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ON THE SCHWARZ SYMMETRY PRINCIPLE IN A MODEL CASE

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Abstract. In this article, we prove that smooth CR diffeomorphisms between two real analytic holomorphically nondegenerate hypersurfaces, one of which is rigid and polynomial, extend to be locally biholomorphic. It turns out that the result can be generalized to not totally degenerate mappings, in the sense of Baouendi and Rothschild.

Introduction

Since the fundamental work of Baouendi, Jacobowitz and Treves [BJT], no particular attention was given to the analog of the Schwarz symmetry principle in the complex euclidean space in the case of non essentially finite real analytic hypersurfaces, not to mention [MEY], [MM]. However, in view of the recent deep work of Baouendi, Huang and Rothschild [BHR], it can be easily conjectured that the local Schwarz symmetry principle holds for a $\mathcal{C}^\infty$-smooth CR diffeomorphism $f : M \to M'$, between holomorphically nondegenerate real analytic hypersurfaces M and M', which is holomorphic in one side of M, and that this is the *optimal sufficient condition* to get analyticity of a smooth CR mapping. In this paper, we give a short and elegant geometric proof of a precise and general statement in the case M' is polynomial and rigid. We do not assume that M is algebraic, so our result does not follow from [BHR].

1. Smooth CR diffeomorphisms

Let M' be a real analytic hypersurface in $\mathbb{C}^n$ and assume that its equation in coordinates $t = (w, z)$, $t \in \mathbb{C}^n$, $w = (w_1, ..., w_{n-1}) \in \mathbb{C}^{n-1}$, $z \in \mathbb{C}$, is in the special form

$$M' : \qquad \bar{z} = z + i\rho'(w, \bar{w}) = z + i \sum_{\alpha \in \mathbb{N}^{n-1}, |\alpha| \leq N_0} \rho'_\alpha(w)\bar{w}^\alpha, \tag{1.1}$$

where the function $\rho'(w, \bar{w})$ is a *polynomial* in the variables $w, \bar{w}$, and $N_0 \in \mathbb{N}$. Choose coordinates t in $\mathbb{C}^n$ near M. Such an equation is usually called *polynomial rigid*. Then one has the following remarkable statement.

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Theorem 1.1. *Let M be a real analytic hypersurface in $\mathbb{C}^n$, let $p \in M$, let f be a holomorphic mapping defined into a side D of M at p, $\mathcal{C}^\infty$ up to M, such that f sends (M,p) CR diffeomorphically into another real analytic hypersurface $(M', f(p))$. Assume that there exist coordinates (w,z) at $f(p)=0$ such that M' has a polynomial like equation as* (1.1) *and let $G(f,\lambda)$ denote $f_n + i\rho'(f^*,\lambda)$, $f^* = (f_1,...,f_{n-1})$. Then*

(a) *$(t,\lambda) \mapsto G(f(t),\lambda)$ extends as a holomorphic function to $\mathcal{O}_t(0)\times\mathcal{O}_\lambda(0)$;*

(b) *In case the coordinates (w,z) are normal,* i.e. *$\rho'(w,0)\equiv 0$, the normal component f_n of f extends as a holomorphic function to $\mathcal{O}_t(0)$;*

(c) *f extends as a holomorphic function to $\mathcal{O}_t(0)$ if M (hence M' too) is holomorphically nondegenerate.*

Proof. Define the *reflection function*

$$G(f(t),\lambda) = f_n(t) + i\rho'(f^*(t),\lambda).$$

Now let $S \subset M$ be a real analytic totally real submanifold of dimension n containing 0. Since there exists H, a well-defined biholomorphism taking S into a piece of $\mathbb{R}^n$ through 0, we can introduce an antiholomorphic reflection mapping σ_S, $\sigma_S|_S = \mathrm{id}|_S$, by taking $\sigma_S(t) := H^{-1}(\overline{H(t)})$. Choose $\mathcal{W}^-$ a wedge of edge S near 0 such that $\mathcal{W}^- \subset D$ and $\sigma_S(\mathcal{W}^-) =: \mathcal{W}^+ \subset U\backslash\overline{D}$, U a neighborhood of 0.

First, we notice that $G(f(t),\lambda)$ is holomorphic over $D\times\mathcal{O}_\lambda(0)$ and $\mathcal{C}^\infty$ over $(D\cup M)\times\mathcal{O}_\lambda(0)$. By the assumption that $f(M)\subset M'$, we have

$$\overline{f_n(t)} = G(f(t),\overline{f^*(t)}), \qquad \text{for } t\in M.$$

Choose a basis $\{L_1,...,L_{n-1}\}$ of the complex tangent bundle $T^{1,0}M$ with analytic coefficients in $(t,\bar t)$. Applying $\overline{L}_j$ to the previous equation, one gets:

$$\overline{L}_j\overline{f_n(t)} = \sum_{k=1}^{n-1}\frac{\partial G}{\partial\lambda_k}\overline{L}_j\overline{f}_k. \tag{1.2}$$

Let $\mathcal{J}$ denote the matrix $(\overline{L}_j\overline{f}_k)_{1\le j,k\le n-1}$ and set $J=\det\mathcal{J}$. Since f is a CR diffeomorphism, $J(t)\neq 0$ for $t\in M$. We now have

$$\left(\frac{\partial G}{\partial\lambda_1},...,\frac{\partial G}{\partial\lambda_{n-1}}\right)^\tau = \mathcal{J}^{-1}(\overline{Lf^*})(\overline{L}_1\overline{f_n},...,\overline{L}_{n-1}\overline{f_n})^\tau \tag{1.3}$$

(τ denotes transposition). Writing (1.3) as $(n-1)$ scalar equations, applying $\overline{L}_j$ to each of them and proceeding in this manner, we see, by induction, that for each multiindex $\beta=(\beta_1,...,\beta_{n-1})$, there are two holomorphic functions $P^{(1)}_\beta$ and $P^{(2)}_\beta$ in the arguments $(t,\bar t,\{\overline{L}^\gamma\overline{f}\}_{|\gamma|\le|\beta|})$ and $(t,\bar t,\{\overline{L}^\gamma\overline{f}\}_{|\gamma|\le 1})$ such that, for each $t\in M$, one has $P^{(2)}_\beta(t,\bar t,\{\overline{L}^\gamma\overline{f}\}_{|\gamma|\le 1})\neq 0$ and

$$\partial^\beta_\lambda G(f(t),\overline{f^*(t)}) = \frac{P^{(1)}_\beta(t,\bar t,\{\overline{L}^\gamma\overline{f}\}_{|\gamma|\le|\beta|})}{P^{(2)}_\beta(t,\bar t,\{\overline{L}^\gamma\overline{f}\}_{|\gamma|\le 1})}. \tag{1.4}$$

Here, L^γ denotes $L_1^{\gamma_1}\cdots L_{n-1}^{\gamma_{n-1}}$. Since $P^{(2)}_\beta$ does not vanish on M, we see that the function $P^{(1)}_\beta/P^{(2)}_\beta$ has a continuous extension to M, which we will denote by $h_\beta(t,\bar t,\{\overline{L}^\gamma\overline{f}\}_{|\gamma|\le|\beta|})$. *Recall* that since ρ' is a *polynomial*, $\partial^\beta_\lambda G$ becomes zero for $|\beta|$ sufficiently large, say $|\beta|\ge N_0+1$. Set, for $t\in\mathcal{W}^+$, $|\beta|\le N_0$, $\tilde h_\beta(t) :=$

$h_\beta(t, \overline{\sigma_S(t)}, \{\overline{L}^\gamma \overline{f(\sigma_S(t))}\}_{|\gamma|\le|\beta|})$. Since $t \mapsto \sigma_S(t)$ is antiholomorphic in t, $\tilde{h}_\beta$ extends as a holomorphic function into $\mathcal{W}^+$ and continuous in $\mathcal{W}^+ \cup S$. Now let

$$\phi_\beta^+ := \frac{1}{\beta!}(\partial_\lambda^\beta \sum_{|\gamma|\le N_0} \tilde{h}_\gamma(t)(\lambda - \overline{f^*(\sigma_S(t))})^\gamma)_{\lambda=0}, \qquad \text{for } t \in \mathcal{W}^+,\ |\beta| \le N_0,$$

$$\phi_\beta^-(t) := \frac{1}{\beta!}(\partial_\lambda^\beta G(f(t), \lambda))_{\lambda=0}, \qquad \text{for } t \in \mathcal{W}^-,\ |\beta| \le N_0.$$

Notice that ϕ_β^+ matches up with ϕ_β^- over S, by (1.4). Then the edge of the wedge theorem implies that there exists a neighborhood V of 0 such that each function equal to ϕ_β^- in $\mathcal{W}^-$, ϕ_β^+ in $\mathcal{W}^+$, extends as a holomorphic function ϕ_β defined in V, which can be filled in by analytic discs with boundaries in $\mathcal{W}^+ \cup \mathcal{W}^-$. Now,

$$\sum_{|\beta|\le N_0} \phi_\beta(t)\lambda^\beta$$

clearly gives the desired holomorphic extension for $G(f(t), \lambda)$ to $\mathcal{O}_t(0) \times \mathcal{O}_\lambda(0)$. The proof of (a) is complete.

If coordinates (w, z) are normal, the relation $G(f(t), 0) = f_n(t) = \phi_0(t)$ shows that the normal component of f extends holomorphically at 0. This gives (b).

Since M' is holomorphically nondegenerate, the complex analytic set at 0

$$\Delta' = \{(w, z) \in \mathbb{C}^n;\ \det \left(\frac{\partial \rho'_{\alpha_j}}{\partial w_j}\right)_{1\le i,j\le n-1} (w) = 0,\ \ \forall\ (\alpha_1, ..., \alpha_{n-1}) \in \mathbb{N}^{(n-1)^2}\}$$

has complex codimension greater than one [S]. Hence $\mathbb{C}^n \backslash \Delta'$ is connected, and $M' \backslash (M' \cap \Delta')$ too, since $T_0 M' = \{\bar{z} = z\}$ and the equations of Δ' depend only on the w-coordinate. Set

$$\mathcal{C} = \{(t, \lambda, \mu) \in \mathbb{C}^n \times \mathbb{C}^n;\ t \in M,\ \mu = \phi_0(t),\ \rho'_\alpha(\lambda) = \rho'_\alpha(f^*(t)),\ \forall\ |\alpha| \ge 1\}.$$

According to (b), $\mathcal{C}$ is a *real analytic* subset of $\mathbb{C}^n$. Furthermore, $\mathcal{C}$ contains the germ at 0 of a $\mathcal{C}^\infty$ smooth $(2n-1)$-dimensional manifold, namely the graph of f,

$$\Gamma = \{(t, f(t)) \in \mathbb{C}^n \times \mathbb{C}^n;\ t \in M\}.$$

Lemma 1.2. *For $p \in M$, if $f(p) \in M' \backslash (M' \cap \Delta')$, then $\mathcal{C} \equiv \Gamma$ in a neighborhood of $(p, f(p))$.*

Proof. Apply the implicit function theorem. □

Since $M \backslash (f^{-1}(M' \cap \Delta'))$ is connected too (recall f is a diffeomorphism), Lemma 1.2 implies that a single irreducible component $\mathcal{C}_1$ of the real analytic set $\mathcal{C}$ contains Γ, $\dim_{\mathbb{R}} \mathcal{C}_1 = 2n-1$. To conclude that f is analytic (hence holomorphically extendable, by complexification), apply the following theorem of Malgrange: *A $\mathcal{C}^\infty$-smooth q-dimensional $(q \ge 1)$ manifold contained in a real analytic set of dimension q is a real analytic manifold* [BHR]. □

Remark 1.3. Under the hypothesis of (c), M' being *rigid* and nondegenerate cannot contain a complex hypersurface through $f(p)$, so M is minimal too and we do not have to assume that f is holomorphic in one side of M at p, because of Trépreau's extension theorem.

Remark 1.4. The authors conjecture that statements (a), (b) are true for *any* mapping f between real analytic hypersurfaces *without* assuming that M' is polynomial rigid nor that $\det J_f(p) \neq 0$.

2. Not totally degenerate CR mappings

Theorem 1.1 is easily seen to be true if one assumes only that the mapping f is not totally degenerate in the sense that $\det(\frac{\partial F_i}{\partial w_j})(w,0) \not\equiv 0$, where $F_1, ..., F_{n-1}$ denote the formal power series of $f_1, ..., f_{n-1}$ at 0. We shall get the following refinement of Theorem 1.1 (*cf.* [MIR]).

Theorem 2.1. *(a), (b) and (c) are valid for $\mathcal{C}^\infty$ f not totally degenerate.*

Proof. Let $\tilde{\mathcal{J}}$ denote the adjoint matrix of $\mathcal{J} = (\bar{L}_j \bar{f}_k)_{1\leq j,k\leq n-1}$, so that $\tilde{\mathcal{J}}\mathcal{J} = \det \mathcal{J}\ \mathrm{Id} = J\ \mathrm{Id}$. Applying $\bar{L}_j$ to the fundamental equation $\bar{f}_n = G(f, \bar{f}^*)$, we get

$$J(\partial_{\lambda_k} G(f, \bar{f}^*))_{1\leq k\leq n-1} = \tilde{\mathcal{J}}(\bar{L}_1 \bar{f}_n, ..., \bar{L}_{n-1}\bar{f}_n)^\tau.$$

Assume, by induction, that for each $\beta \in \mathbb{N}^{n-1}$ with $|\beta| \leq k_0$, there exists a holomorphic *polynomial* g_β such that, on M,

$$J^{2|\beta|-1}\partial_\lambda^\beta G(f, \bar{f}^*) = g_\beta(t, \bar{t}, \{\bar{L}^\gamma \bar{f}\}_{|\gamma|\leq|\beta|}). \tag{2.1}$$

Prove it for $|\beta| = k_0 + 1$. Indeed, applying $\bar{L}_j$, $j = 1, ..., n-1$, to (2.1), we get

$$(2|\beta|-1)J^{2|\beta|-2}\bar{L}_j(J)\partial_\lambda^\beta G(f, \bar{f}^*) + J^{2|\beta|-1}\sum_{k=1}^{n-1}\partial_{\lambda_k}\partial_\lambda^\beta G(f, \bar{f}^*)\bar{L}_j\bar{f}_k$$
$$= g_{j\beta}(t, \bar{t}, \{\bar{L}^\gamma \bar{f}\}_{|\gamma|\leq|\beta|+1}),$$

where the $g_{j\beta}$ are holomorphic polynomials. Multiplying the equation by J, and replacing $J^{2|\beta|-1}\partial_\lambda^\beta G(f, \bar{f}^*)$ by its value given by (2.1), we get

$$J^{2|\beta|}\sum_{k=1}^{n-1}\partial_{\lambda_k}\partial_\lambda^\beta G(f, \bar{f}^*)\bar{L}_j\bar{f}_k = g'_{j\beta}(t, \bar{t}, \{\bar{L}^\gamma \bar{f}\}_{|\gamma|\leq|\beta|+1}), \qquad j = 1, ..., n-1, \tag{2.2}$$

for some polynomial $g'_{j\beta}$. Then (2.1) follows at order $k_0 + 1$ by multiplying (2.2) by $\tilde{\mathcal{J}}$. Recall that *since G is a polynomial*, the equations (2.1) are $0 = 0$, *i.e.* $\partial_\lambda^\beta G \equiv 0$, for $|\beta| \geq N_0 + 1$. Now, according to [BR], there exists $\gamma \in \mathbb{N}^{n-1}$ such that $\bar{L}^\gamma J(0) \neq 0$, and thus also, for each β, there exists $\gamma = \gamma(|\beta|)$ such that $(\bar{L}^\gamma J^{2|\beta|-1})(0) \neq 0$. This implies that for each β with $|\beta| = N_0$, we have

$$\partial_\lambda^\beta G(f, \bar{f}^*) = \partial_\lambda^\beta G(f, 0) = \frac{\bar{L}^\gamma(g_\beta(t, \bar{t}, \{\bar{L}^\gamma \bar{f}\}_{|\gamma|\leq|\beta|}))}{\bar{L}^\gamma(J^{2|\beta|-1})} = h_\beta(t, \bar{t}, \{\bar{L}^\gamma \bar{f}\}_{|\gamma|\leq\Gamma(\beta)}), \tag{2.3}$$

for some holomorphic function h_β near 0 and $\Gamma(\beta) \geq \sup\{\gamma(|\beta|);\ |\beta| = N_0\} + N_0$. Assume by downards induction that, for each $k_0 + 1 \leq |\beta| \leq N_0$, there exists a holomorphic function h_β on a neighborhood of 0 in $\mathbb{C}^{\Gamma(\beta)+2n}$ such that (2.3) is true on M. Prove it for $|\beta| = k_0$. In fact, there exists a holomorphic polynomial d_β such that

$$(\partial_\lambda^\beta G)(f, \bar{f}^*) = (\partial_\lambda^\beta G)(f, 0) + d_\beta(\{(\partial_\lambda^\beta G)(f, 0)\}_{k_0+1\leq|\beta|\leq N_0}, \bar{f}^*),$$

so that, by (2.1) and (2.3), we can write

$$J^{2|\beta|-1}(\partial_\lambda^\beta G)(f,0) = g_\beta(t,\bar{t},\{\bar{L}^\gamma \bar{f}\}_{|\gamma|\le|\beta|}) - \tag{2.4}$$

$$J^{2|\beta|-1} d_\beta(\{h_\beta(t,\bar{t},\{\bar{L}^\gamma \bar{f}\}_{|\gamma|\le\Gamma(\beta)}\}_{k_0+1\le|\beta|\le N_0}, \bar{f}^*),$$

and (2.3) follows by applying some power of $\bar{L}$ to (2.4). Since we obtained the representation (2.3), (a) and (b) follow along the lines of 1.1. (The above calculations are extracted from [BR].)

To prove (c), it is sufficient to show that the set

$$\Delta = \{(w,z) \in M;\ \det\ (L_j f_k)_{1\le j,k\le n-1}(w,z) = 0\}$$

is of Hausdorff codimension at least two, so that $M\backslash\Delta$ is connected. Indeed, let $U \subset M\backslash\Delta$ be a small open set such that $f : U \to f(U) = V$ is CR isomorphic. Then $U\backslash f^{-1}(V \cap \Delta')$ is connected too, so $M_1 = (M\backslash\Delta)/(f^{-1}(\Delta' \cap f(M\backslash\Delta)))$ is connected too and Lemma 1.2 shows that $M_1 \equiv \mathcal{C}$ there.

Lemma 2.2. *The set $\Delta \subset M$ is of Hausdorff dimension $\le 2n-3$ in a neighborhood of the origin.*

Proof. Let δ denote the $\mathcal{C}^\infty$-smooth function on M, $\delta = \det\ (L_j f_k)_{1\le j,k\le n-1}$. Since f is not totally degenerate at 0, we can write (recall δ is flat in $\bar{w}$ at 0)

$$\delta(w,\bar{w},x) = P(w) + \sum_{|\beta|=N+1} w^\beta R_\beta(w,\bar{w},x) + xQ(w,\bar{w},x),$$

for some nonzero polynomial P and $\mathcal{C}^\infty$ functions R_β, Q. We study the zero locus of δ. We can assume that $\delta(w_1,0,\bar{w}_1,0,0) = w_1^N(1 + S(w_1,\bar{w}_1))$, with S $\mathcal{C}^\infty$, flat in $\bar{w}_1$ at 0, $S(0) = 0$. Malgrange's preparation theorem yields that there exist $\mathcal{C}^\infty$ functions q and r on M with $q(0) \ne 0$ and $r(0) \ne 0$ such that (writing $w^* = (w_2,...,w_{n-1})$)

$$(q\delta)(w,\bar{w},x) = u_1^N + \sum_{1\le j\le N} \lambda_j(v_1,w^*,\bar{w}^*,x)u_1^{N-j}$$

and

$$(r\delta)(w,\bar{w},x) = v_1^N + \sum_{1\le j\le N} \mu_j(u_1,w^*,\bar{w}^*,x)v_1^{N-j},$$

for $\mathcal{C}^\infty$ functions λ_j, μ_j, $1 \le j \le N$, vanishing at 0. Fixing small $(w^*,\bar{w}^*,x)$, one sees that there can be at most N^2 solutions to the system of the two equations above. □

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