metric can still be described as a "normal biquotient".
  5. Püttmann [1997] also studied for  $k \neq l$  the curvature of all homogeneous metrics on  $M_{kl}^7$  which up to scaling constitute a 3-parameter family. He has shown that in general the optimal pinching constant is not attained in the 1-parameter Aloff–Wallach family.

*Proof of Proposition 1.* By Proposition 2 it is sufficient to show that the manifolds  $(Q_{11}^7, \hat{g}_\lambda)$  and  $(V_3, \tilde{g}_\lambda)$  are isometric. The subgroup  $\mathbf{T}_{11}$  is the center of  $\mathrm{U}(2)$ , and  $\mathbf{T}_{11} \backslash \mathrm{U}(2) = \mathrm{U}(2)/\mathbf{T}_{11}$  is the Lie group  $\mathrm{SO}(3)$ . Since a Lie algebra isomorphism is a linear isometry provided that both Lie algebras are equipped with their Killing forms, it follows that the homomorphism

$$\varphi: (\mathrm{SU}(3) \times \mathrm{U}(2), \hat{g}_\lambda) \rightarrow \mathrm{SU}(3) \times (\mathrm{U}(2)/\mathbf{T}_{11}) \cong (\mathrm{SU}(3) \times \mathrm{SO}(3), \tilde{g}_\lambda)$$

is a Riemannian submersion. Moreover  $\varphi$  maps the group  $(\{e\} \times \mathbf{T}_{11}) \cdot \mathrm{U}^\bullet(2)$  onto  $\mathrm{U}^\bullet(2)$  and thus it induces an isometry  $(Q_{11}^7, \hat{g}_\lambda) \rightarrow (V_3, \tilde{g}_\lambda)$ .  $\square$

*Proof of Proposition 2.* Let  $\sigma: (\mathrm{SU}(3) \times \mathrm{U}(2), \hat{g}_\lambda) \rightarrow (\mathrm{SU}(3) \times \mathrm{U}(2))/\mathrm{U}^\bullet(2)$  be the projection. Following our conventions we shall also write  $\hat{g}_\lambda$  for the induced metric on the image of  $\sigma$ .

For  $x \in \mathfrak{u}(2)^\perp$  the vector  $(x, 0) \in \mathfrak{su}(3) \times \mathfrak{u}(2)$  is horizontal with respect to  $\sigma$  and thus  $\|x\|_{g_t} = \|F_*(x)\|_{\hat{g}_\lambda}$ . For  $v \in \mathfrak{u}(2)$  the horizontal component  $(v, 0)^h$  of  $(v, 0)$  in

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$(\mathfrak{su}(3) \times \mathfrak{u}(2), \hat{g}_\lambda)$  is given by

$$(v, 0)^h = \left( \frac{2\lambda}{2\lambda + 3} v, \frac{-3}{2\lambda + 3} v \right).$$

Hence,

$$\|F_*(v)\|_{\hat{g}_\lambda}^2 = \|(v, 0)^h\|_{\hat{g}_\lambda}^2 = -\frac{4\lambda^2 + 6\lambda}{(2\lambda + 3)^2} B_{\mathfrak{su}(3)}(v, v) = \|v\|_{g_t}^2.$$

Moreover, the vectors  $F_*(x)$  and  $F_*(v)$  are perpendicular and thus  $F$  is an isometry.

Next we observe that

$$F(gs^{-1}) = (gs^{-1}, e) \cdot \mathbf{U}^\bullet(2) = (g, s) \cdot \mathbf{U}^\bullet(2) \quad \text{for } s \in \mathbb{T}_{kl}.$$

In other words,  $F$  maps the left cosets of  $\mathbb{T}_{kl}$  in  $\mathbf{SU}(3)$  onto the fibers of the projection

$$(\mathbf{SU}(3) \times \mathbf{U}(2))/\mathbf{U}^\bullet(2) \longrightarrow (\{e\} \times \mathbb{T}_{kl}) \backslash (\mathbf{SU}(3) \times \mathbf{U}(2))/\mathbf{U}^\bullet(2).$$

Therefore  $F$  induces an isometry  $(M_{kl}^7, g_t) \rightarrow (Q_{kl}^7, \hat{g}_\lambda)$ .  $\square$

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