THE GROWTH THEOREM OF CONVEX MAPPINGS ON THE UNIT BALL IN $\mathbb{C}^n$

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Abstract. Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{C}^n$. Let $f$ be a normalized biholomorphic convex mapping on the unit ball in $\mathbb{C}^n$ with respect to the norm $\|\cdot\|$. We will give an upper bound of the growth of $f$.

Let $\Omega$ be a domain in $\mathbb{C}^n$ which contains the origin in $\mathbb{C}^n$. A holomorphic mapping $f$ from $\Omega$ to $\mathbb{C}^n$ is said to be normalized, if $f(0) = 0$ and the Jacobian matrix at the origin $Df(0)$ is the identity matrix. Let $B_n$ denote the Euclidean unit ball in $\mathbb{C}^n$. Let $f(z)$ be a normalized biholomorphic convex mapping on $B_n$. Then FitzGerald and Thomas [2], Liu [6] and Suffridge [7] independently used different methods to prove the following growth theorem.

$$\frac{|z|}{1+|z|} \leq |f(z)| \leq \frac{|z|}{1-|z|},$$

where $|\cdot|$ denotes the Euclidean norm. Let

$$B_p = \left\{ z \in \mathbb{C}^n; \|z\|_p = \left( \sum_{i=1}^{n} |z_i|^p \right)^{1/p} < 1 \right\}$$


$$\|f(z)\|_p \leq \frac{\|z\|_p}{1-\|z\|_p}.$$  

They also obtained an upper bound of the growth of normalized biholomorphic convex mappings on the convex complex ellipsoid

$$D(p_1, \ldots, p_n) = \{ z \in \mathbb{C}^n; |z_1|^{p_1} + \cdots + |z_n|^{p_n} < 1 \}$$

with $p_1, \ldots, p_n \geq 1$.

Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{C}^n$ and let $B$ denote the unit ball in $\mathbb{C}^n$ with respect to the norm $\|\cdot\|$. Using the idea of FitzGerald and Thomas [2], we obtain the following theorem.
Theorem. Let $f(z)$ be a normalized biholomorphic convex mapping from $\mathbb{B}$ to $\mathbb{C}^n$. Then
\[ \|f(z)\| \leq \frac{\|z\|}{1 - \|z\|}. \]

Proof. Let $\Delta$ be the unit disc in $\mathbb{C}$. For any fixed $w \in \partial \mathbb{B}$ and $\zeta \in \Delta$, let
\[ f(\zeta w) = \zeta w + \sum_{i=2}^{\infty} d_i \zeta^i. \]

Since $f$ is a holomorphic mapping into $\mathbb{C}^n$, $d_i \in \mathbb{C}^n$. Let $m \geq 2$, $m \in \mathbb{Z}$ be fixed. Let $\varepsilon = \exp(2\pi i/m)$. Then
\[ m - 1 \sum_{k=0}^{\infty} f(\zeta^{1/m} \varepsilon^k w) = m \sum_{i=1}^{\infty} d_m \zeta^i \]
is holomorphic with respect to $\zeta \in \Delta$. Since $f(\mathbb{B})$ is convex,
\[ h(\zeta) = f^{-1}\left(1/m \sum_{k=0}^{\infty} f(\zeta^{1/m} \varepsilon^k w)\right) \]
is well-defined and holomorphic on $\Delta$. Since $f$ is normalized,
\[ f^{-1}(z) = z + O(|z|^2). \]
Therefore, $h(\zeta) = d_m \zeta + O(|\zeta|^2)$. Since $h(\Delta) \subset \mathbb{B}$, we obtain $\|d_m\| \leq (1 - \delta)^{-1}$ by applying the maximum modulus theorem with values in a complex Banach space (cf. Dunford and Schwartz [1]) to the holomorphic mapping $h(\zeta)/\zeta$ on $|\zeta| < 1 - \delta$. Letting $\delta$ tend to 0, we have $\|d_m\| \leq 1$. Then we have
\[ \|f(\zeta w)\| \leq |\zeta| + \sum_{i=2}^{\infty} |\zeta|^i = \frac{|\zeta|}{1 - |\zeta|} = \frac{\|\zeta w\|}{1 - \|\zeta w\|}. \]

Let $D$ be a bounded convex balanced domain in $\mathbb{C}^n$. Then the Minkowski function of $D$ is a norm on $\mathbb{C}^n$ and $D$ is the unit ball with respect to the norm (cf. Jarnicki and Pflug [5]). Then the above theorem holds for $D$. In particular, the theorem gives another growth theorem of convex mappings on $D(p_1, \ldots, p_n)$ with $p_1, \ldots, p_n \geq 1$.

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References