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REMARK ABOUT HEAT DIFFUSION ON PERIODIC SPACES

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Abstract. Let M be a complete Riemannian manifold with a free cocompact $\mathbb{Z}^k$-action. Let $k(t, m_1, m_2)$ be the heat kernel on M. We compute the asymptotics of $k(t, m_1, m_2)$ in the limit in which $t \to \infty$ and $d(m_1, m_2) \sim \sqrt{t}$. We show that in this limit, the heat diffusion is governed by an effective Euclidean metric on $\mathbb{R}^k$ coming from the Hodge inner product on $\mathrm{H}^1(M/\mathbb{Z}^k; \mathbb{R})$.

1. Introduction

Let M be a complete connected oriented n-dimensional Riemannian manifold. Let $k(t, m_1, m_2)$ be the time-t heat kernel on M. The usual ansatz to approximate $k(t, m_1, m_2)$ is to say that

$$k(t, m_1, m_2) \sim P(t, m_1, m_2)\, e^{-\frac{d(m_1,m_2)^2}{4t}} \tag{1.1}$$

where $e^{-\frac{d(m_1,m_2)^2}{4t}}$ is considered to be the leading term and $P(t, m_1, m_2)$ is a correction term which can be computed iteratively. There are results which make this precise. For example [1], if m_1 and m_2 are nonconjugate, then as $t \to 0$,

$$k(t, m_1, m_2) = \sum_{\gamma} \frac{(\det\, d(\exp_{m_1})_{v_\gamma})^{-1/2}}{(4\pi t)^{n/2}}\, e^{-\frac{d(m_1,m_2)^2}{4t}}\, (1 + O(t)). \tag{1.2}$$

Here the sum is over minimal geodesics $\gamma : [0, 1] \to M$ joining m_1 to m_2 of the form $\gamma(s) = \exp_{m_1}(s v_\gamma)$. For another example, if M has bounded geometry, then lower and upper heat kernel bounds [4], [5] imply that (1.1) is a good approximation if $d(m_1, m_2) >> t$, in the sense that $-\ln(k(t, m_1, m_2))$ is well-approximated by $\frac{d(m_1,m_2)^2}{4t}$.

One can ask if the ansatz (1.1) is relevant for other asymptotic regimes. In this paper we look at the case when M has a periodic metric, meaning that $\mathbb{Z}^k$ acts freely by orientation-preserving isometries on M, with $X = M/\mathbb{Z}^k$ compact. We consider the asymptotic regime in which $t \to \infty$ and $d(m_1, m_2) \sim \sqrt{t}$. As the typical time-$t$ Brownian path will travel a distance comparable to $\sqrt{t}$, this is the regime which contains the bulk of the diffusing heat. We show that, in this regime, (1.1) is no longer a valid approximation. Instead, the heat diffusion is governed by

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an effective Euclidean metric on $\mathbb{R}^k$. This metric is constructed using the Hodge inner product on $\mathrm{H}^1(X;\mathbb{R})$.

To state the precise result, let $\mathcal{F}$ be a fundamental domain in M for the $\mathbb{Z}^k$-action. Given $\mathbf{v} \in \mathbb{Z}^k$, put

$$k(t,\mathbf{v}) = \int_{\mathcal{F}} k(t,m,\mathbf{v}\cdot m)\, d\,\mathrm{vol}(m). \tag{1.3}$$

This is independent of the choice of fundamental domain $\mathcal{F}$.

The covering $M \to X$ is classified by a map $\nu : X \to B\mathbb{Z}^k$, defined up to homotopy, which is π_1-surjective. It induces a surjection $\nu_* : \mathrm{H}_1(X;\mathbb{R}) \to \mathbb{R}^k$ and an injection $\nu^* : (\mathbb{R}^k)^* \to \mathrm{H}^1(X;\mathbb{R})$. Let $\langle \cdot,\cdot\rangle_{H^1(X;\mathbb{R})}$ be the Hodge inner product on $\mathrm{H}^1(X;\mathbb{R})$.

Definition 1. The inner product $\langle \cdot,\cdot\rangle_{(\mathbb{R}^k)^*}$ on $(\mathbb{R}^k)^*$ is given by

$$\langle \cdot,\cdot\rangle_{(\mathbb{R}^k)^*} = \frac{(\nu^*)^*\langle \cdot,\cdot\rangle_{H^1(X;\mathbb{R})}}{\mathrm{vol}(X)}. \tag{1.4}$$

The inner product $\langle \cdot,\cdot\rangle_{\mathbb{R}^k}$ is the dual inner product on $\mathbb{R}^k$.

Let $\mathrm{vol}(\mathbb{R}^k/\mathbb{Z}^k)$ be the volume of a lattice cell in $\mathbb{R}^k$, measured with $\langle \cdot,\cdot\rangle_{\mathbb{R}^k}$.

Proposition 1. *Fix $C > 0$. Then in the region $\{(t,\mathbf{v}) \in \mathbb{R}^+ \times \mathbb{Z}^k : \langle \mathbf{v},\mathbf{v}\rangle_{\mathbb{R}^k} \le Ct\}$, as $t \to \infty$ we have*

$$k(t,\mathbf{v}) = \frac{\mathrm{vol}(\mathbb{R}^k/\mathbb{Z}^k)}{(4\pi t)^{k/2}}\, e^{-\langle \mathbf{v},\mathbf{v}\rangle_{\mathbb{R}^k}/(4t)} + O(t^{-\frac{k+1}{2}}) \tag{1.5}$$

uniformly in $\mathbf{v}$.

Example. 1. If $M = \mathbb{R}^k$ with a flat metric $\langle \cdot,\cdot\rangle_{flat}$, then one can check that $\langle \cdot,\cdot\rangle_{\mathbb{R}^k} = \langle \cdot,\cdot\rangle_{flat}$, so one recovers the standard flat-space heat kernel.

2. If $n = 2$, then $\langle \cdot,\cdot\rangle_{H^1(X;\mathbb{R})}$ is conformally-invariant. Hence in this case, the heat kernel asymptotics only depend on $\mathrm{vol}(X)$ and the induced complex structure on X.

One can get similar pointwise estimates on $k(t,m_1,m_2)$ by the same methods. We omit the details.

The result of Proposition 1 is an example of the phenomenon of "homogenization", which has been much-studied for differential operators on $\mathbb{R}^n$. Homogenization means that in an appropriate scaling limit, the solution to a problem is governed by the solution to a spatially homogeneous problem; see [2] and references therein. Thus it is not surprising that the answer in Proposition 1 has a homogeneous form. The point of the present paper is to show how one can compute the exact asymptotics in the general geometric setting.

We remark that when $t \to \infty$ and $d(m_1,m_2) >> t$, the asymptotic expression (1.1) also shows homogenization. This follows from the result of D. Burago [3] that there are a Banach norm $\|\cdot\|$ on $\mathbb{R}^k$ and a constant $c > 0$ such that if $m \in M$ and $\mathbf{v} \in \mathbb{Z}^k$, then $|d(m,\mathbf{v}\cdot m) - \|\mathbf{v}\|| \le c$. Thus as $t \to \infty$, if $d(m_1,m_2) \sim \sqrt{t}$, then the effective geometry is $(\mathbb{R}^k, \langle \cdot,\cdot\rangle_{\mathbb{R}^k})$, while if $d(m_1,m_2) >> t$, then the effective geometry is $(\mathbb{R}^k, \|\cdot\|)$.

It would be interesting if one could extend the results of this paper to the setting in which Γ is a nonabelian discrete group, such as the fundamental group of a closed hyperbolic surface. In this case, the relevant scaling regime should be $t \to \infty$ and

$d(m_1, m_2) \sim t$, as the typical time-t Brownian path on the hyperbolic plane travels a distance comparable to t.

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2. Proof of Proposition 1

We first recall some basic facts about the eigenvalues of a parametrized family of operators [7, Chapter XII].

Let $M_d(\mathbb{C})$ be the vector space of $d \times d$ complex matrices and let $M_d^{sa}(\mathbb{C})$ be the subspace of self-adjoint matrices. Let $f : \mathbb{R}^k \to M_d(\mathbb{C})$ be a real-analytic map. The eigenvalues $\{\lambda_i(x)\}_{i=1}^d$ of $f(x)$ are algebraic functions of x, meaning the roots of a polynomial whose coefficients are real-analytic functions of x, as they are given by $\det(f(x) - \lambda) = 0$. If $\lambda_1(0)$ is a nondegenerate eigenvalue of $f(0)$, then it extends near $x = 0$ to a real-analytic function $\lambda_1(x)$.

If $k = 1$ and f takes values in $M_d^{sa}(\mathbb{C})$, then the eigenvalues of f form d real-analytic functions $\{\lambda_i(x)\}_{i=1}^d$ on $\mathbb{R}$. Of course, these functions may cross. If $k > 1$ and f takes values in $M_d^{sa}(\mathbb{C})$, then it may not be true that the eigenvalues form real-analytic functions on $\mathbb{R}^k$. This can be seen in the example $f(x_1, x_2) = \begin{pmatrix} 0 & x_1 - ix_2 \\ x_1 + ix_2 & 0 \end{pmatrix}$. Its eigenvalues are $\pm\sqrt{x_1^2 + x_2^2}$, which are not the union of two smooth functions on $\mathbb{R}^2$. However, if $\gamma(s)$ is a real-analytic curve in $\mathbb{R}^2$, then the eigenvalues of $f(\gamma(s))$ do form real-analytic functions in s.

If f is instead an appropriate real-analytic family of operators on a Hilbert space, then one has similar results. We refer to [7, Chapter XII.2] for the precise requirements.

To prove Proposition 1, we use the method of [6, Section VI]. The Pontryagin dual of $\mathbb{Z}^k$ is $T^k = (\mathbb{R}^k)^*/2\pi(\mathbb{Z}^k)^*$. Given $\theta \in T^k$, let $\rho(\theta) : \mathbb{Z}^k \to U(1)$ be the corresponding representation and let $E(\theta)$ be the flat line bundle on X associated to the representation $\pi_1(X) \stackrel{\nu_*}{\to} \mathbb{Z}^k \stackrel{\rho(\theta)}{\to} U(1)$. Let $\triangle_\theta$ be the Laplacian on $L^2(X; E(\theta))$. Then Fourier analysis gives

$$k(t, \mathbf{v}) = \int_{T^k} e^{i\theta\cdot\mathbf{v}} \ \mathrm{Tr}\left(e^{-t\triangle(\theta)}\right) \frac{d^k\theta}{(2\pi)^k}. \tag{2.1}$$

Now $\mathrm{Ker}(\triangle(\theta)) = 0$ if $\theta \neq 0$ and $\mathrm{Ker}(\triangle(0)) \cong \mathbb{C}$ consists of the constant functions on X.

In order to write all of the operators $\triangle(\theta)$ as acting on the same Hilbert space, let $\{\tau^j\}_{j=1}^k$ be a set of harmonic 1-forms on X which gives an integral basis of $(\mathbb{Z}^k)^* \subset (\mathbb{R}^k)^* \subseteq \mathrm{H}^1(X; \mathbb{R})$. Let $e(\tau^j)$ denote exterior multiplication by τ^j on $C^\infty(X)$ and let $i(\tau^j)$ denote interior multiplication by τ^j on $\Omega^1(X)$. Putting

$$d(\theta) = d + i \sum_{j=1}^k \theta_j e(\tau^j) \tag{2.2}$$

and

$$d^*(\theta) = d^* - i \sum_{j=1}^k \theta_j i(\tau^j), \tag{2.3}$$

$\triangle(\theta)$ is unitarily equivalent to the self-adjoint operator $d^*(\theta)d(\theta)$ (which we shall also denote by $\triangle(\theta)$) acting on $L^2(X)$. Because $\triangle(\theta)$ is quadratic in θ, it is easy to see that $\{\triangle(\theta)\}_{\theta\in T^k}$ is an analytic family of type (A) in the sense of [7, Chapter XII.2], so we can apply analytic eigenvalue perturbation theory. In particular, if $\{\lambda_i(\theta)\}_{i\in\mathbb{Z}^+}$ are the eigenvalues of $\triangle(\theta)$, arranged in increasing order and repeated if there is a multipicity greater than one, then $\lambda_1(\theta) \geq 0$ and $\lambda_1(\theta) = 0$ if and only if $\theta = 0$, in which case it is a nondegenerate eigenvalue. Thus λ_1 extends to a real-analytic function in a neighborhood of $\theta = 0$. So for sufficiently small $\epsilon > 0$, there is a neighborhood $U \subseteq T^k$ of $0 \in T^k$ such that

1. If $\theta \notin U$, then $\lambda_1(\theta) > \epsilon$.
2. Restricted to U, λ_1 is a real-analytic function which represents a nondegenerate eigenvalue and $\lambda_2 > \epsilon$.

From (2.1), we have

$$k(t,\mathbf{v}) = \int_{T^k} e^{i\theta\cdot\mathbf{v}} \sum_{i=1}^{\infty} e^{-t\lambda_i(\theta)} \frac{d^k\theta}{(2\pi)^k}. \tag{2.4}$$

Then it is easy to show that

$$k(t,\mathbf{v}) = \int_{U} e^{i\theta\cdot\mathbf{v}} e^{-t\lambda_1(\theta)} \frac{d^k\theta}{(2\pi)^k} + O(e^{-\epsilon t/2}), \tag{2.5}$$

uniformly in $\mathbf{v}$.

Lemma 1. *The Taylor's series of $\lambda_1(\theta)$ near $\theta = 0$ starts off as*

$$\lambda_1(\theta) = \langle\theta,\theta\rangle_{(\mathbb{R}^k)^*} + O(|\theta|^3). \tag{2.6}$$

Proof. It suffices to compute $\frac{d\lambda_1(s\vec{w})}{ds}\Big|_{s=0}$ and $\frac{d^2\lambda_1(s\vec{w})}{ds^2}\Big|_{s=0}$ for all $\vec{w} \in (\mathbb{R}^k)^*$. For simplicity, denote $\triangle(s\vec{w})$ by $\triangle(s)$ and $\lambda_1(s\vec{w})$ by $\lambda(s)$. As $\lambda(s)$ is nonnegative and $\lambda(0) = 0$, we must have $\lambda'(0) = 0$. Let $\psi(s)$ denote a nonzero eigenfunction with eigenvalue $\lambda(s)$; we can assume that it is real-analytic in s with $\psi(0) = 1$. Differentiation of $\triangle(s)\psi(s) = \lambda(s)\psi(s)$ gives

$$\triangle'(0)\psi(0) + \triangle(0)\psi'(0) = 0 \tag{2.7}$$

and

$$\triangle''(0)\psi(0) + 2\triangle'(0)\psi'(0) + \triangle(0)\psi''(0) = \lambda''(0)\psi(0). \tag{2.8}$$

Taking the inner product of (2.8) with $\psi(0)$ gives

$$\langle\psi(0),\triangle''(0)\psi(0)\rangle + 2\langle\psi(0),\triangle'(0)\psi'(0)\rangle = \lambda''(0)\langle\psi(0),\psi(0)\rangle. \tag{2.9}$$

Let G be the Green's operator for $\triangle(0)$. From (2.7),

$$\psi'(0) = c\psi(0) - G\triangle'(0)\psi(0) \tag{2.10}$$

for some constant c. Changing $\psi(s)$ to $e^{-cs}\psi(s)$, we may assume that $c = 0$. Substituting (2.10) into (2.9) gives

$$\langle\psi(0),\triangle''(0)\psi(0)\rangle - 2\langle\psi(0),\triangle'(0)G\triangle'(0)\psi(0)\rangle = \lambda''(0)\langle\psi(0),\psi(0)\rangle. \tag{2.11}$$

It remains to compute $\langle\psi(0),\triangle''(0)\psi(0)\rangle$ and $\langle\psi(0),\triangle'(0)G\triangle'(0)\psi(0)\rangle$. Put $D(s) = d_{s\vec{w}}$ and $D^*(s) = d^*_{s\vec{w}}$. Then $\triangle(s) = D^*(s)D(s)$. From (2.2) and (2.3), $D(s)$ and $D^*(s)$ are linear in s, with

$$D'(0) = i\sum_{j=1}^{k} w_j\, e(\tau^j) \tag{2.12}$$

and

$$(D^*)'(0) = -i \sum_{j=1}^{k} w_j \, i(\tau^j). \tag{2.13}$$

Then

$$\begin{aligned} \langle \psi(0), \triangle''(0)\psi(0)\rangle &= 2\langle \psi(0), (D^*)'(0)D'(0)\psi(0)\rangle \\ &= 2\big|D'(0)\psi(0)\big|^2_{H^1(X;\mathbb{C})} \\ &= 2\big|\sum_{j=1}^{k} w_j \, \tau^j\big|^2_{H^1(X;\mathbb{C})}. \end{aligned} \tag{2.14}$$

Now

$$\begin{aligned} \triangle'(0)\psi(0) &= [(D^*)'(0)D(0) + D^*(0)D'(0)]\, \psi(0) \\ &= d^* \left(-i \sum_{j=1}^{k} w_j \, \tau^j \right) = 0. \end{aligned} \tag{2.15}$$

Substituting (2.14) and (2.15) into (2.11) and using the fact that $\langle \psi(0), \psi(0)\rangle = \mathrm{vol}(X)$, the lemma follows. □

Continuing with the proof of Proposition 1, by Morse theory and Lemma 1, we can find a change of coordinates near $0 \in T^k$ with respect to which λ_1 becomes quadratic. That is, if $B_r(0)$ denotes the ball of radius r in $(\mathbb{R}^k)^*$, we can find an $r > 0$, a neighborhood U of $0 \in T^k$ and a diffeomorphism $\phi : B_r(0) \to U$ such that $\phi(0) = 0$, $d\phi_0 = \mathrm{Id}$ and $\lambda_1(\phi(x)) = \langle x, x\rangle_{(\mathbb{R}^k)^*}$. Then there is some $\alpha > 0$ such that as $t \to \infty$,

$$k(t, \mathbf{v}) = \int_{B_r(0)} e^{i\phi(x)\cdot \mathbf{v}} \, e^{-t\langle x,x\rangle_{(\mathbb{R}^k)^*}} \det(d\phi_x) \, \frac{d^k x}{(2\pi)^k} + O(e^{-\alpha t}), \tag{2.16}$$

uniformly in $\mathbf{v}$. Multiplying by a cutoff function on $(\mathbb{R}^k)^*$, we can write

$$\begin{aligned} k(t, \mathbf{v}) &= \int_{(\mathbb{R}^k)^*} e^{i\phi(x)\cdot \mathbf{v}} \, e^{-t\langle x,x\rangle_{(\mathbb{R}^k)^*}} \, g(x) \, \frac{d^k x}{(2\pi)^k} + O(e^{-\alpha' t}) \\ &= t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} e^{i\phi(\frac{x}{\sqrt{t}})\cdot \mathbf{v}} \, e^{-\langle x,x\rangle_{(\mathbb{R}^k)^*}} \, g\left(\frac{x}{\sqrt{t}}\right) \, \frac{d^k x}{(2\pi)^k} + O(e^{-\alpha' t}) \end{aligned} \tag{2.17}$$

for some $g \in C_0^\infty\left((\mathbb{R}^k)^*\right)$ with $g(0) = 1$ and some $\alpha' > 0$. (Here ϕ has been extended to become a map $\phi : (\mathbb{R}^k)^* \to (\mathbb{R}^k)^*$ which is the identity outside of a compact set.)

We have now reduced to a stationary-phase-type integral. Let

$$g(x) = 1 + (\nabla g)(0) \cdot x + E(x) \tag{2.18}$$

be the beginning of the Taylor's expansion of g. We can write

(2.19)

$$\begin{aligned} t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} & e^{i\phi(\frac{x}{\sqrt{t}})\cdot \mathbf{v}} \, e^{-\langle x,x\rangle_{(\mathbb{R}^k)^*}} \, g\left(\frac{x}{\sqrt{t}}\right) \frac{d^k x}{(2\pi)^k} \\ &= t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} e^{i\frac{x}{\sqrt{t}}\cdot \mathbf{v}} \, e^{-\langle x,x\rangle_{(\mathbb{R}^k)^*}} \left[1 + (\nabla g)(0)\cdot \frac{x}{\sqrt{t}} + E\left(\frac{x}{\sqrt{t}}\right)\right] \frac{d^k x}{(2\pi)^k} \\ &\quad + t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} e^{i\frac{x}{\sqrt{t}}\cdot \mathbf{v}} \left[e^{i\left(\phi(\frac{x}{\sqrt{t}}) - \frac{x}{\sqrt{t}}\right)\cdot \mathbf{v}} - 1\right] e^{-\langle x,x\rangle_{(\mathbb{R}^k)^*}} \, g\left(\frac{x}{\sqrt{t}}\right) \frac{d^k x}{(2\pi)^k}. \end{aligned}$$

Recall that the measure $\frac{d^k x}{(2\pi)^k}$ on $(\mathbb{R}^k)^*$ derives from the product measure on $T^k = (\mathbb{R}^*/2\pi\mathbb{Z}^*)^k$. Let $\langle \cdot, \cdot \rangle_{prod}$ be the standard product Euclidean metric on $(\mathbb{R}^*)^k$. Let Q be the self-adjoint operator on $(\mathbb{R}^k)^*$ such that $\langle x, x\rangle_{(\mathbb{R}^k)^*} = \langle x, Qx\rangle_{prod}$. Then a standard calculation gives

(2.20)

$$t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} e^{i\frac{x}{\sqrt{t}}\cdot \mathbf{v}} \, e^{-\langle x,x\rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k} = \frac{(\det Q)^{-1/2}}{(4\pi t)^{k/2}} \, e^{-\langle \mathbf{v},\mathbf{v}\rangle_{\mathbb{R}^k}/(4t)}.$$

On the other hand,

$$(\det Q)^{-1/2} = \mathrm{vol}(\mathbb{R}^k/\mathbb{Z}^k). \tag{2.21}$$

By symmetry,

$$t^{-\frac{k}{2}} \int_{(\mathbb{R}^k)^*} e^{i\frac{x}{\sqrt{t}}\cdot \mathbf{v}} \, e^{-\langle x,x\rangle_{(\mathbb{R}^k)^*}} (\nabla g)(0)\cdot \frac{x}{\sqrt{t}} \frac{d^k x}{(2\pi)^k} = 0. \tag{2.22}$$

Let $c > 0$ be such that $|E(x)| \leq c\langle x, x\rangle_{(\mathbb{R}^k)^*}$ for all $x \in (\mathbb{R}^k)^*$. Then

$$\begin{aligned} &\left| \int_{(\mathbb{R}^k)^*} e^{i\frac{x}{\sqrt{t}}\cdot \mathbf{v}} \, e^{-\langle x,x\rangle_{(\mathbb{R}^k)^*}} E\left(\frac{x}{\sqrt{t}}\right) \frac{d^k x}{(2\pi)^k} \right| \\ &\qquad \leq \frac{c}{t} \int_{(\mathbb{R}^k)^*} \langle x,x\rangle_{(\mathbb{R}^k)^*} \, e^{-\langle x,x\rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k}. \end{aligned} \tag{2.23}$$

Finally,

(2.24)

$$\begin{aligned} &\left| \int_{(\mathbb{R}^k)^*} e^{i\frac{x}{\sqrt{t}}\cdot \mathbf{v}} \left[e^{i\left(\phi(\frac{x}{\sqrt{t}}) - \frac{x}{\sqrt{t}}\right)\cdot \mathbf{v}} - 1\right] e^{-\langle x,x\rangle_{(\mathbb{R}^k)^*}} \, g\left(\frac{x}{\sqrt{t}}\right) \frac{d^k x}{(2\pi)^k} \right| \\ &\qquad \leq \| g \|_\infty \int_{(\mathbb{R}^k)^*} 2 \left| \sin\left(\frac{1}{2}\left[\phi\left(\frac{x}{\sqrt{t}}\right) - \frac{x}{\sqrt{t}}\right]\cdot \mathbf{v}\right)\right| e^{-\langle x,x\rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k}. \end{aligned}$$

We can find a constant $c' > 0$ such that

$$2\left| \sin\left(\frac{1}{2}\left[\phi(x) - x\right]\cdot \mathbf{v}\right)\right| \leq c' \, \langle x, x\rangle_{(\mathbb{R}^k)^*} \ \| \mathbf{v} \|_{\mathbb{R}^k} \tag{2.25}$$

for all $x \in (\mathbb{R}^k)^*$ and $\mathbf{v} \in \mathbb{Z}^k$. Then

(2.26)
$$\| g \|_\infty \int_{(\mathbb{R}^k)^*} 2 \left| \sin \left(\frac{1}{2} \left[\phi \left(\frac{x}{\sqrt{t}} \right) - \frac{x}{\sqrt{t}} \right] \cdot \mathbf{v} \right) \right| e^{-\langle x,x \rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k}$$
$$\leq \frac{c'}{\sqrt{t}} \frac{\| \mathbf{v} \|_{\mathbb{R}^k}}{\sqrt{t}} \| g \|_\infty \int_{(\mathbb{R}^k)^*} \langle x, x \rangle_{(\mathbb{R}^k)^*} e^{-\langle x,x \rangle_{(\mathbb{R}^k)^*}} \frac{d^k x}{(2\pi)^k}.$$

By assumption,

(2.27)
$$\frac{\| \mathbf{v} \|_{\mathbb{R}^k}}{\sqrt{t}} \leq \sqrt{C}.$$

The proposition follows from combining equations (2.17)–(2.27).

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