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THE GROWTH THEOREM OF CONVEX MAPPINGS ON THE UNIT BALL IN $\mathbb{C}^n$

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ABSTRACT. Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{C}^n$. Let f be a normalized biholomorphic convex mapping on the unit ball in $\mathbb{C}^n$ with respect to the norm $\|\cdot\|$. We will give an upper bound of the growth of f.

Let Ω be a domain in $\mathbb{C}^n$ which contains the origin in $\mathbb{C}^n$. A holomorphic mapping f from Ω to $\mathbb{C}^n$ is said to be normalized, if $f(0) = 0$ and the Jacobian matrix at the origin $Df(0)$ is the identity matrix. Let $\mathbb{B}^n$ denote the Euclildean unit ball in $\mathbb{C}^n$. Let $f(z)$ be a normalized biholomorphic convex mapping on $\mathbb{B}^n$. Then FitzGerald and Thomas [2], Liu [6] and Suffridge [7] independently used different methods to prove the following growth theorem.

$$\frac{|z|}{1+|z|} \leq |f(z)| \leq \frac{|z|}{1-|z|},$$

where $|\cdot|$ denotes the Euclidean norm. Let

$$B_p = \left\{ z \in \mathbb{C}^n; \|z\|_p = \left(\sum_{i=1}^{n} |z_i|^p \right)^{1/p} < 1 \right\}$$

for $p \geq 1$. Gong and Liu [4] gave the following upper bound of the growth of normalized biholomorphic convex mappings on B_p.

$$\|f(z)\|_p \leq \frac{\|z\|_p}{1-\|z\|_p}.$$

They also obtained an upper bound of the growth of normalized biholomorphic convex mappings on the convex complex ellipsoid

$$D(p_1, \dots, p_n) = \{z \in \mathbb{C}^n; |z_1|^{p_1} + \dots + |z_n|^{p_n} < 1\}$$

with $p_1, \dots, p_n \geq 1$.

Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{C}^n$ and let $\mathbb{B}$ denote the unit ball in $\mathbb{C}^n$ with respect to the norm $\|\cdot\|$. Using the idea of FitzGerald and Thomas [2], we obtain the following theorem.

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Theorem. *Let $f(z)$ be a normalized biholomorphic convex mapping from $\mathbb{B}$ to $\mathbb{C}^n$. Then*

$$\|f(z)\| \le \frac{\|z\|}{1-\|z\|}.$$

Proof. Let Δ be the unit disc in $\mathbb{C}$. For any fixed $w \in \partial\mathbb{B}$ and $\zeta \in \Delta$, let

$$f(\zeta w) = \zeta w + \sum_{i=2}^{\infty} d_i \zeta^i.$$

Since f is a holomorphic mapping into $\mathbb{C}^n$, $d_i \in \mathbb{C}^n$. Let $m \ge 2$, $m \in \mathbb{Z}$ be fixed. Let $\varepsilon = \exp(2\pi i/m)$. Then

$$\sum_{k=0}^{m-1} f(\zeta^{1/m}\varepsilon^k w) = m \sum_{i=1}^{\infty} d_{mi}\zeta^i$$

is holomorphic with respect to $\zeta \in \Delta$. Since $f(\mathbb{B})$ is convex,

$$h(\zeta) = f^{-1}\left(\frac{1}{m}\sum_{k=0}^{m-1} f(\zeta^{1/m}\varepsilon^k w)\right)$$

is well-defined and holomorphic on Δ. Since f is normalized,

$$f^{-1}(z) = z + O(|z|^2).$$

Therefore, $h(\zeta) = d_m\zeta + O(|\zeta|^2)$. Since $h(\Delta) \subset \mathbb{B}$, we obtain $\|d_m\| \le (1-\delta)^{-1}$ by applying the maximum modulus theorem with values in a complex Banach space (cf. Dunford and Schwartz [1]) to the holomorphic mapping $h(\zeta)/\zeta$ on $|\zeta| < 1-\delta$. Letting δ tend to 0, we have $\|d_m\| \le 1$. Then we have

$$\|f(\zeta w)\| \le |\zeta| + \sum_{i=2}^{\infty} |\zeta|^i = \frac{|\zeta|}{1-|\zeta|} = \frac{\|\zeta w\|}{1-\|\zeta w\|}.$$

$\square$

Let D be a bounded convex balanced domain in $\mathbb{C}^n$. Then the Minkowski function of D is a norm on $\mathbb{C}^n$ and D is the unit ball with respect to the norm (cf. Jarnicki and Pflug [5]). Then the above theorem holds for D. In particular, the theorem gives another growth theorem of convex mappings on $D(p_1, \dots, p_n)$ with $p_1, \dots, p_n \ge 1$.

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