PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 4, April 1999, Pages 1127–1131
S 0002-9939(99)04645-6

PROJECTIONS FROM $L(E,F)$ ONTO $K(E,F)$

PIERRE DAVID SAPHAR

(Communicated by Dale E. Alspach)

Abstract. Let E and F be two infinite dimensional real Banach spaces. The following question is classical and long-standing. Are the following properties equivalent?

a) There exists a projection from the space $L(E,F)$ of continuous linear operators onto the space $K(E,F)$ of compact linear operators.

b) $L(E,F) = K(E,F)$.

The answer is positive in certain cases, in particular if E or F has an unconditional basis. It seems that there are few results in the direction of a general solution. For example, suppose that E and F are reflexive and that E or F has the approximation property. Then, if $L(E,F) \neq K(E,F)$, there is no projection of norm 1, from $L(E,F)$ onto $K(E,F)$. In this paper, one obtains, in particular, the following result:

Theorem. *Let F be a real Banach space which is reflexive (resp. with a separable dual), of infinite dimension, and such that F^* has the approximation property. Let λ be a real scalar with $1 < \lambda < 2$. Then F can be equivalently renormed such that, for any projection P from $L(F)$ onto $K(F)$, one has $\|P\| \geq \lambda$. One gives also various results with two spaces E and F.*

1. Notation

All the Banach spaces in this work are real. A Banach space E is considered without special notation, as a subspace of its bidual E^{**}. For two Banach spaces E and F, we denote by $L(E,F)$ (resp. $K(E,F)$) the space of continuous (resp. compact) linear operators from E to F. If $E = F$, we simply write $L(E,F) = L(E)$, $K(E,F) = K(E)$. We denote by (AP) (resp. (MAP)) the standard notions of approximation property (resp. metric approximation property); see [7]. We denote by π (resp. ε) the projective (resp. injective) tensor norm on $E \otimes F$. For basic facts on tensor products see, for example, [9], chap. IX, 2. Let E be a Banach space and X a closed subspace of the dual E^*. We denote by $r(X)$ the characteristic of X, which is defined to be the greatest constant r such that $\sup_{x^* \in X; \|x^*\| \leq 1} |\langle x, x^* \rangle| \geq r\|x\|$, for every $x \in E$. See [2], for various properties of the characteristic.

Received by the editors October 30, 1996 and, in revised form, July 28, 1997.

1991 *Mathematics Subject Classification.* Primary 46B20; Secondary 46B28.

Key words and phrases. Space of continuous linear operators, space of compact operators, projection.

This research was supported by the fund for the promotion of Research at the Technion.

2. The set $G(E, M)$

Extending the notion of Godun set which is defined and studied in [4] (see also [11]), one sets the following:

Definition 2.1. Let E be a Banach space and M a closed subspace of E^{**} which contains E. We define the set $G(E, M)$ of positive scalars λ such that for any $x^{**} \in M$, there exists a net (x_b) in E which verifies the following properties:

1) $x_b \to x^{**}$ for the weak star topology $\sigma(E^{**}, E^*)$,
2) $\overline{\lim}_b \|x^{**} - \lambda x_b\| \leq \|x^{**}\|$.

By a simple exercise, one can prove that $G(E, E) = [0, 2]$. Moreover, it is clear that the function $M \to G(E, M)$ is decreasing. It follows that for any M, $G(E, M) \subset [0, 2]$.

Lemma 2.2. *Let E be a Banach space, M a subspace of E^{**} which contains E and λ a real scalar, $1 < \lambda$, such that $\lambda \in G(E, M)$. Then, for every $\varphi \in M \backslash \{0\}$, $r(\ker \varphi) \leq \frac{1}{\lambda}$. Moreover, if $M \neq E$, then there is no projection from E onto M with norm $< \lambda$.*

Proof. There exists a net (x_b) of E such that $x_b \to \varphi$ for the weak star topology $\sigma(E^{**}, E^*)$ and $\overline{\lim}_b \|\varphi - \lambda x_b\| \leq \|\varphi\|$. One has

$$r(\ker \varphi) \leq \frac{\|\frac{1}{\lambda}\varphi - x_b\|}{\|x_b\|} = \frac{1}{\lambda} \frac{\|\varphi - \lambda x_b\|}{\|x_b\|}$$

for each b (see [2]), hence

$$r(\ker \varphi) \leq \frac{1}{\lambda} \frac{\overline{\lim}_b \|\varphi - \lambda x_b\|}{\underline{\lim}_b \|x_b\|} \leq \frac{1}{\lambda}.$$

Suppose that $M \neq E$ and let P be a projection from M onto E. One has $M = E \oplus \ker P$. Let φ be an element of $\ker P \backslash \{0\}$ and $x \in E$ of norm 1. For any real α one has:

$$\begin{aligned} &x = P(x + \alpha\varphi), \\ &1 \leq \|P\| \, \|x + \alpha\varphi\|, \\ &1 \leq \|P\| r(\ker \varphi) \quad \text{(see [2])}. \end{aligned}$$

The result follows.

Proposition 2.3. *Let E be a Banach space, M a subspace of E^{**} which contains E and F a closed subspace of E. Let M_1 be a closed subspace of F^{**} which contains F. We identify F^{**} with $F^{\perp\perp}$. Then, if M_1 is a subspace of M, $G(E, M)$ is a subset of $G(F, M_1)$.*

Proof. We use a method of ([8], sublemma, p. 378). Take $\lambda \in G(E, M)$ and let φ be a non-null element of M_1. There exists a net (x_b) of E such that $x_b \to \varphi$ for the weak star topology $\sigma(E^{**}, E^*)$ and $\overline{\lim}_b \|\varphi - \lambda x_b\| \leq \|\varphi\|$. Define the following subsets:

$$\begin{aligned} U &= \{y | y \in F, \|y\| \leq \|\varphi\|\} \quad \text{and} \\ V_b &= \{t | t \in E, t \in \overline{\operatorname*{conv}_{a \geq b}}(x_a)\} \quad \text{for each } b. \end{aligned}$$

It is clear that U and V_b are closed convex subsets of E. For two subsets A and B of E, $d(A, B)$ denotes the distance between A and B. Then, we will show that, for each b, $d(U, V_b) = 0$. Indeed, if $d(U, V_{b_0}) > 0$ for b_0, then there exists, by the Hahn-Banach Theorem, an $x^* \in E^*$ with $\|x^*\| = 1$ and

$$\sup_{y \in U} |\langle y, x^* \rangle| < \inf_{t \in V_{b_0}} |\langle t, x^* \rangle|.$$

By Goldstine's theorem, it follows that

$$|\langle x^*, \varphi \rangle| \leq \sup_{y \in U} |\langle y, x^* \rangle| < \inf_{t \in V_{b_0}} |\langle t, x^* \rangle| \leq |\langle x^*, \varphi \rangle|,$$

which is a contradiction. Then, there exists $y_b \in U$ and $t_b \in V_b$ for each b, so that $\|y_b - t_b\| \to 0$. One checks easily that $t_b \to \varphi$, for the weak star topology $\sigma(E^{**}, E^*)$ and it follows that $y_b \to \varphi$ for the weak star topology $\sigma(F^{**}, F^*)$. Moreover, since $t_b \in V_b$ for each b, it follows that $\overline{\lim}_b \|\varphi - \lambda t_b\| \leq \|\varphi\|$. One concludes that $\overline{\lim}_b \|\varphi - \lambda y_b\| \leq \|\varphi\|$. The result is obtained.

3. Applications to the space $K(E,F)$

Proposition 3.1. *Let F be a Banach space with a separable dual and λ a scalar with $1 < \lambda < 2$. Then F can be equivalently renormed so that*

1) *If F^* has the (AP), for any Banach space E, $\lambda \in G(K(E,F), L(E,F))$. Moreover, if $K(E,F) \neq L(E,F)$, there is no projection from $L(E,F)$ onto $K(E,F)$ with norm $< \lambda$.*

2) *For any reflexive Banach space E, $\lambda \in G(K(E,F), K(E,F)^{**})$.*

Proof. A. We suppose in this part that F has a shrinking basis, $(e_k, f_k)_{k \geq 1}$, with $e_k \in F$ and $f_k \in F^*$. Let $(S_n)_{n \geq 1}$ be the natural projections associated to the basis. By ([1], Lemma 3.4) there exists an equivalent norm on F so that $\|S_n\| = \|I - \lambda S_n\| = 1$, for each $n \geq 1$. In part A, F is equipped with this norm.

1) Let E be a Banach space. Since F^* is separable and with the (AP) one has the following equalities of Banach spaces:

$$E^* \otimes_\varepsilon F = K(E,F),$$
$$(E^* \otimes_\varepsilon F)^* = F^* \otimes_\pi E^{**}, \quad \text{and}$$
$$(E^* \otimes_\varepsilon F)^{**} = L(E^{**}, F^{**}).$$

It is clear that $E^* \otimes_\varepsilon F \subset L(E,F) \subset L(E^{**}, F^{**})$ (we consider $L(E,F)$ as a subspace of $L(E^{**}, F^{**})$ by taking the transformation $T \to T^{**}$). Let T be an element of $L(E,F)$. For any $x^{**} \in E^{**}$ and $y^* \in F^*$, one has:

$$\langle y^*, (S_n T)^{**} x^{**} \rangle = \langle (S_n T)^* y^*, x^{**} \rangle = \langle T^* S_n^* y^*, x^{**} \rangle.$$

Since $S_n^* y^*$ converges to y^* in norm, the term $\langle y^*, (S_n T)^{**} x^{**} \rangle$ converges to $\langle y^*, T^{**}, x^{**} \rangle$. But the sequence $(S_n T)$ is bounded in norm and it follows that $S_n T \to T$ for the weak star topology $\sigma(K(E,F)^{**}, K(E,F)^*)$.

Moreover, $\|T - \lambda S_n T\| \leq \|I - \lambda S_n\| \, \|T\| \leq \|T\|$, for each n.

It follows that $\lambda \in G(K(E,F), L(E,F))$. By Lemma 2.2, one deduces that if $K(E,F) \neq L(E,F)$, then there is no projection from $L(E,F)$ onto $K(E,F)$ with norm $< \lambda$.

2) Let E be a reflexive Banach space. In this case, $K(E,F)^{**} = L(E, F^{**})$. One defines, for each n, $A_n \in K(F^{**}, F)$ by $A_n(y^{**}) = \sum_{k=1}^{n} \langle f_k, y^{**} \rangle e_k$. Let T be an element of $L(E, F^{**})$. One checks, like in 1), that the sequence $(A_n T)$ of $K(E,F)$ converges to T for the weak star topology $\sigma(K(E,F)^{**}, K(E,F)^*)$ and also that $\|T - \lambda A_n T\| \leq \|T\|$, for each n. The result is obtained.

B. We suppose in this part that F has a separable dual. By [10], there exists a Banach space F_1 with a shrinking basis so that F can be isometrically identified with a subspace of F_1. We take the renorming of F_1 which is defined in part A. It gives a renorming of F. In all of part B we use this renorming of F.

1) Suppose that F^* has the (AP) and let E be a Banach space. One has also in this case $K(E,F)^{**} = L(E^{**}, F^{**})$. One checks easily the following inclusions of Banach spaces:

$$\begin{aligned} &K(E,F) \subset L(E,F) \subset L(E^{**}, F^{**}), \\ &K(E,F_1) \subset L(E,F_1) \subset L(E^{**}, F_1^{**}), \\ &K(E,F) \subset K(E,F_1), \qquad L(E,F) \subset L(E,F_1). \end{aligned}$$

Then the result follows by Lemma 2.2, Proposition 2.3 and part A of the proof.

2) Suppose that E is a reflexive Banach space. In this case by Proposition 2.3

$$G(K(E,F_1), K(E,F_1)^{**}) \subset G(K(E,F), K(E,F)^{**}).$$

Then we obtain the result by part A of the proof.

Lemma 3.2. *Let E and F be two reflexive Banach spaces such that $K(E,F)$ is nonreflexive. Then, there exists a separable subspace of F, F_1, which is one complemented in F, such that $K(E,F_1)$ is nonreflexive.*

Proof. By [3], Proposition 1.1, we can identify $K(E,F)^{**}$ with a subspace of $L(E,F)$. Let A be a noncompact element of $K(E,F)^{**}$. There exists a sequence $(x_n)_n$ of E, with $\|x_n\| = 1$, such that the sequence $(Ax_n)_n$ will be without convergent subsequence in F. Define $M = \overline{\operatorname{span}}_n(Ax_n)$. By [6], Proposition 1, there exists a separable subspace F_1 of F which contains M and a projection R from F onto F_1 such that $\|R\| = 1$. By [3], Proposition 1.1, there exists a bounded net (T_b) of elements of $K(E,F)$ such that $T_b \to A$ for the weak star topology $\sigma(L(E,F), E \otimes_\pi F^*)$. It follows that $RT_b \to RA$ for the weak star topology $\sigma(L(E,F_1), E \otimes_\pi F_1^*)$. One deduces that RA is an element of $K(E,F_1)^{**}$. Moreover $RAx_n = Ax_n$, for all n. It follows that RA is noncompact. Then $K(E,F_1)$ is nonreflexive.

Theorem 3.3. *Let E and F be two reflexive Banach spaces such that $K(E,F)$ is nonreflexive, and let λ be a scalar, $1 < \lambda < 2$. Then F can be equivalently renormed so that, for any renorming of E, there is no projection from $L(E,F)$ onto $K(E,F)$ with norm $< \lambda$.*

Proof. Let E and F be two reflexive Banach spaces such that $K(E,F)$ is nonreflexive. By Lemma 3.2, there exists a separable subspace of F, F_1, which is one complemented in F, such that $K(E,F_1)$ is nonreflexive. Let R be the projection of F onto F_1 with $\|R\| = 1$ and i the canonical injection from F_1 to F. We denote by $\| \, \|$ the initial norm on F. Fix a real number λ with $1 < \lambda < 2$. By Proposition 3.1 there exists an equivalent norm on F_1, that we denote by $\| \, \|_1$, such that after this renorming, $\lambda \in G(K(E,F_1), K(E,F_1)^{**})$. We define an equivalent norm on F, $\| \, \|_\lambda$, by the formula $\|x\|_\lambda = \operatorname{Max}\{\|Rx\|_1, \|(I-R)x\|\}$, $x \in F$. In the continuation of the proof F is equipped with the norm $\| \, \|_\lambda$. It is very easy to check that $\|R\|_\lambda = 1$.[1] Let P be a projection from $K(E,F)^{**}$ onto $K(E,F)$ and $U \in K(E,F_1)^{**}$. With the help of [3], Proposition 1.1, one verifies that iU is an element of $K(E,F)^{**}$. We define the transformation P_1 from $K(E,F_1)^{**}$ to $K(E,F_1)$ by $P_1(U) = R \circ P(iU)$.

[1] $\|R\|_\lambda$ is the norm of R when F is equipped with $\| \, \|_\lambda$.

We check that P_1 is a linear projection onto $K(E,F_1)$ and also that $\lambda \le \|P_1\|_\lambda$, by Proposition 3.1 and Lemma 2.2. One deduces that $\lambda \le \|P\|_\lambda$. The same property follows immediately for any projection from $L(E,F)$ onto $K(E,F)$ by [3], Proposition 1.1. Moreover, the construction of the renorming of F does depend on E, but only on the isomorphic class of E and is, in fact, not affected by renorming E.

Remark 3.4. If E and F are two reflexive Banach spaces such that E or F has the (AP) and $K(E,F) \neq L(E,F)$, the conclusions of Theorem 3.3 hold since, in this case, $K(E,F)^{**} = L(E,F)$.

Theorem 3.5. *Let F be a Banach space which is reflexive (resp. with a separable dual) and of infinite dimension. Suppose that F (resp. F^*) has the (AP). Then, for any scalar λ, $1 < \lambda < 2$, F can be equivalently renormed so that there is no projection from $L(F)$ into $K(F)$ with norm $< \lambda$.*

This follows from Proposition 3.1 and Remark 3.4.

Question 3.6. Is it possible to take $\lambda = 2$ in the previous results?

Remark 3.7. The value $\lambda = 2$ seems interesting since one has the following:

Proposition 3.8. *Let E be a Banach space such that $L(E) = K(E) \oplus \operatorname{span}(I)$, and let P be the projection from $L(E)$ onto $K(E)$ with $\ker P = \operatorname{span}(I)$. Then, $\|P\| \le 2$.*

Acknowledgment

The author wishes to thank the referee for very valuable remarks and suggestions.

References

1. P. G. Casazza and N. J. Kalton, *Notes on approximation properties on separable Banach spaces*, in: Geometry of Banach Spaces, London Math. Soc. Lecture Note Ser. 158, Cambridge Univ. Press (1991), 49–63. MR **92d:**46022
2. J. Dixmier, *Sur un théorème de Banach*, Duke Math. J. **15** (1948), 1057–1071. MR **10:**306g
3. G. Godefroy and P. D. Saphar, *Duality in spaces of operators and smooth norms on Banach spaces*, Illinois J. Math. **32** (1988), 673–695. MR **89j:**47026
4. G. Godefroy, N. J. Kalton and P. D. Saphar, *Unconditional ideals in Banach spaces*, Studia Math. **104** (1993), 13–59. MR **94k:**46024
5. K. John, *On the uncomplemented subspace $K(X,Y)$*, Czechoslovak Math. J. **42** (117) (1992), 167–173. MR **93b:**47085
6. J. Lindenstrauss, *On nonseparable reflexive Banach spaces*, Bull. Amer. Math. Soc. **72** (1966), 967–970. MR **34:**4875
7. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, vol. I, Sequence spaces, Springer-Verlag, Berlin, 1977. MR **58:**17766
8. E. Odell and H. P. Rosenthal, *A double dual characterization of separable Banach spaces containing ℓ_1*, Israel J. Math. **20** (1975), 375–384. MR **51:**13654
9. H. H. Schaefer, *Banach lattices and positive operators*, Springer-Verlag, New York, 1974. MR **54:**11023
10. M. Zippin, *Banach spaces with separable duals*, Trans. Amer. Math. Soc. **310** (1988), 371–379. MR **90b:**46028
11. B. V. Godun, *Unconditional bases and spanning basic sequences*, Izv. Vyssh. Uchebn. Zaved. Mat. **24** (1980), 69–72. MR **82h:**46017

Department of Mathematics, Technion–Israel Institute of Technology, Haifa, Israel
E-mail address: `saphar@techunix.technion.ac.il`