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# COMMON FIXED POINTS OF COMMUTING HOLOMORPHIC MAPS IN THE UNIT BALL OF $\mathbb{C}^n$

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Abstract. Let $\mathbb{B}^n$ be the unit ball of $\mathbb{C}^n$ $(n > 1)$. We prove that if $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ are holomorphic self-maps of $\mathbb{B}^n$ such that $f \circ g = g \circ f$, then $f$ and $g$ have a common fixed point (possibly at the boundary, in the sense of $K$-limits). Furthermore, if $f$ and $g$ have no fixed points in $\mathbb{B}^n$, then they have the same Wolff point, unless the restrictions of $f$ and $g$ to the one-dimensional complex affine subset of $\mathbb{B}^n$ determined by the Wolff points of $f$ and $g$ are commuting hyperbolic automorphisms of that subset.

## 0. Introduction

In their papers [3] and [13], Behan and Shields showed that, except for the case of two hyperbolic automorphisms of $\Delta$ (the unit disk of $\mathbb{C}$), two non-trivial commuting holomorphic self-maps of $\Delta$ have the same fixed point in $\Delta$ or the same "Wolff point" in $\partial\Delta$.

In the multi-dimensional case it has been proved (see *e.g.* [1]) that two commuting holomorphic self-maps of $\mathbb{B}^n$ which are continuous up to the boundary $\partial\mathbb{B}^n$ have a common fixed point in $\overline{\mathbb{B}^n}$. In [6] we proved that if two holomorphic self-maps of $\mathbb{B}^n$, $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$, with no fixed points, commute and if there exists a complex geodesic $\varphi : \Delta \to \mathbb{B}^n$ such that $f(\varphi(\Delta)) \subseteq \varphi(\Delta)$, then there exists $\tau \in \partial\mathbb{B}^n$ such that $K\text{-}\lim_{z\to\tau} f(z) = \tau$ and $g$ has restricted $K$-limit $\tau$ at $\tau$.

In this paper, under the only hypothesis that $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ are such that $f \circ g = g \circ f$, we prove that either there exists $z_0 \in \mathbb{B}^n$ such that $f(z_0) = g(z_0) = z_0$, or there exists $\tau \in \partial\mathbb{B}^n$ such that $K\text{-}\lim_{z\to\tau} f(z) = K\text{-}\lim_{z\to\tau} g(z) = \tau$. In particular if $f$ has no fixed points and $\tau(f) \in \partial\mathbb{B}^n$ is its Wolff point, then $\tau(f)$ is a fixed point (in the sense of $K$-limits) for $g$. Finally we show that a Behan Shields-type theorem holds in $\mathbb{B}^n$, that is we prove that two commuting holomorphic maps with no fixed points either have the same Wolff point or they are conjugated to two commuting holomorphic maps whose first components are hyperbolic automorphisms of $\Delta$.

The plan of this paper is the following. In the first section we establish the notations and recall some classical theorems from the iteration theory in $\mathbb{B}^n$; then we define the Wolff point and the boundary dilatation coefficient for holomorphic

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self-maps of $\mathbb{B}^n$. In the second section we state and prove the main theorem (Theorem 2.1) concerning the common fixed points of commuting holomorphic maps of $\mathbb{B}^n$ and (if the maps have no fixed points) we establish an upper bound for the boundary dilatation coefficients at the Wolff point. In the third section we deal with holomorphic maps having no fixed points in $\mathbb{B}^n$. We construct a simple example of commuting maps with different Wolff points which are not automorphisms of $\mathbb{B}^n$ (if $n > 1$), but whose first components are indeed hyperbolic automorphisms of $\Delta$. Then we prove that this is in a certain sense the only case in which two commuting holomorphic self-maps of $\mathbb{B}^n$ may have different Wolff points (see Theorem 3.2).

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## 1. Preliminary results and notations

In this section we state some classical theorems and definitions. The books by Abate [1] and Rudin [12] are suggested as general references. We begin with the following:

**Theorem 1.1** (Hervé [10]). *Let $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$. Then the set of fixed points of $h$, $\mathrm{Fix}(h) := \{z \in \mathbb{B}^n : h(z) = z\}$, is empty or it is a $m$-dimensional affine subset of $\mathbb{B}^n$, i.e. it is the intersection of $\mathbb{B}^n$ with an affine $m$-dimensional complex subspace of $\mathbb{C}^n$.*

Throughout the paper we will say that $h$ *has no fixed points* to mean that $\mathrm{Fix}(h) = \emptyset$.

**Theorem 1.2** (MacCluer [11]). *Let $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ be without fixed points; then there is a unique $x \in \partial\mathbb{B}^n$ such that for every $z \in \mathbb{B}^n$*

$$\frac{|1 - \langle h(z), x\rangle|^2}{1 - \|h(z)\|^2} \leq \frac{|1 - \langle z, x\rangle|^2}{1 - \|z\|^2},$$

*where $\langle\ ,\ \rangle$ denotes the hermitian product in $\mathbb{C}^n$.*

In the classical iteration theory for $\Delta$ one usually associates to any holomorphic self-map a well-defined “fixed point” belonging to $\overline{\Delta}$, the “Wolff point” of the function. In the multi-dimensional case there is no chance to do the same since the set of fixed points contains in general more than one point. But when the fixed points set is empty, then Theorem 1.2 ensures a situation similar to the classical one-dimensional setting. So we introduce the following:

**Definition 1.3.** If $h$ is a holomorphic self-map of $\mathbb{B}^n$ without fixed points, we call *the Wolff point of $h$* the unique $x \in \partial\mathbb{B}^n$ defined by Theorem 1.2.

In this setting, we can state a Wolff Denjoy-type theorem as follows:

**Theorem 1.4** (Hervé [10], MacCluer [11]). *Let $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$. The sequence of iterates of $h$, $\{h^k\}$, is not compactly divergent if and only if $\mathrm{Fix}(h) \neq \emptyset$. In the case that $h$ has no fixed points then $\{h^k\}$ converges to the Wolff point of $h$.*

Another non-standard definition, well-motivated by the comparison with the one-dimensional case, is the following:

**Definition 1.5.** Let $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ and let $x \in \partial\mathbb{B}^n$. The *boundary dilatation coefficient* of $h$ at $x$ is the value $\liminf_{z\to x}(1 - \|h(z)\|) \cdot (1 - \|z\|)^{-1}$.

*Remark* 1.6. As in the one-dimensional case, the boundary dilatation coefficient is strictly positive (one can check this directly or by means of Julia's lemma). In particular it is easy to show (or see [6]) that if $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ has no fixed points and $x \in \partial\mathbb{B}^n$ is its Wolff point, then the boundary dilatation coefficient $\alpha(h)$ of $h$ at $x$ is such that $0 < \alpha(h) \leq 1$.

We recall that a *Korányi region* of vertex $x \in \partial\mathbb{B}^n$ and amplitude $M > 1$ is defined as

$$K(x, M) := \left\{ z \in \mathbb{B}^n : \frac{|\, 1 - \langle z, x \rangle \,|}{1 - \|z\|} < M \right\}.$$

If $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$, we say that $h$ has *$K$-limit* $y$ at $x \in \partial\mathbb{B}^n$ (in short $K\text{-}\lim_{z\to x} h(z) = y$) if, for each $M > 1$ and for each sequence $\{z_m\} \subset K(x, M)$ such that $\lim_{m\to\infty} z_m = x$, we get $\lim_{m\to\infty} h(z_m) = y$.

For $x \in \partial\mathbb{B}^n$, a *$x$-curve* is a continuous curve $\gamma : [0, 1) \to \mathbb{B}^n$ such that $\gamma(t) \to x$ as $t \to 1$. To every $x$-curve we associate its orthogonal projection $\gamma_x = \langle \gamma, x \rangle x$ on $\mathbb{C}x$. A $x$-curve is said to be *special* if $\lim_{t\to 1}(\|\gamma(t) - \gamma_x(t)\|^2) \cdot (1 - \|\gamma_x(t)\|^2)^{-1} = 0$, and *restricted* if it is special and moreover there is $A < \infty$ such that $(\|\gamma_x(t) - x\|) \cdot (1 - \|\gamma_x(t)\|)^{-1} \leq A$ for all $t \in [0, 1)$. We say that a holomorphic self-map $h$ of $\mathbb{B}^n$ has *restricted $K$-limit* $y$ at $x \in \partial\mathbb{B}^n$ if $h(\gamma(t)) \to y$ as $t \to 1$ for any restricted $x$-curve $\gamma$. The relationship between restricted curves and Korányi regions is stated in the following lemma (see, [12], p.170, or [1], p.171).

**Lemma 1.7.** *Let $x \in \partial\mathbb{B}^n$ and let $\gamma : [0, 1) \to \mathbb{B}^n$ be a $x$-curve. Suppose $\gamma$ is special. Then $\gamma$ is restricted if and only if it lies in $K(x, M)$ for some $M > 1$.*

Now we can state the classical Julia-Wolff-Carathéodory type theorem for $\mathbb{B}^n$:

**Theorem 1.8** (Rudin [12], p.177)**.** *Let $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ and let $x \in \partial\mathbb{B}^n$ be such that*

$$\liminf_{z\to x} \frac{1 - \|h(z)\|}{1 - \|z\|} = C < \infty.$$

*Then there exists a unique $y \in \partial\mathbb{B}^n$ such that $h$ has $K$-limit $y$ at $x$ and the following functions are bounded in every Korányi region:*

1) $\dfrac{1 - \langle h(z), y \rangle}{1 - \langle z, x \rangle}$,
2) $\dfrac{h(z) - \langle h(z), y \rangle y}{(1 - \langle z, x \rangle)^{\frac{1}{2}}}$.

*Moreover the function 1) has restricted $K$-limit $C$ at $x$ and the function 2) has restricted $K$-limit 0 at $x$.*

We recall that the point $y \in \partial\mathbb{B}^n$ in Theorem 1.8 is the unique point of $\partial\mathbb{B}^n$ such that

$$\frac{|\, 1 - \langle h(z), y \rangle \,|^2}{1 - \|h(z)\|^2} \leq C \frac{|\, 1 - \langle z, x \rangle \,|^2}{1 - \|z\|^2}$$

for every $z \in \mathbb{B}^n$.

Keeping in mind this remark and using the uniqueness statement of Theorem 1.2 and the estimate in Remark 1.6, one can easily prove this useful characterization of the Wolff point of a map with no fixed points (for other characterizations see [6]):

**Proposition 1.9.** *Let $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ have no fixed points. Then a point $x \in \partial\mathbb{B}^n$ is the Wolff point of $h$ if and only if*

- $K\text{-}\lim\limits_{z\to x} h(z) = x$ *and*
- $\liminf\limits_{z\to x} \dfrac{1-\|h(z)\|}{1-\|z\|} \leq 1.$

In what follows we need this geometric version of the Julia's lemma:

**Lemma 1.10.** *Let $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ and $x \in \partial\mathbb{B}^n$. If $\liminf_{z\to x} \frac{1-\|h(z)\|}{1-\|z\|} < +\infty$, then there exists $y \in \partial\mathbb{B}^n$ such that $h$ maps Korányi regions with vertex $x$ into Korányi regions with vertex $y$.*

*Proof* (see also [12], p. 176). Let $C := \liminf_{z\to x} \frac{1-\|h(z)\|}{1-\|z\|}$. By Remark 1.6, $C > 0$. Let $y \in \partial\mathbb{B}^n$ be the point defined by Theorem 1.8. Fixing $M > 1$ we claim that $h(K(x, M)) \subseteq K(y, R)$ for some $R > 1$. To show this, it suffices to show that $(|1 - \langle h(z), y\rangle|) \cdot (1 - \|h(z)\|)^{-1}$ is bounded uniformly in $z \in K(x, M)$. Write as follows:

$$\frac{|1-\langle h(z), y\rangle|}{1-\|h(z)\|} = \frac{1-\|z\|}{1-\|h(z)\|} \cdot \frac{|1-\langle z, x\rangle|}{1-\|z\|} \cdot \frac{|1-\langle h(z), y\rangle|}{|1-\langle z, x\rangle|}. \tag{1.1}$$

Now $\limsup_{z\to x}(1-\|z\|) \cdot (1-\|h(z)\|)^{-1} = C^{-1}$, $(|1-\langle z, x\rangle|) \cdot (1-\|z\|)^{-1} \leq M$ since $z \in K(x, M)$ and $(|1-\langle h(z), y\rangle|) \cdot (|1-\langle z, x\rangle|)^{-1}$ is bounded in $K(x, M)$ by Theorem 1.8. Then the assertion follows. □

## 2. Common fixed points

Now we are ready to prove that two commuting holomorphic self-maps of $\mathbb{B}^n$ have a common fixed point (possibly at the boundary in the sense of $K$-limits):

**Theorem 2.1.** *Let $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ and $f \circ g = g \circ f$.*

1) *If $\mathrm{Fix}(f) \neq \emptyset$ and $\mathrm{Fix}(g) \neq \emptyset$, then $\mathrm{Fix}(f) \cap \mathrm{Fix}(g) \neq \emptyset$.*
2) *If $\mathrm{Fix}(f) = \emptyset$, then there exists $\tau \in \partial\mathbb{B}^n$ such that $K\text{-}\lim\limits_{z\to\tau} f(z) = K\text{-}\lim\limits_{z\to\tau} g(z) = \tau$.*

Statement 1) is actually a known fact since its proof is the same as in the case in which $f$ and $g$ extend continuously to the boundary. So we only sketch the proof, referring the interested reader to [1], p.186, or [2] for the case of convex domains. Both to prove Theorem 2.1 and for the results of the next section, we need two technical lemmas.

**Lemma 2.2.** *Let $h \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ and $x \in \partial\mathbb{B}^n$ be such that $\liminf_{z\to x}(1-\|h(z)\|)\cdot(1-\|z\|)^{-1} = C < \infty$, and let $y \in \partial\mathbb{B}^n$ be such that $K\text{-}\lim_{z\to x} h(z) = y$. Then the curve $r \mapsto h(rx)$ is a restricted $y$-curve.*

*Proof* (see also [7], p.184). By Theorem 1.8, the curve $r \mapsto h(rx)$ is a $y$-curve and it lies in a suitable Korányi region of vertex $y$ (by Lemma 1.10). So, if we prove that it is special, then Lemma 1.7 implies that it is restricted. The $x$-curve $r \mapsto rx$ is obviously restricted, so Theorem 1.8 implies

$$\frac{\|h(rx) - \langle h(rx), y\rangle y\|}{\sqrt{1-r}} \to 0 \text{ as } r \to 1.$$

Moreover the function $r \mapsto (1-r)\cdot(1-|\langle h(rx),y\rangle|^2)^{-1}$ is bounded because

$$\limsup_{r\to 1} \frac{1-r}{1-|\langle h(rx),y\rangle|^2} = \frac{1}{2}\left(\liminf_{r\to 1}\frac{1-|\langle h(rx),y\rangle|}{1-r}\right)^{-1} \le \frac{1}{2C},$$

and $C>0$ by Remark 1.6. Thus

$$\begin{aligned}\lim_{r\to 1} &\frac{\|h(rx)-\langle h(rx),y\rangle y\|^2}{1-|\langle h(rx),y\rangle|^2} \\ &= \lim_{r\to 1}\left(\frac{\|h(rx)-\langle h(rx),y\rangle y\|}{\sqrt{1-r}}\right)^2 \cdot \frac{1-r}{1-|\langle h(rx),y\rangle|^2} = 0,\end{aligned}$$

and the curve is special. □

**Lemma 2.3** (Estimate at the Wolff point)**.** *Suppose $f,g \in \mathrm{Hol}(\mathbb{B}^n,\mathbb{B}^n)$ have no fixed points and $f\circ g = g\circ f$. Let $\tau \in \partial\mathbb{B}^n$ be the Wolff point of $f$, let $\delta\in\partial\mathbb{B}^n$ be the Wolff point of $g$ and let $L := \liminf_{z\to\delta}(1-\|g(z)\|)\cdot(1-\|z\|)^{-1}$ denote the boundary dilatation coefficient of $g$ at its Wolff point. Then $K\text{-}\lim_{z\to\tau} g(z) = \tau$ and*

$$\liminf_{z\to\tau}\frac{1-\|g(z)\|}{1-\|z\|} \le \frac{1}{L}. \tag{2.1}$$

*Proof.* Let $\alpha := \liminf_{z\to\tau}\frac{1-\|f(z)\|}{1-\|z\|}$ and let $\beta := \liminf_{z\to\tau}\frac{1-\|g(z)\|}{1-\|z\|}$. By Remark 1.6 we obtain $0<\alpha\le 1$, $0<L\le 1$ and $\beta>0$. Fix $z\in\mathbb{B}^n$ and consider the sequence of iterates of $f$, $\{f^k(z)\}$. By Theorem 1.4 it follows that $f^k(z)\to\tau$ as $k\to\infty$. Moreover

$$\begin{aligned}&\left|\log\frac{1-\|g(f^k(z))\|}{1-\|f^k(z)\|}\right| \\ &\quad\le \left|\log\frac{1+\|f^k(z)\|}{1-\|f^k(z)\|} - \log\frac{1+\|g(f^k(z))\|}{1-\|g(f^k(z))\|}\right| + \left|\log\frac{1+\|g(f^k(z))\|}{1+\|f^k(z)\|}\right| \\ &\quad= 2\left|k_n(0,f^k(z)) - k_n(0,g(f^k(z)))\right| + \left|\log\frac{1+\|g(f^k(z))\|}{1+\|f^k(z)\|}\right|,\end{aligned}$$

where $k_n$ is the *Bergman distance* on $\mathbb{B}^n$ (see Bergman [4], [5]). Let us recall how $k_n$ is defined. Let $\Phi_a(z)$ be the automorphism of $\mathbb{B}^n$ given by

$$\Phi_a(z) := \frac{a - \frac{\langle z,a\rangle}{\langle a,a\rangle}a - \sqrt{1-|a|^2}\left(z-\frac{\langle z,a\rangle}{\langle a,a\rangle}a\right)}{1-\langle z,a\rangle}. \tag{2.2}$$

A straightforward computation shows that $\Phi_a(a)=0$ and $\Phi_a(0)=a$. The *Bergman metric* $dk^2$ on $\mathbb{B}^n$ is given by $dk_0^2 := ds^2$ (with $ds^2$ the hermitian metric on $\mathbb{C}^n$) and, for all $a\in\mathbb{B}^n$ and $u,v\in\mathbb{C}^n$, by (here we are identifying $\mathbb{C}^n$ with the tangent space of $\mathbb{B}^n$ at $a$):

$$dk_a^2(u,v) := \left[(\Phi_a)^* ds^2\right](u,v).$$

The distance associated to $dk^2$ is the Bergman distance $k_n$. One of the main properties of $k_n$ is that it is contracted by holomorphic self-maps of $\mathbb{B}^n$ and it is invariant under the action of automorphisms of $\mathbb{B}^n$ (see, *e.g.*, [1], p.163-164, or [12], p.163). Moreover it is easy to see that

$$k_n(0,w) = \frac{1}{2}\log\frac{1+\|w\|}{1-\|w\|} \text{ for all } w\in\mathbb{B}^n.$$

Then, using the aforementioned properties of $k_n$ and because $f \circ g = g \circ f$, we get:

$$\begin{aligned}
2\left|k_n(0, f^k(z)) - k_n(0, g(f^k(z)))\right| &+ \left|\log \frac{1 + \|g(f^k(z))\|}{1 + \|f^k(z)\|}\right| \\
&\leq 2k_n(z, g(z)) + \left|\log \frac{1 + \|f^k(g(z))\|}{1 + \|f^k(z)\|}\right| \\
&= 2k_n(0, \Phi_z(g(z))) + \left|\log \frac{1 + \|f^k(g(z))\|}{1 + \|f^k(z)\|}\right|.
\end{aligned}$$

By Theorem 1.4 it follows that $\lim_{k\to\infty} f^k(w) = \tau$ for each $w \in \mathbb{B}^n$. Then

$$\left|\log \frac{1 + \|f^k(g(z))\|}{1 + \|f^k(z)\|}\right| \to 0 \quad \text{as } k \to \infty$$

and, since $2k_n(0, \Phi_z(g(z))) = \log \dfrac{1 + \|\Phi_z(g(z))\|}{1 - \|\Phi_z(g(z))\|}$, taking the limit for $k \to \infty$ we obtain

$$\liminf_{k\to\infty} \frac{1 - \|g(f^k(z))\|}{1 - \|f^k(z)\|} \leq \frac{1 + \|\Phi_z(g(z))\|}{1 - \|\Phi_z(g(z))\|}. \tag{2.3}$$

In particular (2.3) implies $\beta < +\infty$. Then by Theorem 1.8 there exists $\sigma \in \partial\mathbb{B}^n$ such that $K\text{-}\lim_{z\to\tau} g(z) = \sigma$. Notice that if $f^k(z)$ converges to $\tau$ in a Korányi region, then $\sigma = \tau$ (but in general $\{f^k(z)\}$ doesn't lie in any Korányi region: check the convergence of the iterates of a parabolic automorphism of $\mathbb{B}^n$).

Moreover since $\beta$ and $\alpha$ are bounded, it follows from Lemma 2.2 that the curves $r \mapsto f(r\tau)$ and $r \mapsto g(r\tau)$ are, respectively, a restricted $\tau$-curve and a restricted $\sigma$-curve. Therefore, since $\lim_{r\to 1} f(g(r\tau)) = \lim_{r\to 1} g(f(r\tau)) = \sigma$, it follows that, if $f$ has $K$-limit at $\sigma$, it has to be $\sigma$. So if we prove that $\liminf_{z\to\sigma} \frac{1-\|f(z)\|}{1-\|z\|} \leq 1$, then Proposition 1.9 will yield $\sigma = \tau$. To this aim it suffices to show that $\liminf_{r\to 1} \frac{1-\|f(g(r\tau))\|}{1-\|g(r\tau)\|} \leq 1$. We can write

$$\frac{1 - \|f(g(r\tau))\|}{1 - \|g(r\tau)\|} = \frac{1 - \|f(g(r\tau))\|}{|1 - \langle f(r\tau), \tau\rangle|} \cdot \frac{|1 - \langle f(r\tau), \tau\rangle|}{1 - r} \cdot \frac{1 - r}{1 - \|g(r\tau)\|}. \tag{2.4}$$

Now we shall estimate the factors on the right-hand side of equality (2.4). Since $\beta < \infty$, since $r \mapsto f(r\tau)$ is a restricted $\tau$-curve and since $f$ and $g$ commute, then by Theorem 1.8 we obtain

$$\frac{1 - \|f(g(r\tau))\|}{|1 - \langle f(r\tau), \tau\rangle|} \leq \frac{|1 - \langle g(f(r\tau)), \sigma\rangle|}{|1 - \langle f(r\tau), \tau\rangle|} \to \beta \text{ as } r \to 1.$$

In the same way (taking into account that $r \mapsto r\tau$ is a restricted $\tau$-curve) we get

$$\frac{|1 - \langle f(r\tau), \tau\rangle|}{1 - r} \to \alpha \text{ as } r \to 1.$$

Moreover

$$\limsup_{r\to 1} \frac{1 - r}{1 - \|g(r\tau)\|} = \left(\liminf_{r\to 1} \frac{1 - \|g(r\tau)\|}{1 - r}\right)^{-1} \leq \frac{1}{\beta}.$$

Then taking the limit for $r \to 1$ in equation (2.4) we have:

$$\liminf_{r\to 1} \frac{1 - \|f(g(r\tau))\|}{1 - \|g(r\tau)\|} \leq \beta \cdot \alpha \cdot \frac{1}{\beta} = \alpha \leq 1.$$

It remains to prove that $\beta \leq L^{-1}$. From formula (2.3) it follows that $\beta \leq \frac{1+\|\Phi_z(g(z))\|}{1-\|\Phi_z(g(z))\|}$ for each $z \in \mathbb{B}^n$. Choose $z := r\delta$, with $0 \leq r < 1$. It suffices to show that

$$\|\Phi_{r\delta}(g(r\delta))\| \to \frac{1-L}{1+L} \text{ as } r \to 1.$$

From the definition of $\Phi_{r\delta}(g(r\delta))$—see (2.2)—we obtain

$$\begin{aligned}\Phi_{r\delta}(g(r\delta)) = &\left(\frac{(1-\langle g(r\delta),\delta\rangle)\delta}{1-r} + \frac{(r-1)\delta}{1-r} - \sqrt{1+r}\,\frac{g(r\delta)-\langle g(r\delta),\delta\rangle\delta}{\sqrt{1-r}}\right) \\ &\cdot \frac{1-r}{1-\langle g(r\delta), r\delta\rangle}.\end{aligned} \tag{2.5}$$

By Theorem 1.8, since the curve $r \mapsto r\delta$ is $\delta$-restricted, the first factor on the right-hand side of equality (2.5) tends to $(L-1)\delta$ as $r \to 1$. As for the second factor of equality (2.5), using again Theorem 1.8, we get for $r \to 1$

$$\frac{1-r}{1-\langle g(r\delta), r\delta\rangle} = \frac{1-r}{1-\langle g(r\delta),\delta\rangle} \cdot \frac{1}{r + \frac{1-r}{1-\langle g(r\delta),\delta\rangle}} \to \frac{1}{L} \cdot \frac{1}{1+\frac{1}{L}} = \frac{1}{1+L}.$$

Therefore $\|\Phi_{r\delta}(g(r\delta))\| \to (1-L)(1+L)^{-1}$. □

*Proof of Theorem 2.1.* 1) Since $f \circ g = g \circ f$, then

$$f(z) = f(g(z)) = g(f(z)) \text{ for any } z \in \mathrm{Fix}(g). \tag{2.6}$$

Then $f(\mathrm{Fix}(g)) \subseteq \mathrm{Fix}(g)$ and by Theorem 1.1 it follows that $\mathrm{Fix}(g) \simeq \mathbb{B}^m$ for some $m \in \mathbb{N}$. Hence $f$ has a fixed point in $\mathrm{Fix}(g)$ since, otherwise, we could find a compactly divergent sequence $f^k(z)$, but this is impossible because of Theorem 1.4.

2) Suppose $\mathrm{Fix}(g) \neq \emptyset$. Take $z \in \mathrm{Fix}(g)$. If $\hat{f}$ denotes the restriction of $f$ to $\mathrm{Fix}(g)$, by equation (2.6) it follows that $\hat{f}$ is a well-defined holomorphic self-map of $\mathrm{Fix}(g)$ with no fixed points (and then $\mathrm{Fix}(g)$ has dimension at least one). Let $\tau$ be the Wolff point of $f$. Since the Wolff point of $\hat{f}$ coincides with the Wolff point of $f$, then $\tau \in \partial\mathrm{Fix}(g)$. Now, using a sequence which lies in $\mathrm{Fix}(g)$ and converges to $\tau$, it follows that the boundary dilatation coefficient of $g$ at $\tau$ is less than or equal to 1 and thus, by Theorem 1.8, $K\text{-}\lim_{z\to\tau} g(z) = \tau$. Then the assertion holds because $\tau$ is the Wolff point of $f$ and by Proposition 1.9 it follows that $K\text{-}\lim_{z\to\tau} f(z) = \tau$.

The remaining case, that is when $f$ and $g$ are without fixed points, follows directly from Lemma 2.3. □

## 3. On the Wolff points of commuting maps

In the previous section we proved that two commuting holomorphic maps $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ always have a common fixed point which can lie in $\mathbb{B}^n$, or in $\partial\mathbb{B}^n$ in the sense of $K$-limits. In the case that one of the maps—say $f$—has no fixed points, then in the proof of assertion 2) of Theorem 2.1 we showed that the Wolff point of $f$ has to be a "boundary fixed point" (in the sense of $K$-limits) also for $g$. So if $g$ has no fixed points either, $f$ and $g$ have two boundary fixed points, the Wolff point of $f$ and the Wolff point of $g$, which can even coincide. If the boundary dilatation coefficient of $g$ at its Wolff point is equal to 1, it follows from Lemma 2.3 and Proposition 1.9 that $f$ and $g$ have the same Wolff point. But, what happens in general? Let us examine the following example:

**Example.** Let $F(z) := (\gamma(z_1), 0, \dots, 0)$ and let $G(z) := (\gamma^{-1}(z_1), 0, \dots, 0)$, where $\gamma$ is a hyperbolic automorphism of $\Delta$, *e.g.*

$$\gamma(\xi) := (\cosh t\, \xi + \sinh t) \cdot (\sinh t\, \xi + \cosh t)^{-1} \quad \text{with } t \in \mathbb{R} - \{0\}.$$

Then $F$ and $G$ have no fixed points and commute; furthermore the Wolff point of $F$ is $e_1$ and the Wolff point of $G$ is $-e_1$. The boundary dilatation coefficient of $G$ at $e_1$ is exactly the inverse of its boundary dilatation coefficient at $-e_1$.

So in general, if $n > 1$, we can find commuting holomorphic maps with no fixed points and with different Wolff points, which are not automorphisms of $\mathbb{B}^n$. Nevertheless the previous example gives us an idea of how things work in dimension greater than one. Before going ahead, let us recall the following:

**Lemma 3.1** (Behan [3]). *Let $\eta \in \mathrm{Hol}(\Delta, \Delta)$ be such that $\lim_{r\to 1} \eta(r) = 1$, $\lim_{r\to -1} \eta(r) = -1$. If $d_\eta(x)$ is the boundary dilatation coefficient of $\eta$ at $x \in \partial\Delta$, $d_\eta(x) := \liminf_{\xi\to x} \frac{1-|\eta(\xi)|}{1-|\xi|}$, we have*

$$d_\eta(1) \cdot d_\eta(-1) \geq 1. \tag{3.1}$$

*Moreover the equality holds in* (3.1) *if and only if $\eta$ is a (hyperbolic) automorphism of $\Delta$.*

Now, suppose that $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ have no fixed points and that $f \circ g = g \circ f$. If $f$ and $g$ do not have the same Wolff point, then both $f$ and $g$ have two boundary fixed points and the product of the boundary dilatation coefficients of $f$ (and $g$) at the fixed points is less than or equal to 1 (by Lemma 2.3). So we can hope to use estimate (3.1) and prove that the restriction of $f$ to a suitable subset $\Gamma$ of $\mathbb{B}^n$ (biholomorphic to $\Delta$) is a hyperbolic automorphism of $\Gamma$. Well, that is what we shall do:

**Theorem 3.2.** *If $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ have no fixed points and $f \circ g = g \circ f$, then:*

- *either $f$ and $g$ have the same Wolff point or*
- *there exists $\varphi \in \mathrm{Aut}(\mathbb{B}^n)$ such that, by setting $\tilde f := \varphi^{-1} \circ f \circ \varphi$ and $\tilde g := \varphi^{-1} \circ g \circ \varphi$, it follows that:*
  1. *The maps $z_1 \mapsto \tilde f_1(z_1, 0, \dots, 0)$ and $z_1 \mapsto \tilde g_1(z_1, 0, \dots, 0)$ are two commuting hyperbolic automorphisms of $\Delta$,*
  2. *$\tilde f_2(z_1, 0, \dots, 0) = \dots = \tilde f_n(z_1, 0, \dots, 0) = 0$ and $\tilde g_2(z_1, 0, \dots, 0) = \dots = \tilde g_n(z_1, 0, \dots, 0) = 0$.*

We recall that a *complex geodesic* of $\mathbb{B}^n$ is a injective holomorphic map $\varphi : \Delta \to \mathbb{B}^n$ such that $\varphi(\Delta)$ is a one-dimensional affine subset of $\mathbb{B}^n$. We can now rephrase Theorem 3.2 in the following equivalent way:

**Theorem 3.3.** *Suppose $f, g \in \mathrm{Hol}(\mathbb{B}^n, \mathbb{B}^n)$ have no fixed points and $f \circ g = g \circ f$. Suppose that the Wolff point of $f$ is different from the Wolff point of $g$ and let $\varphi : \Delta \to \mathbb{B}^n$ be the complex geodesic whose closure contains the Wolff points of $f$ and $g$. Then $f(\varphi(\Delta)) = \varphi(\Delta)$, $g(\varphi(\Delta)) = \varphi(\Delta)$ and $f|_{\varphi(\Delta)}$, $g|_{\varphi(\Delta)}$ are commuting hyperbolic automorphisms of $\varphi(\Delta)$.*

*Proof of Theorem 3.2.* Since the group of automorphisms of $\mathbb{B}^n$ acts doubly transitively on $\partial\mathbb{B}^n$, if the Wolff point of $g$ does not coincide with the Wolff point of $f$, then we can suppose that, up to a conjugation in $\mathrm{Aut}(\mathbb{B}^n)$, $e_1$ is the Wolff point of $g$ and $-e_1$ is the Wolff point of $f$. If $L$ is the boundary dilatation coefficient of $g$ at $e_1$ $(0 < L \leq 1)$, then by estimate (2.1) the boundary dilatation coefficient of $g$ at $-e_1$ is less than or equal to $L^{-1}$. Consider the map $\eta : \Delta \to \Delta$ defined by $\eta(\xi) := g_1(\xi, 0, \dots, 0)$. Then $\eta \in \mathrm{Hol}(\Delta, \Delta)$ and since $g$ has $K$-limit $e_1$ at $e_1$ and $-e_1$ at $-e_1$, it follows that $\lim_{r\to 1} \eta(r) = 1$ and $\lim_{r\to -1} \eta(r) = -1$. Moreover, using the notation of Lemma 3.1, $d_\eta(1) \leq L$ and $d_\eta(-1) \leq L^{-1}$ because, by Theorem 1.8:

$$d_\eta(1) = \liminf_{\xi\to 1} \frac{1 - |\eta(\xi)|}{1 - |\xi|} \leq \liminf_{r\to 1} \frac{1 - |g_1(re_1)|}{1 - r} \leq \lim_{r\to 1} \frac{|1 - g_1(re_1)|}{1 - r} = L.$$

The same holds for $d_\eta(-1)$. Therefore $d_\eta(1) \cdot d_\eta(-1) \leq 1$, and by estimate (3.1) we get $d_\eta(1) \cdot d_\eta(-1) = 1$ so that, by Lemma 3.1, $\eta$—and hence $g_1(\xi e_1)$—is a hyperbolic automorphism.

It remains to prove that $g_2(\xi e_1) = \ldots = g_n(\xi e_1) = 0$. To do this, notice that for each $\theta \in [0, 2\pi]$ we have $\liminf_{z\to e^{i\theta} e_1} \frac{1-\|g(z)\|}{1-\|z\|} \leq \liminf_{r\to 1} \frac{1-\left|g_1(re^{i\theta})\right|}{1-r} < +\infty$, since $g_1(\xi e_1)$ is an automorphism. Now Theorem 1.8 implies that for each $\theta$ and for each $j = 2, \dots, n$ we have $\lim_{r\to 1} g_j(re^{i\theta}) = 0$, since $\left|g_1(e^{i\theta})\right| \equiv 1$. Then $g_j(\xi e_1) \equiv 0$ for $j = 2, \dots, n$. Since the same holds for $f$ and $f \circ g = g \circ f$, we get the assertion. □

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