PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 4, April 1999, Pages 1201–1208
S 0002-9939(99)04654-7

LIPSCHITZ PRECOMPACTNESS FOR CLOSED NEGATIVELY CURVED MANIFOLDS

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(Communicated by Christopher Croke)

ABSTRACT. We prove that, given a integer $n \geq 3$ and a group π, the class of closed Riemannian n-manifolds of uniformly bounded negative sectional curvatures and with fundamental groups isomorphic to π is precompact in the Lipschitz topology. In particular, the class breaks into finitely many diffeomorphism types.

§1. INTRODUCTION

According to the Mostow rigidity theorem, the isometry type of a closed locally symmetric negatively curved Riemannian manifold is uniquely determined by its fundamental group. This is no longer true for manifolds of variable sectional curvature. The purpose of this note is to observe that there are essentially finitely many possibilities for the geometry and topology of such manifolds provided the sectional curvature is pinched between two negative fixed constants.

1.1. Theorem. *For any number $b \in [-1, 0)$ and a group π, the class $\mathcal{M}_{n,b,\pi}$ of closed Riemannian manifolds of dimension $n \geq 3$ with sectional curvatures in $[-1, b]$ and fundamental groups isomorphic to π is precompact in the Lipschitz topology.*

Recall that the class of all compact Riemannian manifolds of a given dimension has the so-called Lipschitz topology, namely, two manifolds M and N are said to be ϵ-*close* if there exists a diffeomorphism $f : M \to N$ such that both f and f^{-1} are e^{ϵ}-Lipschitz. A class of manifolds is called *precompact* if for any positive ϵ, every sequence of manifolds in the class has a subsequence whose members are mutually ϵ-close. A landmark theorem of Gromov asserts the Lipschitz precompactness of the class of closed Riemannian manifolds of uniformly bounded diameters and sectional curvatures and with a uniform lower bound on the injectivity radii. Here is an immediate corollary of Lipschitz precompactness.

1.2. Corollary. *For any number $b \in [-1, 0)$, a positive integer n, and a group π, there exist positive numbers $D > d$ depending only on b and π such that any manifold from $\mathcal{M}_{n,b,\pi}$ has diameter in $[d, D]$. The same conclusion is true for volume and injectivity radius.*

Received by the editors July 30, 1997.
1991 *Mathematics Subject Classification.* Primary 53C20, 53C23; Secondary 20F32, 57R55.
Key words and phrases. Lipschitz convergence, negatively curved manifold.

Note that the corollary fails if $b = 0$. Indeed, given a closed negatively curved manifold M, the rescaled manifold $k \cdot M$ has sectional curvature within $[1, 0)$ for large k, while the diameters are unbounded. Combining 1.2 with the Gromov-Fukaya pinching theorem [F] we get

1.3. Corollary. *Given a group π and a positive integer n, there exists $\epsilon \in (0, 1)$ such that any manifold from $\mathcal{M}_{n,-1+\epsilon,\pi}$ is diffeomorphic to a manifold of constant negative sectional curvature.*

Topology of closed negatively curved manifolds seems to be encoded in the fundamental group. It is a very particular case of the Borel conjecture that any two closed homotopy equivalent negatively curved manifolds must be homeomorphic. However, in higher dimensions there are homeomorphic negatively curved manifolds that are not diffeomorphic [FJ1, 2]. In fact, Farrell and Jones proved that given a closed real hyperbolic manifold N of dimension $n \geq 5$ and $\epsilon > 0$, there is a finite cover $N_\epsilon \to N$ such that for any smooth homotopy n-sphere Σ the connected sum $N_\epsilon \# \Sigma$ has a Riemannian metric of sectional curvature within $[-1, -1 + \epsilon]$. Moreover, given a pair of nondiffeomorphic homotopy spheres Σ_1 and Σ_2, the manifolds $N_\epsilon \# \Sigma_1$ and $N_\epsilon \# \Sigma_2$ are not diffeomorphic. The following result is another standard application of Lipschitz precompactness.

1.4. Corollary. *For any number $b \in [-1, 0)$ and a group π there exist at most finitely many nondiffeomorphic closed Riemannian manifolds with sectional curvatures in $[-1, b]$ and fundamental groups isomorphic to π.*

In dimensions ≥ 5 a much stronger statement is true. Namely, a deep theorem of Farrell and Jones implies that any manifold homotopy equivalent to a closed nonpositively curved manifold M must be homeomorphic to M [FJ3] Since a manifold of dimension ≥ 5 can have only finitely many smooth structures [KS], every homotopy type contains finitely many nondiffeomorphic closed nonpositively curved manifolds of dimension ≥ 5.

No results similar to this theorem of Farrell and Jones are available in dimension four because it is unknown in this dimension if the topological surgery works for the word-hyperbolic fundamental groups. In general, one expects most closed 4–manifolds to have more than one smooth structure.

According to the (as yet unproved) Geometrization Conjecture of Thurston, any closed negatively curved 3–manifold should admit a metric of constant negative curvature. In particular, if the conjecture is true, any two homotopy equivalent closed negatively curved 3–manifolds must be diffeomorphic, due to the Mostow rigidity theorem. Notice that for closed Haken 3–manifolds with no curvature assumptions, homotopy equivalence implies diffeomorphism [W]. The same is true for any closed irreducible 3–manifold that is homotopy equivalent to a 3–manifold of constant negative curvature [G], [GMT].

We finally sketch the proof of Theorem 1.1. In the case of pinched negative sectional curvature, arguments based on the Margulis lemma provide a universal lower bound on the injectivity radius, so there is no collapse. This guarantees precompactness in the *pointed* Lipschitz topology where limit points may be noncompact manifolds. It turns out to be technically convenient to think of closed manifolds from $\mathcal{M}_{n,b,\pi}$ as Hadamard manifolds equipped with the isometric, free, cocompact actions of π. Using a method due to Bestvina and Paulin and a famous result of Rips, Bestvina and Feighn on small actions on real trees, one can show that these actions do not "diverge". This implies that whenever a sequence $M_k \in \mathcal{M}_{n,b,\pi}$ converges in the pointed Lipschitz topology to a manifold M, the group $\pi_1(M)$ contains a subgroup isomorphic to π. Counting cohomological dimension, one can deduce that M is compact, and hence M_k converges to M in the (nonpointed) Lipschitz topology.

The author is grateful to Mladen Bestvina and Bernhard Leeb for helpful discussions on the subject of this paper and to Bill Goldman for friendly support.

§2. Two types of convergence

2.1. Equivariant pointed Lipschitz topology. Let Γ_k be a discrete subgroup of the isometry group of a complete Riemannian manifold X_k and p_k be a point of X_k. The class of all such triples $\{(X_k, p_k, \Gamma_k)\}$ can be given the so-called equivariant pointed Lipschitz topology [F]; when Γ_k is trivial this reduces to the usual pointed Lipschitz topology. For convenience of the reader we give here some definitions borrowed from [F].

For a group Γ acting on a pointed metric space (X, p, d) the set $\{\gamma \in \Gamma : d(p, \gamma(p)) < r\}$ is denoted by $\Gamma(r)$. An open ball in X of radius r with center at p is denoted by $B_r(p, X)$.

For $i = 1, 2$, let (X_i, p_i) be a pointed complete metric space with the distance function d_i and let Γ_i be a discrete group of isometries of X_i. In addition, assume that X_i is a C^∞–manifold. Take any $\epsilon > 0$.

Then a quadruple $(f_1, f_2, \phi_1, \phi_2)$ of maps $f_i : B_{1/\epsilon}(p_i, X_i) \to B_{1/\epsilon}(p_{3-i}, X_{3-i})$ and $\phi_i : \Gamma_i(1/3\epsilon) \to \Gamma_{3-i}$ is called an *ϵ–Lipschitz approximation* between the triples (X_1, p_1, Γ_1) and (X_2, p_2, Γ_2) if the following seven conditions hold:

- f_i is a diffeomorphism onto its image;
- for each $x_i \in B_{1/3\epsilon}(p_i, X_i)$ and every $\gamma_i \in \Gamma_i(1/3\epsilon)$, $f_i(\gamma_i(x_i)) = \phi_i(\gamma_i)(f_i(x_i))$;
- for every $x_i, x_i' \in B_{1/\epsilon}(p_i, X_i)$, $e^{-\epsilon} < d_{3-i}(f_i(x_i), f_i(x_i'))/d_i(x_i, x_i') < e^{\epsilon}$;
- $f_i(B_{1/\epsilon}(p_i, X_i)) \supset B_{(1/\epsilon)-\epsilon}(p_{3-i}, X_{3-i})$ and $\phi_i(\Gamma_i(1/3\epsilon)) \supset \Gamma_{3-i}(1/3\epsilon - \epsilon)$;
- $f_i(B_{(1/\epsilon)-\epsilon}(p_i, X_i)) \supset B_{1/\epsilon}(p_{3-i}, X_{3-i})$ and $\phi_i(\Gamma_i(1/3\epsilon - \epsilon)) \supset \Gamma_{3-i}(1/3\epsilon)$;
- $f_{3-i} \circ f_i|_{B_{(1/\epsilon)-\epsilon}(p_i, X_i)} = \mathrm{id}$ and $\phi_{3-i} \circ \phi_i|_{\Gamma_i(1/3\epsilon-\epsilon)} = \mathrm{id}$;
- $d_{3-i}(f_i(p_i), p_{3-i}) < \epsilon$.

We say a sequence of triples (X_k, p_k, Γ_k) *converges* to (X, p, Γ) in the equivariant pointed Lipschitz topology if, for any $\epsilon > 0$, there is $k(\epsilon)$ such that for all $k > k(\epsilon)$, there exists an ϵ–Lipschitz approximation between (X_k, p_k, Γ_k) and (X, p, Γ).

Notice that if all the groups Γ_k are trivial, then Γ is trivial; in this case we say that that (X_k, p_k) converges to (X, p) in the pointed Lipschitz topology. Note that if X_k is a complete Riemannian manifold for all k, then the space X is necessarily a C^∞–manifold with a complete $C^{1,\alpha}$–Riemannian metric [GW].

If X_k and X are compact, then (X_k, p_k) converges to (X, p) in the pointed Lipschitz topology if and only if, for any $\epsilon > 0$ there is $k(\epsilon)$ such that for all $k > k(\epsilon)$, there exists a diffeomorphism $f : X_k \to X$ such that both f and f^{-1} are e^ϵ–Lipschitz. In this case we just say that the sequence X_k converges to X in the Lipschitz topology.

2.2. Pointwise convergence topology. Suppose that, for some $p_k \in X_k$, the sequence (X_k, p_k) converges to (X, p) in the pointed Lipschitz topology.

We say that a sequence $x_k \in X_k$ *converges* to $x \in X$ if, for some ϵ, the sequence of ϵ–Lipschitz approximations (f_k, g_k) between (X_k, p_k) and (X, p) has the property that $d(f_k(x_k), x) = d_k(x_k, g_k(x)) \to 0$ as $k \to \infty$. Trivial example: if (X_k, p_k) converges to (X, p) in the pointed Lipschitz topology, then p_k converges to p.

Given a sequence of isometries $\gamma_k \in \mathrm{Isom}(X_k)$ we say that γ_k *converges*, if for any $x_k \in X_k$, $\gamma_k(x_k)$ converges. The limiting transformation γ of X is necessarily an isometry.

We say that a sequence of actions (X_k, p_k, ρ_k) *converges* to an action (X, p, ρ) in the *pointwise convergence topology* if $\rho_k(\gamma)$ converges to $\rho(\gamma)$ for every $\gamma \in \pi$.

Clearly, if π is generated by a finite set S, then in order to prove $\rho_k \to \rho$, it suffices to check that $\rho_k(\gamma)$ converges to $\rho(\gamma)$ for every $\gamma \in S$.

Let $d_k : X_k \times X_k \to \mathbb{R}$ be the distance function. It is easy to check that a sequence of actions (X_k, p_k, ρ_k) is *precompact* in the pointwise convergence topology (i.e. every subsequence of (X_k, p_k, ρ_k) has a converging subsequence) if for any $\gamma \in \pi$ and any (or, equivalently, some) $x_k \in X_k$, the sequence $d_k(x_k, \rho_k(\gamma)(x_k))$ is bounded (the proof is in the spirit of [KN, 4.7]). Again, if π is generated by a finite subset S it is enough to verify the above for elements of S only.

2.3. Motivating example. Let X be a complete Riemannian manifold. Consider the isometry group $\mathrm{Isom}(X)$ of X and let π be a group.

The space $\mathrm{Hom}(\pi, \mathrm{Isom}(X))$ has a natural topology (which is usually called "algebraic topology" or "pointwise convergence topology"), namely ρ_k is said to converge to ρ if, for each $\gamma \in \pi$, $\rho_k(\gamma)$ converges to $\rho(\gamma)$ in the Lie group $\mathrm{Isom}(X)$. Note that if π is finitely generated, this topology on $\mathrm{Hom}(\pi, \mathrm{Isom}(X))$ coincides with the compact-open topology.

Certainly, for any $p \in X$, the constant sequence (X, p) converges to itself in pointed Lipschitz topology. Then, obviously, the sequence (X, p, ρ_k) converges in the pointwise convergence topology (as defined in 2.2) if and only if $\rho_k \in \mathrm{Hom}(\pi, \mathrm{Isom}(X))$ converges in the algebraic topology.

2.4. Lemma. *Let $\rho_k : \pi \to Isom(X_k)$ be a sequence of free, isometric actions of a discrete group π on complete Riemannian n-manifolds X_k.*

If the sequence $(X_k, p_k, \rho_k(\pi))$ converges in the equivariant pointed Lipschitz topology to (X, p, Γ) and (X_k, p_k, ρ_k) converges to (X, p, ρ) in the pointwise convergence topology, then

(1) *Γ acts freely, and*
(2) *ρ is injective, and*
(3) *$\rho(\pi) \subset \Gamma$.*

Proof. (1) Assume $\gamma \in \Gamma$ and $\gamma(x) = x$. Choose $\epsilon \in (0, 1/10)$ so that there is an ϵ–approximation $(f_k, g_k, \phi_k, \tau_k)$ of (X_k, p_k, ρ_k) and (X, p, Γ) and $x \in B(p, \epsilon/10)$. Then $g_k(x) = g_k(\gamma(x)) = \tau_k(\gamma)(g_k(x))$. Since $\rho_k(\pi)$ acts freely, $\tau_k(\gamma) = \mathrm{id}$. By the same argument $\tau_k(\mathrm{id}) = \mathrm{id}$. Hence $\mathrm{id} = \phi_k(\tau_k(\mathrm{id})) = \phi_k(\mathrm{id}) = \phi_k(\tau_k(\gamma)) = \gamma$ as desired.

(2) Assume $\rho(\gamma) = \mathrm{id}$. Fix any $\epsilon \in (0, 1/10)$. Take $x \in B(p, \epsilon/10)$ and consider an ϵ–approximation $(f_k, g_k, \phi_k, \tau_k)$ of (X_k, p_k, ρ_k) and (X, p, Γ). We have $g_k(x) \to x$ and $\rho_k(\gamma)(g_k(x)) \to \rho(\gamma)(x) = x$. Note that $d(x, \phi_k(\rho_k(\gamma))(x))$ is equal to

$$d(f_k(g_k(x)), f_k(\rho_k(\gamma)(g_k(x)))) < e^{\epsilon} d_k(g_k(x), \rho_k(\gamma)(g_k(x))) \underset{k\to\infty}{\longrightarrow} > 0.$$

Therefore, $\phi_k(\rho_k(\gamma)) = \mathrm{id}$, because Γ is a discrete subgroup that acts freely. Hence $\rho_k(\gamma) = \tau_k(\phi_k(\rho_k(\gamma))) = \tau_k(\mathrm{id}) = \mathrm{id}$. Since ρ_k is injective, $\gamma = \mathrm{id}$ as claimed.

(3) We need to show that $\rho(\gamma) \in \Gamma$, for any $\gamma \in \pi$. We can assume $\rho(\gamma) \neq \mathrm{id}$. Choose $\epsilon \in (0, 1/10)$ so that the ball $B(p, 1/11\epsilon)$ contains $\rho(\gamma)(p)$ and consider an ϵ–approximation $(f_k, g_k, \phi_k, \tau_k)$ of (X_k, p_k, ρ_k) and (X, p, Γ).

Then for all large enough k, $\rho_k(\gamma) \in B(p_k, 1/10\epsilon)$. Look at $\tau_k(\rho_k(\gamma)) \in \Gamma(1/9\epsilon)$. Since the set $\Gamma(1/9\epsilon)$ is finite, we can pass to a subsequence to assume that $\tau_k(\rho_k(\gamma))$ is equal to $\gamma_\epsilon \in \Gamma(1/9\epsilon)$; thus $\rho_k(\gamma) = \phi_k(\gamma_\epsilon)$.

Take an arbitrary $x \in B(p, 1/9\epsilon)$. Then $g_k(x) \to x$ and, hence, $\rho_k(\gamma)(g_k(x))$ converges to $\rho(\gamma)(x)$. Notice that $\rho_k(\gamma)(g_k(x)) = \phi_k(\gamma_\epsilon)(g_k(x)) \to \gamma_\epsilon(x)$. So $\rho(\gamma)(x) = \gamma_\epsilon(x)$ for any $x \in B(p, 1/9\epsilon)$.

Thus, for any small enough ϵ, we have found $\gamma_\epsilon \in \Gamma$ that is equal to $\rho(\gamma)$ on the ball $B(p, 1/9\epsilon)$. Since Γ acts freely, $\gamma_\epsilon = \gamma_{\epsilon'}$ for all $\epsilon' \leq \epsilon$, that is the element $\gamma_\epsilon \in \Gamma$ is independent of ϵ. Thus $\rho(\gamma) = \gamma_\epsilon$ everywhere and hence $\rho(\gamma) \in \Gamma$. □

§3. Main lemma

3.1. Proposition. *Assume that π is a finitely presented discrete group, that is not virtually nilpotent and does not have a nontrivial decomposition into an amalgamated product or an HNN-extension over a virtually nilpotent group.*

Let $\rho_k : \pi \to Isom(X_k)$ be an arbitrary sequence of free and isometric actions of π on Hadamard n-manifolds X_k. Assume that the sectional curvatures of X_k lie in $[-1, b]$ for $b < 0$.

Then, for some $p_k \in X_k$, $(X_k, p_k, \rho_k(\pi))$ is precompact in the equivariant pointed Lipschitz topology and (X_k, p_k, ρ_k) is precompact in the pointwise convergence topology.

Proof. Let $S \subset \pi$ be a finite subset that generates π. For $x \in X_k$, we denote by $D_k(x)$ the diameter of the set $\rho_k(S)(x)$. Set $D_k = \inf_{x \in X_k} D_k(x)$.

Suppose D_k is unbounded. Then it follows from a work of Bestvina and Paulin [Bes], [P1], [P2] (cf. [KL]) that there exists an action of π on a real tree with no global fixed point and virtually nilpotent arc stabilizers. For completeness we briefly review this construction. The rescaled pointed Hadamard manifold $\frac{1}{D_k} \cdot X_k$ has sectional curvature $\leq b \cdot D_k \to -\infty$ as $k \to \infty$. Find $q_k \in X_k$ such that $D_k(q_k) \leq D_k + 1/k$. Consider the sequence of triples $(\frac{1}{D_k} \cdot X_k, q_k, \rho_k)$. Repeating an argument of Paulin [P2, §4], we can pass to a subsequence that converges to a triple $(X_\infty, q_\infty, \rho_\infty)$. (For the definition of the convergence see [P1], [P2]. Paulin calls it "convergence in the Gromov topology".)

The limit space X_∞ is a length space of curvature $-\infty$, that is a real tree. Because of rescaling, the limit space has a natural isometric action ρ_∞ of π. By the Margulis lemma the stabilizer of any non-degenerate segment is virtually nilpotent (cf. [P1]). One can check that the action ρ_∞ has no global fixed point [P1], [P2]. Then it is a standard fact that there exists a unique π–invariant subtree T of X_∞ that has no proper π–invariant subtree. In fact T is the union of all the axes of all hyperbolic elements in π.

Note that any increasing sequence of virtually nilpotent subgroups of π is stationary. Indeed, since a virtually nilpotent group is amenable, the union U of an increasing sequence $U_1 \subset U_2 \subset U_3 \subset \dots$ of virtually nilpotent subgroups is also an amenable group. If the fundamental group of a complete manifold of pinched negative curvature is amenable, it must be finitely generated [BS], [Bow]. In particular, U is finitely generated; hence $U_n = U$ for some n. Thus, the π–action on the tree T is stable [BF, Proposition 3.2(2)].

We summarize that the π–action on T is stable, has virtually nilpotent arc stabilizers and no proper π–invariant subtree. Therefore, the Rips machine [BF, Theorem 9.5] produces a splitting of π over a virtually solvable group. Any amenable subgroup of π is virtually nilpotent [BS], [Bow]; hence π splits over a virtually nilpotent group. This is a contradiction with the assumption that D_k is unbounded.

Thus the sequence $D_k(q_k)$ is bounded. As we observed in 1.2, the sequence (X_k, q_k, ρ_k) is precompact in the pointwise convergence topology since the sequence $D_k(q_k)$ is bounded. Note that for any $p_k \in X_k$ such that $d_k(p_k, q_k)$ is bounded, the sequence (X_k, p_k, ρ_k) is also precompact in the pointwise convergence topology.

We now find $p_k \in X_k$ such that $d_k(p_k, q_k)$ is bounded and (X_k, p_k, ρ_k) is precompact in the equivariant pointed Lipschitz topology.

Let $\mu_n > 0$ be the Margulis constant for dimension n. Set $r_k = D_k(q_k) + 1$. Note that r_k is a bounded sequence of positive numbers such that, for any $\gamma \in S$, $\rho_k(\gamma)(B(q_k, r_k)) \cap B(q_k, r_k) \neq \emptyset$.

Assume that for some k and for every point x of the r_k–ball centered at q_k, there exists $\gamma \in \pi$ such that $d_k(x, \rho_k(\gamma)(x)) < \mu_n/2$. Then the whole ball $B(q_k, r_k)$ projects into the thin part $\{\text{InjRad} < \mu_n/2\}$ under the projection $\pi_k : X_k \to X_k/\rho_k(\pi)$. Thus the ball $B(p, r_k)$ lies in a connected component W of the π_k–preimage of the thin part of $X_k/\rho_k(\pi)$. According to [BGS, p. 111] the stabilizer of W in $\rho_k(\pi)$ is virtually nilpotent and, moreover, the stabilizer contains every element $\gamma \in \rho_k(\pi)$ with $\gamma(W) \cap W \neq \emptyset$. Therefore, the whole group $\rho_k(\pi)$ stabilizes W. Hence $\rho_k(\pi)$ must be virtually nilpotent. As ρ_k is injective, π is virtually nilpotent. A contradiction.

Thus, for every k, there exists $p_k \in B(q_k, r_k)$ such that the injectivity radius of $X_k/\rho_k(\pi)$ at the point $\pi_k(p_k)$ is at least $\mu_n/2$. Thus, by a theorem of Fukaya [F], $(X, p_k, \rho_k(\pi))$ is precompact in the equivariant pointed Lipschitz topology. □

3.2. Remark. It is well-known that the fundamental group of a closed negatively curved manifold of dimension ≥ 3 is not virtually nilpotent [Y] and does not split over a virtually nilpotent group (the latter is an easy group cohomology exercise, see for example [Bel]).

§4. Main theorem

4.1. Theorem. *Let $\rho_k : \pi \to Isom(X_k)$ be a sequence of free, isometric and cocompact actions of a discrete group π on complete Riemannian n-manifolds X_k.*

Assume that for some $p_k \in X_k$, $(X_k, p_k, \rho_k(\pi))$ converges in the equivariant pointed Lipschitz topology to (X, p, Γ) and that (X_k, p_k, ρ_k) converges in the pointwise convergence topology to (X, p, ρ). Suppose that the manifolds X_k and X are contractible.

Then the sequence of manifolds $\{X_k/\rho_k(\pi)\}$ converges to X/Γ in the Lipschitz topology.

Proof. Since $\rho(\pi) \subset \Gamma$, the quotient map $X \to X/\Gamma$ factors through $X/\rho(\pi)$. The fundamental group of $X/\rho(\pi)$ is isomorphic to π because X is contractible and ρ is an isomorphism onto its image.

The manifolds $X_k/\rho_k(\pi)$ and $X/\rho(\pi)$ are aspherical and with fundamental groups isomorphic to π. Hence, $X_k/\rho_k(\pi)$ and $X/\rho(\pi)$ are homotopy equivalent. Since $X_k/\rho_k(\pi)$ is compact, so is $X/\rho(\pi)$ (look at $\mathbb{Z}_2$–top–dimensional homology). Therefore, X/Γ is compact. Hence there exists a ball $B(p, r) \subset X$ that projects onto X/Γ. Take any ϵ so that $r < 1/3\epsilon$.

Take k so large that there is an ϵ–Lipschitz approximation $(\tilde{f}_k, \tilde{g}_k, \phi_k, \tau_k)$ of $(X_k, p_k, \rho_k(\pi))$ and (X, p, Γ). The equivariant diffeomorphism $\tilde{g}_k : B(p, 1/3\epsilon) \to X_k$ descends to a smooth embedding g_k of a closed manifold X/Γ into the closed manifold X_k/Γ_k. Similarly, $\tilde{f}_k$ descends to a smooth embedding $f_k : X_k/\Gamma_k \to X/\Gamma$ which is the inverse of g_k. By construction g_k and f_k are ϵ–Lipschitz. Therefore, X_k/Γ_k converges to X/Γ in the Lipschitz topology. □

4.2. Corollary. *Let $\rho_k : \pi \to Isom(X_k)$ be a sequence of free, isometric and cocompact actions of a discrete group π on Hadamard n-manifolds X_k.*

Assume that for some $p_k \in X_k$, $(X_k, p_k, \rho_k(\pi))$ converges in the equivariant pointed Lipschitz topology to (X, p, Γ) and that (X_k, p_k, ρ_k) converges in the pointwise convergence topology to (X, p, ρ).

Then the sequence of manifolds $\{X_k/\rho_k(\pi)\}$ converges to X/Γ in the Lipschitz topology.

Proof. According to 4.1, it suffices to show that X is contractible. Indeed, any spheroid in X lies in the diffeomorphic image of a metric ball in X_j. Any metric ball in a Hadamard manifold is contractible. Thus $\pi_*(X) = 1$ as desired. □

4.3. Corollary. *For any number $b \in [-1, 0)$ and a group π, the class $\mathcal{M}_{n,b,\pi}$ of closed Riemannian manifolds of dimension $n \geq 3$ with sectional curvatures in $[-1, b]$ and fundamental groups isomorphic to π is precompact in the Lipschitz topology.*

Proof. Combine 3.1, 3.2 and 4.2. □

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