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FIXED POINT ITERATION FOR PSEUDOCONTRACTIVE MAPS

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Abstract. Let K be a compact convex subset of a real Hilbert space, H; $T : K \to K$ a continuous pseudocontractive map. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}$ and $\{c_n^{'}\}$ be real sequences in [0,1] satisfying appropriate conditions. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}_{n=1}^{\infty}$ iteratively by $x_{n+1} = a_n x_n + b_n T y_n + c_n u_n; y_n = a_n^{'} x_n + b_n^{'} T x_n + c_n^{'} v_n,\ n \geq 1$, where $\{u_n\}, \{v_n\}$ are arbitrary sequences in K. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T. A related result deals with the convergence of $\{x_n\}_{n=1}^{\infty}$ to a fixed point of T when T is Lipschitz and pseudocontractive. Our theorems also hold for the slightly more general class of *continuous hemicontractive* nonlinear maps.

1. Introduction

Let H be a Hilbert space. A mapping $T : H \to H$ is said to be *pseudocontractive* (see e.g., [1], [2]) if

$$||Tx - Ty||^2 \leq ||x - y||^2 + ||(I - T)x - (I - T)y||^2, \forall x, y \in H \tag{1}$$

and is said to be *strongly pseudocontractive* if there exists $k \in (0, 1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + k||(I - T)x - (I - T)y||^2, \forall x, y \in H. \tag{2}$$

Let $F(T) := \{x \in H : Tx = x\}$ and let K be a nonempty subset of H. A map $T : K \to K$ is called *hemicontractive* if $F(T) \neq \emptyset$ and

$$||Tx - x^*||^2 \leq ||x - x^*||^2 + ||x - Tx||^2\ \forall\ x \in H, x^* \in F(T).$$

It is easy to see that the class of pseudocontractive maps with fixed points is a subclass of the class of hemicontractions. The following example, due to Rhoades [25], shows that the inclusion is proper. For $x \in [0, 1]$, define $T : [0, 1] \to [0, 1]$ by $Tx = (1 - x^{\frac{2}{3}})^{\frac{3}{2}}$. It is shown in [25] that T is not Lipschitz and so cannot be nonexpansive. A straightforward computation (see e.g., [28]) shows that T is pseudocontractive. For the importance of fixed points of pseudocontractions the reader may consult [1].

In the last ten years or so, numerous papers have been published on the iterative approximation of fixed points of Lipschitz *strongly* pseudocontractive (and correspondingly Lipschitz *strongly* accretive) maps using the *Mann iteration process* (see e.g., [17]). Results which had been known only in *Hilbert spaces* and only for *Lipschitz maps* have been extended to more general Banach spaces (see e.g., [3]–[6], [7], [8], [9], [10], [11], [12], [13], [16], [21]–[24], [25], [26]–[28], [29], [30], [31], [32] and the

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references cited therein) and to more general classes of maps (see e.g., [4]–[6], [7], [8], [9], [10], [11], [12], [13], [14], [16], [18]–[20], [21]–[24], [26]–[28], [29], [30], [31], [32] and the references cited therein). This success, however, has not carried over to arbitrary *Lipschitz pseudocontraction* T even when the domain of the operator T is a *a compact convex subset of a Hilbert space*. In fact, it is still an open question whether or not the Mann iteration process converges under this setting. In 1974, Ishikawa introduced an iteration process which, in some sense, is more general than that of Mann and which converges, under this setting, to a fixed point of T. He proved the following theorem.

Theorem I (Ishikawa [15])**.** *If K is a compact convex subset of a Hilbert space H, $T : K \mapsto K$ is a Lipschitzian pseudocontractive map and x_0 is any point in K, then the sequence $\{x_n\}_{n\geq 0}$ converges strongly to a fixed point of T, where x_n is defined iteratively for each positive integer $n \geq 0$ by*

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T[(1-\beta_n)x_n + \beta_n T x_n] \tag{3}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying the conditions

$$(i)\ 0 \leq \alpha_n \leq \beta_n < 1;\ \ (ii)\ \lim_{n\to\infty} \beta_n = 0;\ \ (iii)\ \sum_{n\geq 0} \alpha_n\beta_n = \infty.$$

Since its publication in 1974, Theorem I, as far as we know, has never been extended to more general Banach spaces. In [18], Qihou extended the theorem to the slightly more general class of Lipschitz *hemicontractions* and in [19] he proved, under the setting of Theorem I, that the convergence of the recursion formula (3) to a fixed point of T when T is a continuous hemicontractive map, *under the additional hypothesis that the number of fixed points of T is finite*. The iteration process (3) is generally referred to as the *Ishikawa iteration process* in light of [15]. Another iteration process which has been studied extensively in connection with fixed points of pseudocontractive maps is the following: For K a convex subset of a Banach space E, and $T : K \to K$, the sequence $\{x_n\}_{n=1}^{\infty}$ is defined iteratively by $x_1 \in K$,

$$x_{n+1} = (1-c_n)x_n + c_n T x_n, n \geq 1, \tag{4}$$

where $\{c_n\}$ is a real sequence satisfying the following conditions: (i) $0 \leq c_n < 1; (ii)\ \lim c_n = 0; (iii)\ \sum_{n=1}^{\infty} c_n = \infty$. The iteration process (4) is generally referred to as the *Mann iteration process* in light of [17].

In 1995, Liu [16] introduced what he called *Ishikawa and Mann iteration processes with errors* as follows:

(a) For K a nonempty subset of E and $T : K \to E$, the sequence $\{x_n\}$ defined by $x_1 \in K$, $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T y_n + u_n$; $y_n = (1-\beta_n)x_n + \beta_n T x_n + v_n, n \geq 1$ where, $\{\alpha_n\}, \{\beta_n\}$ are sequences in [0,1] satisfying appropriate conditions and $\sum ||u_n|| < \infty$, $\sum ||v_n|| < \infty$ is called the *Ishikawa Iteration process with errors*.

(b) With K, E and T as in part (a), the sequence $\{x_n\}$ defined by $x_1 \in K$, $x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n + u_n, n \geq 1$ where $\{\alpha_n\}$ is a sequence in [0,1] satisfying appropriate conditions and $\sum ||u_n|| < \infty$, is called the *Mann iteration process with errors.*

While it is known that consideration of error terms in iterative processes is an important part of the theory, it is also clear that the iteration processes with errors introduced by Liu in (a) and (b) are unsatisfactory. The occurrence of errors is random so that the conditions imposed on the error terms in (a) and (b) which imply, in particular, that they tend to zero as n tends to infinity are, therefore, unreasonable. Recently, Yuguang Xu [30] introduced the following more satisfactory definitions.

[A] Let K be a nonempty convex subset of E and $T : K \to K$ a mapping. For any given $x_1 \in K$, the sequence $\{x_n\}_{n=1}^{\infty}$ defined iteratively by

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n ; y_n = a_n^{'} x_n + b_n^{'} T x_n + c_n^{'} v_n , n \geq 1$$

where $\{u_n\}, \{v_n\}$ are bounded sequences in K and $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}$ and $\{c_n^{'}\}$ are sequences in [0,1] such that $a_n+b_n+c_n = a_n^{'}+b_n^{'}+c_n^{'} = 1 \; \forall \, n \geq 1$ is called the *Ishikawa iteration sequence with errors*.

[B] If, with the same notations and definitions as in [A], $b_n^{'} = c_n^{'} = 0$, for all integers $n \geq 1$, then the sequence $\{x_n\}_{n=1}^{\infty}$ now defined by $x_1 \in K, x_{n+1} = a_n x_n + b_n T x_n + c_n u_n, n \geq 1$, is called the *Mann iteration sequence with errors*. We remark that if K is bounded (as is generally the case), the error terms u_n, v_n are *arbitrary* in K.

It is our purpose in this paper to extend Theorem I to the *Ishikawa iteration process with errors in the sense of* [A] and to the slightly more general class of Lipschitz *hemicontractions*. Our theorem will include Theorem 2 of Qihou [18] as a special case. Furthermore, we shall prove a theorem similar to Theorem I for *continuous hemicontractions*.

2. Preliminaries

We shall make use of the following results.

Lemma 1 ([29]). *Suppose that $\{\rho_n\}, \{\sigma_n\}$ are two sequences of nonnegative numbers such that for some real number $N_0 \geq 1$, $\rho_{n+1} \leq \rho_n + \sigma_n \; \forall n \geq N_0$.*

(*a*) *If $\sum \sigma_n < \infty$, then, $\lim \rho_n$ exists.*
(*b*) *If $\sum \sigma_n < \infty$ and $\{\rho_n\}$ has a subsequence converging to zero, then $\lim \rho_n = 0$.*

We shall also use the following well-known identity for Hilbert spaces, H:

$$\begin{aligned} ||(1-\lambda)x + \lambda y||^2 &= (1-\lambda)||x||^2 + \lambda||y||^2 \\ &\quad - \lambda(1-\lambda)||x-y||^2 \; \forall x, y \in H, \lambda \in [0,1]. \end{aligned} \tag{5}$$

3. Main theorems

We prove the following theorems.

Theorem 1. *Let K be a compact convex subset of a real Hilbert space, H; $T : K \to K$ a continuous hemicontractive map. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}$ and $\{c_n^{'}\}$ be real sequences in $[0,1]$ satisfying the following conditions:*

(*i*) $a_n + b_n + c_n = a_n^{'} + b_n^{'} + c_n^{'} = 1 \; \forall \, n \geq 1$;
(*ii*) $\lim b_n = \lim b_n^{'} = 0$;
(*iii*) $\sum c_n < \infty ; \sum c_n^{'} < \infty$;
(*iv*) $\sum \alpha_n \beta_n = \infty ; \sum \alpha_n \beta_n \delta_n < \infty$, *where* $\delta_n := ||Tx_n - Ty_n||^2$;
(*v*) $0 \leq \alpha_n \leq \beta_n < 1 \; \forall \, n \geq 1$, *where* $\alpha_n := b_n + c_n ; \beta_n := b_n^{'} + c_n^{'}$.

For arbitrary $x_1 \in K$, define the sequence $\{x_n\}_{n=1}^{\infty}$ iteratively by

$$x_{n+1} = a_n x_n + b_n T y_n + c_n u_n ; y_n = a_n^{'} x_n + b_n^{'} T x_n + c_n^{'} v_n , n \geq 1,$$

where $\{u_n\}, \{v_n\}$ *are arbitrary sequences in* K. *Then,* $\{x_n\}_{n=1}^{\infty}$ *converges strongly to a fixed point of* T.

Proof. The existence of a fixed point of T follows from Schauder's fixed point theorem. So $F(T) \neq \emptyset$. Let $x^* \in K$ be a fixed point of T. Using the identity (5), we obtain the following estimates: For some constants $M_1 \geq 0, M_2 \geq 0$,

$$\begin{aligned} \|y_n - x^*\|^2 &= \|(1-\beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*) - c_n'(Tx_n - v_n)\|^2 \\ &\leq (\|(1-\beta_n)(x_n - x^*) + \beta_n(Tx_n - x^*)\| + c_n'\|Tx_n - v_n\|)^2 \\ &\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|Tx_n - x^*\|^2 \\ &\quad -\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 + c_n' M_1 \end{aligned} \tag{6}$$

$$\begin{aligned} \|y_n - Ty_n\|^2 &= \|(1-\beta_n)(x_n - Ty_n) + \beta_n(Tx_n - Ty_n) - c_n'(Tx_n - v_n)\|^2 \\ &\leq (1-\beta_n)\|x_n - Ty_n\|^2 + \beta_n\|Tx_n - Ty_n\|^2 \\ &\quad -\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 + c_n' M_2 \end{aligned} \tag{7}$$

so using the fact that T is hemicontractive with the estimates above, we obtain,

$$\begin{aligned} \|Tx_n - x^*\|^2 &\leq \|x_n - x^*\|^2 + \|x_n - Tx_n\|^2 \\ \|Ty_n - x^*\|^2 &\leq \|y_n - x^*\|^2 + \|y_n - Ty_n\|^2 \\ &\leq (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|Tx_n - x^*\|^2 \\ &\quad -\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 + (1-\beta_n)\|x_n - Ty_n\|^2 \\ &\quad +\beta_n\|Tx_n - Ty_n\|^2 - \beta_n(1-\beta_n)\|x_n - Tx_n\|^2 + M_3 c_n' \\ &\leq \|x_n - x^*\|^2 - \beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 \\ &\quad +(1-\beta_n)\|x_n - Ty_n\|^2 + \beta_n\|Tx_n - Ty_n\|^2 + M_3 c_n' \end{aligned}$$

where $M_3 = M_1 + M_2$. Therefore, for some constant $M_4 \geq 0$,

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|Ty_n - x^*\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - Ty_n\|^2 + c_n M_4 \\ &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n[\|x_n - x^*\|^2 \\ &\quad - \beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 + (1-\beta_n)\|x_n - Ty_n\|^2 \\ &\qquad\qquad +\beta_n\|Tx_n - Ty_n\|^2 + M_3 c_n'] \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - Ty_n\|^2 + c_n M_4 \\ &= \|x_n - x^*\|^2 - \alpha_n\beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 \\ &\quad + \alpha_n\beta_n\|Tx_n - Ty_n\|^2 - \alpha_n(\beta_n - \alpha_n)\|x_n - Ty_n\|^2 \\ &\quad + M_5(c_n + c_n') \end{aligned}$$

where $M_5 = \max\{M_3, M_4\}$, since $\alpha_n \leq 1$. Thus, since $\beta_n \geq \alpha_n$, we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - \alpha_n\beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 \\ &\quad +\alpha_n\beta_n\|Tx_n - Ty_n\|^2 + M_5(c_n + c_n'). \end{aligned} \tag{8}$$

Since K is compact and T is continuous, $\{\|x_n - Tx_n\|\}$ is a bounded sequence. Let $\liminf_{n\to\infty}\|x_n - Tx_n\| = \delta \geq 0$.

Claim. $\delta = 0$.

Suppose the claim is false, that is, $\delta > 0$. Then, there exists an integer $N_1 > 0$ such that $\|x_n - Tx_n\| \geq \frac{\delta}{2}$ $\forall n \geq N_1$. Observe that $\|x_n - y_n\| = \beta_n\|x_n - Tx_n\| +$

$c_n^{'}||Tx_n - v_n|| \leq \mathrm{diam}(K)(\beta_n + c_n^{'}) \to 0 \mathrm{as} n \to \infty$. Compactness of K and continuity of T imply that $\|Tx_n - Ty_n\| \to 0$ as $n \to \infty$. Thus, there exists an integer $N_2 > 0$ such that $\|Tx_n - Ty_n\| \leq \frac{\delta}{6}\ \forall n \geq N_2$. By conditions (ii) and (iii), there exists an integer $N_3 > 0$ such that $\beta_n \leq \frac{1}{3}\ \forall n \geq N_3$. Let $N = \max\{N_1, N_2, N_3\}$. Then, $\forall n \geq N$, inequality (8) now yields

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - \alpha_n\beta_n(1-2\beta_n)\frac{\delta^2}{4} + \alpha_n\beta_n\frac{\delta^2}{36} + M_5(c_n + c_n^{'}) \\ &\leq \|x_n - x^*\|^2 - \frac{\delta^2}{18}\alpha_n\beta_n + M_5(c_n + c_n^{'}); \end{aligned} \tag{9}$$

$$\Rightarrow \lambda\alpha_n\beta_n \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M_5(c_n + c_n^{'});\ \lambda = \frac{\delta^2}{18} \tag{10}$$

$$\Rightarrow \lambda\sum_{j=N}^{n}\alpha_j\beta_j \leq \|x_N - x^*\|^2 - \|x_{n+1} - x^*\|^2 + M\sum_{j=N}^{n}(c_j + c_j^{'}).$$

Since the right hand side is finite, this implies that $\sum_{j\geq N}\alpha_j\beta_j < \infty$ which contradicts hypothesis (iv). Hence, $\liminf_{n\to\infty}\|x_n - Tx_n\| = 0$ and the claim is established. By compactness of K this immediately implies that there is a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to a fixed point of T, say x^*.

(For the rest of the argument, x^* refers to this fixed point.) Let

$$\Psi_n = \|x_n - x^*\|^2;\quad \sigma_n = \alpha_n\beta_n\|Tx_n - Ty_n\|^2 + M_5(c_n + c_n^{'})$$

and observe that $\Psi_n \geq 0,\ \sigma_n \geq 0,\ \forall n \geq 1$ and $\sum_{n\geq 1}\sigma_n < \infty$ by conditions (iii) and (iv). Then, inequality (8) yields

$$\Psi_{n+1} \leq \Psi_n + \sigma_n,\quad \forall n \geq 0. \tag{11}$$

It now follows from Lemma 1 that $\Psi_n \to 0$ as $n \to \infty$, i.e., $x_n \to x^*$ as $n \to \infty$. The proof is complete. □

Corollary 1. *Let K be a compact convex subset of a real Hilbert space H, $T : K \to K$ a Lipschitz hemicontractive map. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}$ and $\{c_n^{'}\}$ satisfy conditions (i)-(iii) of Theorem 1. Let (iv) $\sum\alpha_n\beta_n = \infty$, (v) $0 \leq \alpha_n \leq \beta_n < 1\ \forall n \geq 1$, where α_n, β_n are as defined in Theorem 1. For arbitrary $x_1 \in K$, define the sequence $\{x_n\}_{n=1}^{\infty}$ iteratively by*

$$x_{n+1} = a_nx_n + b_nTy_n + c_nu_n; y_n = a_n^{'}x_n + b_n^{'}Tx_n + c_n^{'}v_n,$$

where $\{u_n\}, \{v_n\}$ are arbitrary sequences in K. Then, $\{x_n\}_{n=1}^{\infty}$ converges strongly to a fixed point of T.

Proof. Since T is hemicontractive, $F(T) \neq \emptyset$. Let $x^* \in F(T)$. As in the proof of Theorem 1, we obtain inequality (8). Let $L > 0$ denote the Lipschitz constant of T. Then,

$$||Tx_n - Ty_n|| \leq L||y_n - x_n|| \leq L\{\beta_n||x_n - Tx_n|| + c_n^{'}M_6\}$$

for some constant $M_6 \geq 0$. Thus, for some constant $M_7 \geq 0$,

$$||Tx_n - Ty_n||^2 \leq L^2\beta_n^2||x_n - Tx_n||^2 + c_n^{'}M_7$$

and so inequality (8) now yields, for some constant $M_8 \geq 0$,

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 - \alpha_n\beta_n(1 - 2\beta_n - L^2\beta_n^2)||x_n - Tx_n||^2 + c_n^{'}M_8.$$

Since $\lim \beta_n = 0$, there exists an integer $N_8 \geq 1$ such that $2\beta_n + L^2\beta_n^2 \leq \frac{1}{2}$ so that

$$||x_{n+1} - x^*||^2 \leq ||x_n - x^*||^2 - \frac{1}{2}\alpha_n\beta_n||x_n - Tx_n||^2 + c_n^{'}M_8.$$

The rest of the argument now follows as in the proof of Theorem 1 to yield that $x_n \to x^*$ as $n \to \infty$. The proof is complete. □

Remark 1. Corollary 1 is an extension of Theorem I to the more general *Ishikawa iteration sequence with errors in terms of* [A] and to the slightly more general class of nonlinear Lipschitz *hemicontractions*. The corollary also extends Theorem 2 of Qihou [18] to the more general iterative scheme with errors.

Corollary 2. *Let* $K, H, \{a_n\}, \{b_n\}, \{c_n\}, \{a_n^{'}\}, \{b_n^{'}\}, \{c_n^{'}\}, \alpha_n, \beta_n$ *be as in Theorem 1. Let* $T : K \to H$ *be a continuous hemicontractive map. Let* $P_K : H \to K$ *be the projection operator of* H *onto* K*. Then, the sequence* $\{x_n\}_{n=1}^{\infty}$ *defined iteratively by*

$$x_{n+1} = P_K z_n;\ z_n = a_n x_n + b_n Ty_n + c_n u_n;$$

$$y_n = P_K \omega_n;\ \omega_n = a_n^{'}x_n + b_n^{'}Tx_n + c_n^{'}v_n,\ n \geq 1$$

where $\{u_n\}, \{v_n\}$ *are arbitrary sequences in* K*, converges strongly to a fixed point of* T*.*

Proof. The operator P_K is nonexpansive (see e.g., [2]). K is a Chebyshev subset of H so that, P_K is a single-valued map. Hence, we have the following estimates: For some constants $M_9 \geq 0, M_{10} \geq 0$,

$$\begin{aligned} \|y_n - x^*\|^2 &= \|P_K w_n - P_K x^*\|^2 \leq \|w_n - x^*\|^2 \\ &= (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|Tx_n - x^*\|^2 \\ &\quad -\beta_n(1-\beta_n)\|x_n - Tx_n\|^2 + c_n^{'}M_9, \end{aligned}$$

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|P_K z_n - P_K x^*\|^2 \\ &\leq \|z_n - x^*\|^2 \\ &= \|x_n - x^*\|^2 - \alpha_n\beta_n(1-2\beta_n)\|x_n - Tx_n\|^2 \\ &\quad +\alpha_n\beta_n\|Tx_n - Ty_n\|^2 + M_{10}(c_n + c_n^{'}). \end{aligned}$$

The set $K \cup T(K)$ is compact and so the sequence $\{\|x_n - Tx_n\|\}$ is bounded. The rest of the argument follows exactly as in the proof of Theorem 1 and the proof is complete. □

Remarks. 1. Conditions similar to our condition (iv) have been imposed in the literature. Reich [22] imposed the condition $\sum_{n=0}^{\infty} c_n^2||Tx_n||^2 < \infty$ (where c_n is a real sequence in $(0,1)$ satisfying appropriate conditions) to prove the convergence of the *Mann iteration process* to the solution of an operator equation involving strongly accretive operator T defined on a *uniformly smooth Banach space.*

2. A prototype for our parameters is

$$a_n = 1 - \frac{1}{\sqrt{(n+1)}} = a_n^{'};\ b_n = \frac{1}{\sqrt{(n+1)}} - \frac{1}{(n+1)^2} = b_n^{'};$$

$$c_n = \frac{1}{(n+1)^2} = c_n^{'}$$

for all integers $n \geq 1$. Observe that if T is Lipschitz, then

$$\begin{aligned}\alpha_n\beta_n\delta_n &= \sum \frac{1}{(n+1)}\delta_n \leq L^2 \sum \frac{1}{(n+1)}(\beta_n + c_n^{'})[\mathrm{diam}(K)]^2 \\ &= L^2 \sum \frac{1}{(n+1)}\left(\frac{1}{\sqrt{(n+1)}} + \frac{1}{(n+1)^2}\right)[\mathrm{diam}(K)]^2 < \infty.\end{aligned}$$

3. Extensions of our theorems to *set-valued maps* are basically mere repetitions of our arguments (once a single-valued selection has been made) and are therefore omitted.

In connection with the iterative approximation of fixed points of pseudocontractions, the following question is still open.

Question. Does the Mann iteration process always converge for continuous pseudocontractions, or for even Lipschitz pseudocontractions?

Let E be a Banach space and K be a nonempty compact convex subset of E. Let $T : K \to K$ be a Lipschitz pseudocontractive map. Under this setting, even for $E = H$, a Hilbert space, the answer to the above question is not known. There is, however, an example of a *discontinuous* pseudocontractive map T with a unique fixed point for which the Mann iteration process does not always converge to the fixed point of T.

Example (Hicks and Kubicek [14])**.** Let H be the complex plane and $K := \{z \in H : |z| \leq 1\}$. Define $T : K \to K$ by

$$T(re^{i\theta}) = \begin{cases} 2re^{i(\theta+\frac{\pi}{3})}, & \text{for } 0 \leq r \leq \frac{1}{2}, \\ e^{i(\theta+\frac{2\pi}{3})}, & \text{for } \frac{1}{2} < r \leq 1. \end{cases}$$

Then, zero is the only fixed point of T. It is shown in [12] that T is pseudocontractive and that with $c_n = \frac{1}{n+1}$, the sequence $\{z_n\}$ defined by$z_{n+1} = (1 - c_n)z_n + c_nTz_n, z_0 \in K, n \geq 1$, does not converge to zero. Since the T in this example is not continuous, the above question remains open.

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