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UNIQUENESS OF NON-ARCHIMEDEAN ENTIRE FUNCTIONS SHARING SETS OF VALUES COUNTING MULTIPLICITY

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ABSTRACT. A set is called a unique range set for a certain class of functions if each inverse image of that set uniquely determines a function from the given class. We show that a finite set is a unique range set, counting multiplicity, for non-Archimedean entire functions if and only if there is no non-trivial affine transformation preserving the set. Our proof uses a theorem of Berkovich to extend, to non-Archimedean entire functions, an argument used by Boutabaa, Escassut, and Haddad to prove this result for polynomials

A well-known theorem of R. Nevanlinna (*cf.* [Nev]) says that if f and g are two meromorphic functions such that $f^{-1}(a_i) = g^{-1}(a_i)$ for five distinct points $a_1, \ldots, a_5$ on the Riemann sphere, then either both f and g are constant, or $f \equiv g$. A similar result, but with only two values a_i, and with meromorphic functions replaced by polynomials, can be found in [A-S], where Adams and Straus also show that the statement remains true if f and g are even allowed to be non-Archimedean entire functions. Throughout this work, the expression non-Archimedean entire function will mean a formal power series in one variable with coefficients in an algebraically closed field K of characteristic zero, complete with respect to a (possibly trivial) non-Archimedean absolute value, and such that the power series has infinite radius of convergence. This theorem of Adams and Straus fits into a principle of the first author of the present work, which states that most theorems that are true for polynomials will also be true for non-Archimedean entire functions, if stated appropriately. See [Ch] for a geometric conjecture based, in part, on this principle.

The theorem of Nevanlinna quoted above says that if $f^{-1}(a_i) = g^{-1}(a_i)$ for some values a_i, then f and g must be equal. Rather than consider the values one at a time, we can weaken the hypothesis by considering the values together as a set. Namely, given a set of values S, we say that two functions f and g **share** S, **ignoring multiplicity,** if $f^{-1}(S) = g^{-1}(S)$. One can also take multiplicity into account. Namely, if given a function f, we let

$$E(f, S) = \{(z, m) \in K \times \mathbf{Z} : f(z) = a \in S \text{ and } f(z) = a \text{ with multiplicity } m\},$$

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where K is the field of definition of f, and $\mathbf{Z}$ denotes the set of integers, then f and g **share** S, **counting multiplicity,** if $E(f,S) = E(g,S)$. A set S is called a **unique range set**, counting or ignoring multiplicity, for a certain function class, if whenever two non-constant functions in that class share the set S, counting or ignoring multiplity, then the two functions must be identically equal. For short, we will use URS, URSCM, and URSIM as abbreviations for "unique range set," "unique range set, counting multiplicity," and "unique range set, ignoring multiplicity," respectively.

Gross [Gr] introduced the concept and terminology of a unique range set, and asked whether unique range sets of finite cardinality exist. Gross and Yang [G-Y] were the first to construct a non-trival URSCM for complex entire functions; their set was infinite, but discrete. Since then, people have constructed URS's with finite cardinality, and recent progress in determining the minimal cardinality of URS's for complex entire and meromorphic functions has been fast-paced. See, for example, the work of Li-Yang [L-Y], and of Frank-Reinders [F-R]. Also, see the book (written in Chinese) by Yi and Yang [Y-Y].

A central question is what are necessary and sufficient conditions for a set S to be a URSCM or URSIM as various classes of functions are considered? In the case of complex meromorphic or entire functions, one is still a long way away from characterizing unique range sets in terms of necessary and sufficient conditions. In fact, up till now, only in the case of polynomials did one have a nice characterization of unique range sets. Following Boutabaa, Escassut, and Haddad [B-E-H], we call a subset S of a field K **affinely rigid** if no non-trivial affine transformation of K preserves S. Actually, Boutabaa, Escassut, and Haddad now use the word "stiff" in place of "affinely rigid," but we prefer to keep the terminology "affinely rigid." By an elementary and elegant argument, Boutabaa, Escassut, and Haddad [B-E-H] showed that if K is an algebraically closed field of characteristic zero, then a finite set S is a URSCM for polynomials if and only if S is affinely rigid. In fact, Ostrovskii, Pakovitch, and Zaidenberg [O-P-Z] have shown that precisely the same sets turn out to be unique range sets for polynomials *ignoring* multiplicity, provided one considers only polynomials of a fixed degree.

The various constructions of unique range sets for functions of a complex variable found in the literature tend to make use of Nevanlinna's theory of value distribution. Since an analogue of this theory exists for non-Archimedean meromorphic functions, these constructions also give examples of URS's for non-Archimedean functions, as in the work of Hu-Yang [H-Y]. In the work [B-E-H] cited above, Boutabaa, Escassut, and Haddad constructed URSCM's of arbitrary finite cardinality ≥ 3 for non-Archimedean entire functions. They also succeeded in characterizing the URSCM's of cardinality three by showing that a three element set is a URSCM for non-Archimedean entire functions if and only if it is affinely rigid, as in the case of polynomials. This fits the aforementioned principle that theorems true for polynomials should remain true for non-Archimedean entire functions. It was then natural to suspect that a finite set would be a URSCM for non-Archimedean entire functions if and only if it were a URSCM for polynomials, which is what we show here. In fact, except for our application of a theorem of Berkovich, our proof involves little more than pointing out that the elementary proof used by Boutabaa, Escassut, and Haddad [B-E-H] in their treatment of polynomials extends to the non-Archimedean entire case, and thus the more complicated portions of [B-E-H] can be avoided.

Theorem 1. *Let K be an algebraically closed field of characteristic zero, complete with respect to a non-Archimedean absolute value. A finite set S in K is a unique range set, counting multiplicity, for non-Archimedean entire functions over K if and only if S is affinely rigid.*

We remark that the characteristic zero hypothesis is necessary since, for example in characteristic three, the polynomials x^2 and x^2+x+1 share, counting multiplicities, the set $S = \{-1, \zeta, -\zeta\}$ where $\zeta^2 = -1$, and S is not preserved by any non-trivial affine transformation. We leave this fact as an exercise for the reader. In fact, J. F. Voloch has pointed out to the first author that no set with three elements can be a URS for polynomials in characteristic three. The interesting problem of giving a nice characterization of unique range sets for polynomials in positive characteristic remains open.

The proof of Theorem 1 breaks up into three parts. The first part is essentially Berkovich's Theorem. The remaining two parts are nearly as in [B-E-H], but we recall the proofs here for the convenience of the reader.

Theorem 2. *Let f and g be non-constant non-Archimedean entire functions. Let $F(x, y)$ be a polynomial in two variables with coefficients in K. Suppose $F(f, g) = 0$. Then there exist a non-Archimedean entire function h and two polynomials $p(z)$ and $q(z)$ such that $f = p(h)$ and $g = q(h)$.*

Proof. Let $F_0(x, y)$ be an irreducible factor of F such that $F_0(f, g) = 0$. Since f and g are not constant, by Berkovich's non-Archimedean Picard Theorem ([Ber], Theorem 4.5.1, see also [Ch]), $F_0(x, y) = 0$ is a rational curve (*i.e.* an algebraic curve of genus 0), and since K is algebraically closed, can therefore be rationally parametrized. In other words, there exist rational functions $r(t), s(t)$, and $R(x, y)$ such that $t = R(x, y)$, and $F_0(r(t), s(t)) = 0$. Let $h = R(f, g)$, so that $f = r(h)$, and $g = s(h)$. Now, since f and g are entire, the non-Archimedean meromorphic function h must omit $r^{-1}(\infty)$ and $s^{-1}(\infty)$. However, non-Archimedean meromorphic functions can omit at most one point in $K \cup \{\infty\}$. Thus, $r^{-1}(\infty)$ must equal $s^{-1}(\infty)$, and consists of exactly one point. Therefore, after making a projective linear change in coordinates, we can assume $r^{-1}(\infty) = s^{-1}(\infty) = \infty$, and that h omits ∞. In other words, we may assume r and s are polynomials, and h is entire. □

Lemma 3. *Let h be a non-constant, non-Archimedean entire function, and let p, q, and P be polynomials with coefficients in K. Assume that $n = \deg P \geq 1$, that $d = \deg p \geq \deg q$, and that $d \geq 1$. Suppose that C is a non-zero constant such that $P(p(h)) = CP(q(h))$. Then, $p(h) = \theta q(h) + \xi$, where θ and ξ are in K, and moreover, $\theta^n = C$.*

Proof. Let K_0 be the field generated by the coefficients of p, q, and P. By extending K if necessary, we can assume that K contains elements which are transcendental over K_0. Since h is not constant, we can choose z_0 in K so that $\zeta = h(z_0)$ is transcendental over K_0. Write

$$p(h(z_0)) = \sum_{j=0}^{d} a_j \zeta^j \quad \text{and} \quad q(h(z_0)) = \sum_{j=0}^{d} b_j \zeta^j.$$

Expand out $P(p(h(z_0)))$ and $CP(q(h(z_0)))$, and collect terms according to the power of ζ that appears. Note that only the degree n term in P will produce powers of ζ greater than $d(n-1)$. By comparing coefficients of the term involving ζ^{dn}, we see that $\deg q$ also equals d, and that $a_d = \theta b_d$, for some constant θ with $\theta^n = C$. Now, since we are in characteristic zero, comparing the coefficients of the remaining terms where the power of ζ is larger than $d(n-1)$ shows that $a_j = \theta b_j$ for all $j \geq 1$. Setting $\xi = a_0 - \theta b_0$ completes the proof of the theorem. □

Lemma 4. *Let S be an affinely rigid subset of K. Let g be a non-constant non-Archimedean entire function, and let $f = Ag + B$, where A and B are elements of K, and $A \neq 0$. If $f^{-1}(S) = g^{-1}(S)$, then $f = g$.*

Proof. Because g is non-Archimedean entire, and non-constant, every point in K, and hence in S, is in the image of g. Thus, $g(g^{-1}(S)) = S$. If $f^{-1}(S) = g^{-1}(S)$, then $f(g^{-1}(S)) \subseteq S$. Thus,

$$S \supseteq f(g^{-1}(S)) = Ag(g^{-1}(S)) + B = AS + B.$$

Therefore, the affine transformation $z \mapsto Az + B$ is a non-trivial affine transformation preserving S, contradicting our assumption that S is affinely rigid. □

Proof of Theorem 1. Write $S = \{s_1, \ldots, s_n\}$, and let f and g be two non-constant, non-Archimedean entire functions which share S, counting multiplicity. Let $P(X)$ be the polynomial

$$P(X) = (X - s_1) \cdots (X - s_n).$$

Because f and g share S, counting multiplicity, $P(f)$ and $P(g)$ are non-Archimedean entire functions which have precisely the same zeros, counting multiplicity. Thus $P(f)/P(g)$ is a non-Archimedean entire function which is never zero, and so must be a non-zero constant C. Letting $F(x, y) = P(x) - CP(y)$, we have $F(f, g) = 0$. Thus, by Theorem 2, we have a non-Archimedean entire function h, and two polynomials p and q so that $f = p(h)$ and $g = q(h)$. By Lemma 3, we then have that $f = Ag + B$, where A and B are constants, and $A^n = C$. Lemma 4 then tells us that either $f \equiv g$, or there is a non-trivial affine transformation of K which preserves S. Thus, if S is affinely rigid, then S is a URSCM for non-Archimedean entire functions. For the other implication, it is easy to see that if S is not affinely rigid, then S cannot be a unique range set. □

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