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ON THE NUMBER OF SOLUTIONS OF AN ALGEBRAIC EQUATION ON THE CURVE $y = e^x + \sin x$, $x > 0$, AND A CONSEQUENCE FOR O-MINIMAL STRUCTURES

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ABSTRACT. We prove that every polynomial $P(x, y)$ of degree d has at most $2(d + 2)^{12}$ zeros on the curve $y = e^x + \sin(x)$, $x > 0$. As a consequence we deduce that the existence of a uniform bound for the number of zeros of polynomials of a fixed degree on an analytic curve does not imply that this curve belongs to an o-minimal structure.

MAIN RESULT

The aim of this note is to estimate the number of real solutions of the system

$$P(x, y) = 0, \quad y = e^x + \sin(x), \quad x > 0, \tag{1}$$

where $P(x, y)$ is a non–zero polynomial of degree d. As we prove below we have the following bound.

Theorem 1. *The number of solutions of the system (1) is not greater than* $A(d) = 2(d + 2)^{12}$.

Clearly the condition $x > 0$ in (1) cannot be omitted. For instance the x–axis intersects the graph $y = e^x + \sin(x)$ infinitely many times for $x < 0$.

Using Theorem 1 it is easy to construct a global analytic function with a similar behavior.

Corollary 2. *The number of solutions of the system*

$$P(x, y) = 0, \quad y = e^{x^2} + \sin(x^2), \tag{2}$$

is less than or equal to $2A(2d)$.

Proof. Take a polynomial $P(x, y)$ of degree d and eliminate the variable x from the equations $P(x, y) = 0$, $x^2 = v$. The resultant $R(v, y) = \mathrm{Res}_x(P(x, y), x^2 - v)$ is a polynomial of degree $\leq 2d$ in variables v, y.

If (x, y) is a solution of the system (2), then (v, y), where $v = x^2$, is a solution of the system

$$R(v, y) = 0, \quad y = e^v + \sin(v), \quad v \geq 0.$$

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Therefore by Theorem 1 the number of solutions of system (2) is not greater than $2A(2d)$. □

In the end of this note we give an application of Theorem 1 to the theory of o-minimal structures. This application was our original motivation to study the above question.

Proof of the Theorem

First we will show that it is enough to prove Theorem 1 for $P(x,y)=0$ being smooth. We use the following

Lemma 3. *Let $f:(0,\infty)\to\mathbb{R}$ be an analytic function such that $\#\{x : f(x)=\epsilon\}\le A$ for sufficiently small $|\epsilon|\neq 0$. Then $\#\{x : f(x)=0\}\le A$.*

Proof. We adopt here the proof of Lemma 4.2.6 in [BR]. Suppose first that $\#\{x : f(x)=0\}$ is finite. We divide the set of zeros of f into: n_1 zeros of odd multiplicity, n_2 local minima and n_3 local maxima.

Taking $\epsilon>0$ if $n_2\ge n_3$ and $\epsilon<0$ if $n_2<n_3$ it is easily seen that

$$\#\{x : f(x)=0\}=n_1+n_2+n_3\le \#\{x : f(x)=\epsilon\}\le A\ .$$

Note that the same argument shows that $\#\{x : f(x)=0\}$ is finite provided A is also. □

Now fix a polynomial $P(x,y)$ of degree d and assume that 0 is a critical value of P. Since the polynomials have only finitely many critical values, for sufficiently small $|\epsilon|\neq 0$, the curve $P(x,y)=\epsilon$ is smooth. Suppose that the estimate of Theorem 1 holds true for $P(x,y)-\epsilon$, that is $\#\{x : f(x)=\epsilon,\, x>0\}\le A(d)$ for $f(x)=P(x,e^x+\sin(x))$. By Lemma 3 $\#\{x : f(x)=0,\, x>0\}\le A(d)$, and therefore Theorem 1 holds true for $P(x,y)$ as well.

From now on we assume that $P(x,y)=0$ is a smooth curve.

Step 1. A bound on the number of connected components of the set

$$C=\{\,(x,y) : P(x,y)=0,\ e^x-1\le y\le e^x+1\,\}.$$

First we estimate the number of intersections of the curve $P(x,y)=0$ with the graphs $y=e^x+1$ and $y=e^x-1$. This is precisely the number of solutions of equations $P(x,e^x+1)=0$ and $P(x,e^x-1)=0$. By Khovansky's theorem, see for instance [BR, 4.1.1],

$$\#\{x : P(x,e^x+1)=0\}+\#\{x : P(x,e^x-1)=0\}\le 2d(d+1).$$

Since the curve $P(x,y)=0$ is smooth, it is a finite union of topological ovals and intervals. By Harnack's theorem [BR, 4.4.4, 5.3.2] their number is not greater than $(d^2-d+2)/2$. Hence $\#(\text{connected components of } C)\le (d^2-d+2)/2+2d(d+1)=(5d^2+3d+2)/2$.

Step 2. A bound on the "size" of a connected component of C.

Denote

$$L=d(d+1)^2(d+2)^2\ln 2/4+(d+1)^4(d+2)^3/8.$$

This part of the proof is based on

Lemma 4. *Assume that points $(0,y_1)$, (x_2,y_2) belong to the same component of C. Then $x_2\le L$.*

The proof of Lemma 4 is given in the next section. Lemma 4 has the following consequence.

Corollary 5. *Assume that points* (a, c_1), (b, c_2), *where* $0 \le a \le b$, *belong to the same component of* C. *Then* $b - a \le L$.

Proof. Assume that points (a, c_1), (b, c_2), $0 \le a \le b$, belong to the same component of the set C. It is easy to see that there exist (possibly different than (a, c_1), (b, c_2)) points (a, d_1), $(b, d_2) \in C$ and a connected set $D \subset C \cap \{ (x, y) : a \le x \le b \}$ joining them. We shall write D and C in a new system of coordinates $\bar{x}$, $\bar{y}$ such that $x = \bar{x} + a$, $y = e^a \bar{y}$.

For every point $(x, y) \in D$ we have

$$P(x, y) = 0, \quad e^x - 1 \le y \le e^x + 1, \quad a \le x \le b.$$

Therefore in the new coordinates

$$P(\bar{x} + a, e^a \bar{y}) = 0, \quad e^{\bar{x}+a} - 1 \le e^a \bar{y} \le e^{\bar{x}+a} + 1, \quad a \le \bar{x} + a \le b,$$

and consequently

$$\bar{P}(\bar{x}, \bar{y}) = 0, \quad e^{\bar{x}} - 1 \le \bar{y} \le e^{\bar{x}} + 1, \quad 0 \le \bar{x} \le b - a,$$

where $\bar{P}(\bar{x}, \bar{y}) = P(\bar{x} + a, e^a \bar{y})$.

Set $\bar{C} = \{ (\bar{x}, \bar{y}) : \bar{P}(\bar{x}, \bar{y}) = 0, e^{\bar{x}} - 1 \le \bar{y} \le e^{\bar{x}} + 1 \}$ and $\bar{D} = \{ (\bar{x}, \bar{y}) : (\bar{x} + a, e^a \bar{y}) \in D \}$. As we have already checked, $\bar{D}$ is a subset of $\bar{C}$. It is also clear that $\bar{D}$ is connected and that points $(0, e^{-a} d_1)$, $(b - a, e^{-a} d_2)$ belong to $\bar{D}$. Consequently, $b - a \le L$ follows easily from Lemma 4. □

Step 3. A bound on the number of intersections of a connected component $D \subset C$ with the graph $y = e^x + \sin(x)$, $x > 0$.

Let D be a fixed connected component of C. From Corollary 5 it follows that there are constants $0 \le a \le b$, $b - a \le L$ such that for all $(x, y) \in D$, $x > 0$, we have $a \le x \le b$.

In particular, the number of intersections of D with the graph $y = e^x + \sin(x)$, $x > 0$, is not greater than the number of solutions of the system

$$P(x, e^x + \sin(x)) = 0, \quad a \le x \le b.$$

By a theorem of Khovansky [K2, 1.4] this number is $\le 4d(d + 2)^2(b - a)/\pi \le 4d(d + 2)^2 L/\pi$.

Now we are ready to finish the proof. It is clear that all solutions of (1) belong to C. In Step 3 we have estimated the number of solutions of (1) which belong to a given connected component of C. In Step 1 we have bounded a number of connected components of C. Summing up, the number of solutions of (1) is less than or equal to

$$(4d(d + 2)^2 L/\pi)((5d^2 + 3d + 2)/2) =$$

$$d(d + 1)^2(d + 2)^4(5d^2 + 3d + 2)(d \ln 4 + (d + 1)^2(d + 2))/\pi \le 2(d + 2)^{12}.$$

Proof of Lemma 4

Suppose, contrary to our claim, that there exists a connected component D of the set $C = \{ (x,y) : P(x,y) = 0,\ e^x - 1 \le y \le e^x + 1 \}$ joining two points $(0, y_1)$ and (x_2, y_2) such that $x_2 > L$ where $L = d(d+1)^2(d+2)^2 \ln 2/4 + (d+1)^4(d+2)^3/8$.

Put $n = (d+1)(d+2)/2$, $t = nd\ln 2 + n^2(d+1)$. We have $L = nt$. Let v_i be the vertical segment $\{ (x,y) : x = it, e^x - 1 \le y \le e^x + 1 \}$. For each $i = 1, \dots, n$ we have $0 < it < x_2$. Since D is connected, it must intersect each segment v_i $(i = 1, \dots, n)$. Thus for each $i = 1, \dots, n$ there exists ϵ_i such that $P(it, e^{it} + \epsilon_i) = 0$ and $|\epsilon_i| \le 1$.

Writing the polynomial P as a sum of monomials $P(x,y) = \sum_{k+l \le d} a_{kl} x^k y^l$ we get a square system of n linear equations

$$\sum_{k+l \le d} a_{kl} (it)^k (e^{it} + \epsilon_i)^l = 0, \qquad i = 1, \dots, n, \tag{3}$$

with respect to coefficients a_{kl}.

To get a contradiction it is enough to check that the determinant of this system does not vanish. Indeed, in this case the system (3) has only the zero solution $P \equiv 0$.

To compute this determinant we arrange the set of indices $\{(k,l) \in \mathbb{N}_0^2 : k+l \le d\}$ in a sequence $\{(\alpha_i, \beta_i)\}_{1 \le i \le n}$ ordered as follows: *if* $i < j$, *then* $\beta_i < \beta_j$ *or* $\beta_i = \beta_j$ *and* $\alpha_i < \alpha_j$.

This sequence splits in a natural way into $d+1$ subsequences. In each of them the numbers β_i are constant. More precisely, there exists a partition $N_0 \cup \dots \cup N_d = \{1, \dots, n\}$ such that

$$\beta_j = i, \quad 0 \le \alpha_j \le d - i \qquad \text{for } j \in N_i, \quad i = 0, \dots, d.$$

The determinant $D = \det\big((it)^{\alpha_j} (e^{it} + \epsilon_i)^{\beta_j}\big)_{1 \le i \le n, 1 \le j \le n}$ of the system (3) is by definition equal to

$$\begin{aligned} D &= \sum_{\sigma \in \mathrm{Perm}\{1,\dots,n\}} \operatorname{sgn}(\sigma) (\sigma(1)t)^{\alpha_1} (e^{\sigma(1)t} + \epsilon_{\sigma(1)})^{\beta_1} \cdots (\sigma(n)t)^{\alpha_n} (e^{\sigma(n)t} + \epsilon_{\sigma(n)})^{\beta_n} \\ &= t^s \sum_{\sigma \in \mathrm{Perm}\{1,\dots,n\}} \operatorname{sgn}(\sigma) \sigma(1)^{\alpha_1} \cdots \sigma(n)^{\alpha_n} (e^{\sigma(1)t} + \epsilon_{\sigma(1)})^{\beta_1} \cdots (e^{\sigma(n)t} + \epsilon_{\sigma(n)})^{\beta_n}. \end{aligned}$$

Here $s = \sum_{i=1}^n \alpha_i$. Write $D = Wt^s$ i.e.

$$W = \sum_{\sigma \in \mathrm{Perm}\{1,\dots,n\}} \operatorname{sgn}(\sigma) \sigma(1)^{\alpha_1} \cdots \sigma(n)^{\alpha_n} (e^{\sigma(1)t} + \epsilon_{\sigma(1)})^{\beta_1} \cdots (e^{\sigma(n)t} + \epsilon_{\sigma(n)})^{\beta_n}.$$

Put $K = \max_{\sigma \in \mathrm{Perm}\{1,\dots,n\}} \sum_{i=1}^n \sigma(i)\beta_i$ and let $S_K \subset \mathrm{Perm}\{1, \dots, n\}$ be the set of permutations satisfying the condition: $\sigma \in S_K$ *iff* $\sum_{i=1}^n \sigma(i)\beta_i = K$.

Denote

$$W_1 = \sum_{\sigma \in S_K} \operatorname{sgn}(\sigma) \sigma(1)^{\alpha_1} \cdots \sigma(n)^{\alpha_n} (e^{\sigma(1)t})^{\beta_1} \cdots (e^{\sigma(n)t})^{\beta_n}.$$

For every $\sigma \in S_K$ we have $(e^{\sigma(1)t})^{\beta_1} \cdots (e^{\sigma(n)t})^{\beta_n} = e^{Kt}$ and consequently $W_1 = e^{Kt} W_2$, where

$$W_2 = \sum_{\sigma \in S_K} \operatorname{sgn}(\sigma) \sigma(1)^{\alpha_1} \cdots \sigma(n)^{\alpha_n}.$$

Let us introduce a notation. Consider a non–empty subset A of $\{1,\ldots,n\}$. By $\mathrm{Perm}(A)$ we denote the set of all $\sigma\in\mathrm{Perm}\{1,\ldots,n\}$ such that $\sigma(i)=i$ for $i\in\{1,\ldots,n\}\setminus A$.

In further computations the following description of S_K will be useful.

Lemma 5. *Every permutation $\sigma\in S_K$ admits a decomposition $\sigma=\sigma_0\cdots\sigma_d$ where $\sigma_i\in\mathrm{Perm}(N_i)$ for $i=0,\ldots,d$. Moreover, such a decomposition is unique.*

We omit a purely combinatorial proof of this lemma. By Lemma 5

$$W_2=\prod_{k=0}^{d}\sum_{\sigma_k\in\mathrm{Perm}(N_k)}\mathrm{sgn}(\sigma_k)\prod_{i\in N_k}\sigma_k(i)^{\alpha_i}=\prod_{k=0}^{d}\det(i^j)_{i\in N_k,0\le j\le d-k}.$$

Each determinant in this product is the classical Vandermonde determinant of pairwise distinct integers and hence is a non–zero integer. Therefore W_2 being their product is a non–zero integer. As a consequence

$$|W_1|\ge e^{Kt}. \tag{4}$$

Now we estimate the difference $W-W_1$. From definitions of W and W_1 follows that this number is a sum of at most $2^s n!$ terms of the form

$$\pm\sigma(1)^{\alpha_1}\cdots\sigma(n)^{\alpha_n}e^{K't}\epsilon_1^{\gamma_1}\cdots\epsilon_n^{\gamma_n}.$$

Here $s=\sum_{i=1}^n\alpha_i$ and $\gamma_1,\ldots,\gamma_n$ are non–negative integers, $K'<K$. The absolute value of each term of the sum is not greater than $(n!)^d e^{(K-1)t}$. Hence

$$|W-W_1|\le 2^s(n!)^{(d+1)}e^{(K-1)t}. \tag{5}$$

We have two obvious inequalities: $s=\sum_{i=1}^n\alpha_i\le nd$ and $n!<e^{n^2}$. Hence $2^s(n!)^{(d+1)}<e^{s\ln 2}(e^{n^2})^{d+1}\le e^{nd\ln 2+n^2(d+1)}=e^t$.

By (4) and (5) we have

$$\begin{aligned}|W|&\ge|W_1|-|W-W_1|\ge e^{Kt}-2^s(n!)^{(d+1)}e^{(K-1)t}\\&=e^{(K-1)t}(e^t-2^s(n!)^{(d+1)})>0.\end{aligned}$$

The last inequality shows that the determinant D of the system (3) is non–zero and gives us a contradiction, as desired.

Motivations

Theorem 1 should be understood in the context of Khovansky's theory [K1], [K2]. Our motivation and inspiration for this problem comes from the theory of o-minimal structures. By an o-minimal structure on $(\mathbb{R},+,\cdot)$ we mean a collection $\mathcal{M}=\bigcup_{n\in\mathbb{N}}\mathcal{M}_n$, where each $\mathcal{M}_n$ is a family of subsets of $\mathbb{R}^n$ such that:

(1) each $\mathcal{M}_n$ is closed under finite set-theoretical operations;
(2) if $A\in\mathcal{M}_n$ and $B\in\mathcal{M}_m$, then $A\times B\in\mathcal{M}_{n+m}$;
(3) let $A\in\mathcal{M}_{n+m}$ and $\pi:\mathbb{R}^{n+m}\to\mathbb{R}^n$ be a projection on the first n coordinates; then $\pi(A)\in\mathcal{M}_n$;
(4) every semialgebraic subset of $\mathbb{R}^n$ belongs to $\mathcal{M}_n$;
(5) $\mathcal{M}_1$ consists of all finite unions of open intervals and points.

O-minimal structures, invented by model theorists, are natural and important extensions of semialgebraic (or more general subanalytic) geometry. We mention here only two important examples of o-minimal structures; more details and examples can be found in [DM]. Wilkie [W] proved (using results of Khovansky [K1]) that by adding to semialgebraic sets the graph of an exponential function one gets an o-minimal structure (called $\mathbb{R}_{\exp}$). A similar extension of global subanalytic sets was done by L. van den Dries, A. Macintyre, D. Marker in [DMM].

Let $\mathcal{M}$ be an o-minimal structure on $(\mathbb{R}, +, \cdot)$. The following important finiteness property (see [DM], [vD]) can be obtained from a result of Pillay, Steinhorn, and Knight [PS], [KPS]:

Theorem KPS. *Let $\mathcal{M}$ be an o-minimal structure. Suppose that $A \in \mathcal{M}_{n+m}$ and denote by $\pi : \mathbb{R}^{n+m} \to \mathbb{R}^n$ the projection on the first n coordinates. Then there exists $N \in \mathbb{N}$ such that for each $x \in \mathbb{R}^n$ the fiber $\pi^{-1}(x) \cap A$ has at most N connected components.*

Let us consider the following problem:

Let $f : \mathbb{R} \to \mathbb{R}$ (or more generally $f : (a, \infty) \to \mathbb{R}$) be an analytic function. What conditions on f would guarantee that the graph of f belongs to an o-minimal structure?

Using the fact that the space of polynomials in 2 variables of degree $\leq d$ is of finite dimension, we get easily from theorem KPS the following necessary condition: $(*)$ *For each $d \in \mathbb{N}$ there is $A(d) \in \mathbb{N}$ such that if $P(x,y)$ is a non–zero polynomial of degree d, then the number of isolated solutions of the system*

$$P(x,y) = 0, \quad y = f(x), \quad x > a,$$

is not greater than $A(d)$.

One may conjecture that $(*)$ is also a sufficient condition, but this is not the case. Actually $f(x) = e^x + \sin x,\ x > 0$, is a counter-example. Indeed, by Theorem 1, f satisfies $(*)$ with $A(d) \sim 2d^{12}$. Suppose, contrary to our claim, that the graph of f belongs to some o-minimal structure $\mathcal{M}$. This would imply (see [DM]) that the derivative f' belongs to $\mathcal{M}$. Hence the graph of $\sin x - \cos x = f(x) - f'(x),\ x > 0$, is in $\mathcal{M}$. But this is impossible since $\{x \in \mathbb{R} : \sin x - \cos x = 0,\ x > 0\}$ cannot belong to $\mathcal{M}_1$.

By a similar argument $g(x) = e^{x^2} + \sin(x^2),\ x \in \mathbb{R}$, does not belong to any o-minimal structure even though it clearly satisfies condition $(*)$.

Note that, by the Bezout theorem, if f is algebraic, then the function $d \to A(d)$ can be bounded by a linear one. Actually the converse is also true. To show this suppose that $A(d) \leq const(d+1)$. Then, for d sufficiently large, $A(d) < B(d) - 1$, where $B(d) = \frac{1}{2}(d+1)(d+2)$ is the dimension of the space of polynomials of degree $\leq d$. Take $B(d) - 1$ points on the graph of f and a nonzero polynomial $P(x,y)$, $\deg P \leq d$, which vanishes at these points. Then, by the definition of $A(d)$, P has to vanish on the graph of f, that is f is algebraic. On the other hand, by Khovansky [K2], if f is pffafian (e.g. $f = e^x$), then $A(d)$ can be bounded by a quadratic function.

In general, from the fact that f belongs to some o-minimal structure we cannot deduce anything about $A(d)$. More precisely, if we are given a sequence $\mathbb{N} \ni d \to a(d) \in \mathbb{N}$, then there exist an analytic function $f : (a, \infty) \to \mathbb{R}$, subanalytic at the infinity, and an increasing sequence $k \to d_k$ of integers such that

$$a(d_k) \leq A(d_k)$$

for all $k \in \mathbb{N}$. We sketch only the idea of construction. One can easily construct by induction: a sequence $b_k \in \mathbb{N}$, two sequences $\varepsilon_k > 0$, $\eta_k > 0$, and a sequence of polynomials $P_k = c_{1+b_k} t^{1+b_k} + \cdots + c_{b_{k+1}} t^{b_{k+1}}$ such that:

(1) $\|P_k\| \le \varepsilon_k$,
(2) if $r : (0,1) \to \mathbb{R}$ is continuous, $\sup_{t \in (0,1)} |r(t)| \le \eta_k$, then

$$\#\{t \in (0,1) : P_k(t) + r(t) = 0\} \ge a(4b_k),$$

(3) $\sum_{k>n} \varepsilon_k \ < \ \eta_n$ for all $n \in \mathbb{N}$,

where $\|\cdot\|$ is the sum of absolute values of coefficients. Now, put

$$g(t) = \sum_{k=1}^{\infty} P_k(t).$$

We can take P_k so small that the radius of convergence of the series is > 1. Finally put $f(x) = g\left(\frac{x}{\sqrt{x^2+1}}\right)$, $x > 0$. Let

$$q_k(t, y) = y - \sum_{n=1}^{k-1} P_n(t), \qquad k > 2.$$

Clearly q_k is of degree $\le b_k$ and it has at least $a(4b_k)$ zeros on the graph of $g(t)$, for $t \in [0,1)$. It easy to find a polynomial $Q_k(x,y)$ of degree $\le d_k = 4b_k$ which vanishes on the zeros of $q_k\left(\frac{x}{\sqrt{x^2+1}}, y\right)$. Since Q_k has at least $a(d_k)$ zeros on the graph of f, it follows that $a(d_k) \le A(d_k)$, as desired.

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