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A SPACE ON WHICH DIAMETER-TYPE PACKING MEASURE IS NOT BOREL REGULAR

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ABSTRACT. We construct a separable metric space on which 1-dimensional diameter-type packing measure is not Borel regular.

1. INTRODUCTION

In [11] Taylor and Tricot introduced a new measure on $\mathbf{R}^n$, which they named *packing measure*. This measure was intended as a type of dual to Hausdorff measure, where the idea of economical coverings by sets of small diameter was replaced by that of extravagant packings by balls of small diameter.

For clarity in what follows we now provide a definition of what we shall refer to as *diameter-type packing measure*. Our notation differs from that of [11], and we do not restrict our attention to $\mathbf{R}^n$. We also follow recent practice (see [3, 4, 5, 6], and [9, 5.10]) in using closed balls rather than open in our definition of packing measure.

By a *packing* of a subset S of a metric space X we mean a finite or countable collection of closed balls $\{B(x_i, r_i) : x_i \in S\}$ such that for each $i \neq j$,

$$B(x_i, r_i) \cap B(x_j, r_j) = \emptyset.$$

A *δ-packing* is a packing such that for each i, diam $B(x_i, r_i) \leq \delta$.

If h is a *Hausdorff function*, that is, a non-decreasing function from $\mathbf{R}^+$ to $\mathbf{R}^+$ with $h(0+) = 0$, then $\mathcal{P}^h(S)$, the *diameter-type h-packing measure of S*, may be defined thus:

$$\begin{aligned} P^h_\delta(S) &= \sup\left\{\sum h(\text{diam } B(x_i, r_i)) : \{B(x_i, r_i)\} \text{ a } \delta\text{-packing of } S\right\}, \\ P^h_0(S) &= \lim_{\delta\to 0} P^h_\delta(S), \\ \mathcal{P}^h(S) &= \inf\left\{\sum_1^\infty P^h_0(S_i) : S \subset \bigcup_1^\infty S_i\right\}. \end{aligned}$$

In [4] it was noted that this definition led to a problem, namely that one could not be sure that packing measure, thus defined, was Borel regular, that is, that

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every set was contained inside a Borel set of equal measure. A possible solution proposed there was to modify this definition by replacing the final line by

$$\mathcal{P}^h(S) = \inf \left\{ \sum_1^\infty P_0^h(S_i) : S \subset \bigcup_1^\infty S_i, \text{ and } S_i \text{ Borel} \right\}.$$

Haase also introduced *radius-type packing measure* (see [2]), where he replaced $\operatorname{diam} B(x_i, r_i)$ by $2r_i$ throughout Taylor and Tricot's definition. We write $R_\delta^h(S)$, $R_0^h(S)$ and $\mathcal{R}^h(S)$ for the radius-type analogues of $P_\delta^h(S)$, $P_0^h(S)$ and $\mathcal{P}^h(S)$, and $\mathcal{P}^s$ and $\mathcal{R}^s$ for the measures resulting from the Hausdorff functions $h(r) = r^s$.

Radius-type packing measures are now often used. However, Haase's suggested modification to the definition of diameter-type packing measure has not entered common use. This may be because the problem of non-Borel regularity does not arise on Euclidean spaces, provided h is left-continuous (see Theorem 1.2 below). Also, a useful (and immediate) fact about packing measures is that $\mathcal{P}^h(S) \le P_0^h(S)$ for every set S; if we modify packing measure as suggested above, then there is no longer any reason why this inequality should hold.

In this paper we give a construction which shows that if closed balls are used instead of open in the definition of diameter-type packing measures, then some such modification is indeed needed in order to ensure Borel regularity on non-Euclidean spaces.

The added step in the definition of packing measures as compared to Hausdorff measures makes our result seem perhaps a little less surprising. For example, in [10] the question of the measurability of the packing and Hausdorff measure and dimension functions is investigated. It is shown there that while the Hausdorff dimension and measure functions are indeed measurable with respect to the Borel field generated by the compact subsets of a separable metric space, the packing measure and dimension functions are not.

Lemma 1.1. *Let X be a metric space and let $S \subseteq X$. If h is a left-continuous Hausdorff function, then*

$$R_0^h(S) = R_0^h(\text{Clos } S).$$

If also $X = \mathbf{R}^n$, then

$$P_0^h(S) = P_0^h(\text{Clos } S).$$

The proof is easy.

Theorem 1.2. *Let X be a metric space and let $S \subseteq X$. If h is a left-continuous Hausdorff function, then $\mathcal{R}^h$ is a Borel regular measure. If also $X = \mathbf{R}^n$, then $\mathcal{P}^h$ is a Borel regular measure.*

Proof. Lemma 1.1 ensures that, under these assumptions, each set is contained in an $F_{\sigma\delta}$ set of the same measure (see for example [9, 5.10]). □

However, the statement that the pre-measure P_0^h of each set is the same as that of its closure is not necessarily true, even for h continuous, on spaces other than $\mathbf{R}^n$; an example was provided in [7, 1.4.4]. This note takes the idea of that example, and extends it to provide a construction of a separable metric space with a non-Borel subset of $\mathcal{P}^1$ measure zero, such that every Borel subset of the metric space containing that set has infinite $\mathcal{P}^1$ measure. In this construction, the highly discrete metric will allow us to control the number of disjoint balls which may be centred in certain subsets of our metric space.

2. The construction

Our metric space X is constructed as follows.

First we construct a code space Y with a discrete metric. The measure μ on Y, which is the image of s-dimensional Hausdorff measure on the middle third Cantor set C ($s = \log 2/\log 3$) under the homeomorphism via the natural coding of C, will play an auxiliary role.

In Lemma 2.1 we find a non-Borel subset S of Y. We then take the next step in building up our metric space X by adding to Y countably many copies Z^k of $Y \setminus S$ and suitably extending our metric. Our aim here is to extend the metric space Y to a space where the 1-dimensional packing measure of S is zero, but that of any Borel set containing S is infinite. The role of these sets Z^k is to ensure that balls of certain radii centred in $Y \setminus S$ have larger diameter than balls of the same radii centred in S itself. The final step in constructing X will then be to add families of sequences to our metric space and to extend our metric once more. The purpose of these sequences is to ensure that there are not too many disjoint balls of certain diameters centred in S.

In Lemma 2.2 we see that if a subset V of $Y \setminus S$ has positive μ-measure, then it must satisfy $P_0^1(V) = \infty$. In Lemma 2.1 it was established that if B is a Borel subset of Y containing S, and $B \subseteq \bigcup_i B_i$, then one of the B_i's must contain such a set V. In Lemmas 2.3 and 2.4 we see that a Borel subset B of X which contains S must have infinite $\mathcal{P}^1$ measure, and that S itself has zero $\mathcal{P}^1$ measure. These two lemmas lead directly to our main result, which is Theorem 2.5.

As the first step in our construction, we choose a positive sequence $(d_j)_j$ with $d_{j+1} \le d_j/2$, and an increasing sequence $(m_j)_j$ of integers, and write $n_j = 2^{m_j}$, $N_j = \prod_{i=1}^j n_i$, and $M_j = \sum_{i=1}^j m_i$ (so $N_j = 2^{M_j}$), such that

$$\sum_{i=1}^{\infty} N_k d_{k+1} < \infty, \tag{1}$$

$$\lim_{k\to\infty} N_k d_k = \infty. \tag{2}$$

To see that this is possible choose $d_1 > 0$ and a positive integer m_1. Suppose $d_1, \dots, d_k$ and $m_1, \dots, m_k$ have been chosen. Then we can choose $d_{k+1} \le d_k/2$ sufficiently small such that $N_k d_{k+1} < 2^{-(k+1)}$, and m_{k+1} sufficiently large such that $N_{k+1} d_{k+1} > k + 1$.

We write

$$Y = \{(i_1, i_2, \dots) : 1 \le i_j \le n_j \text{ for each } j \ge 1\},$$

with

$$\operatorname{dist}((i), (j)) = 2d_k,$$

where k is least such that $i_k \ne j_k$. We shall write

$$Y_{i_1,\dots,i_l} = \{(i_1, i_2, \dots, i_l, j_{l+1}, j_{l+2}, \dots) : 1 \le j_{l+q} \le n_{l+q} \text{ for each } q \ge 1\},$$

and refer to such sets as *cylinder sets of* Y. We note that Y is homeomorphic to the middle third Cantor set C. To see this, for each $k \ge 1$ write $\varphi_{k,1}, \dots, \varphi_{k,n_k}$ for the $n_k = 2^{m_k}$ similitudes from C onto the cylinder sets of C of diameter 3^{-m_k}. We may then define a homeomorphism from Y to C by $f((i_1, i_2, \dots)) = \bigcap_{k=1}^{\infty} \varphi_{1,i_1} \circ \dots \circ$

$\varphi_{k,i_k}(C)$. We shall use this homeomorphism to define a (Borel regular) measure μ on Y by setting

$$\mu(S) = \mathcal{H}^s(f(S)) \text{ for each } S \subseteq Y,$$

where $s = \log 2/\log 3$.

Lemma 2.1. *There is a non-Borel set $S \subset Y$ such that if B is a Borel set with $S \subset B \subset Y$, and $\{B_i\}$ are subsets of Y with $B \subset \bigcup_{i=1}^{\infty} B_i$, then there is an i with $\mu(B_i \setminus S) > 0$.*

Proof. Since Y is a complete metric space without isolated points, and since $0 < \mu(Y) < \infty$, we may find $S \subset Y$ such that if K is a closed subset of Y with $\mu(K) > 0$, then K intersects both S and its complement (see [1, 2.2.4]). This property ensures that for each such K, $K \cap S$ is not μ-measurable, and in particular, since Borel subsets of Y are μ-measurable, that S is not a Borel subset of Y.

If B is a Borel subset of Y containing S, then B contains some closed set D such that $\mu(D) > 0$ (otherwise B, like S, would not be μ-measurable). Then $\mu(D \setminus S) > 0$, otherwise $D \setminus S$, and therefore $D \cap S$, would be μ-measurable, contradicting the choice of S. If $B \subset \bigcup_{i=1}^{\infty} B_i$, we have $D \setminus S \subset \bigcup_{i=1}^{\infty}(B_i \setminus S)$, so there is an i with $\mu(B_i \setminus S) > 0$, proving the lemma. □

Choose $S \subset Y$ according to Lemma 2.1. Let $Z = Y \setminus S$. For each $k \geq 1$, let

$$Z^k = Z \times \{d_k\}.$$

As for Y, we write

$$Z^k_{i_1,\dots,i_l} = \{((i_1, i_2, \dots, i_l, j_{l+1}, j_{l+2}, \dots), d_k) : 1 \leq j_{l+q} \leq n_{l+q} \text{ for each } q \geq 1\}.$$

We extend the metric on Y to a metric on $Y \cup \bigcup_{k=1}^{\infty} Z^k$ by putting

$$\begin{aligned} \operatorname{dist}((x, d_k), y) &= \operatorname{dist}(x, y) + d_k, \\ \operatorname{dist}((x, d_k), (z, d_j)) &= \operatorname{dist}(x, z) + |d_k - d_j|, \end{aligned}$$

for each $y \in Y$, $x, z \in Z$, and $k, j \geq 1$. Note that these definitions ensure that if $y \in Z$, then $B(y, d_k) \cap Z^k_{i_1,\dots,i_k}$ contains the single point (y, d_k), and diam $B(y, d_k) \geq d_k$. However, if $y \in S$, then $B(y, d_k) \cap Z^k = \emptyset$, so a ball centred in S which intersects Z^k (and hence has diameter at least d_k) must have radius strictly greater than d_k.

Recall that our goal is to extend Y to a space where the 1-dimensional packing measure of S is zero, but that of any Borel set containing S is infinite. The final step, therefore, will be to add to our metric space sequences of points whose distances from points in $Y_{i_1,\dots,i_{k-1}}$ approach d_k; this will ensure that any two balls of radius strictly greater than d_k centred in $Y_{i_1,\dots,i_{k-1}}$ must intersect each other, but will have no effect on balls of radius d_k or less. Distances from points in these sequences to each other, and to points outside $Y_{i_1,\dots,i_{k-1}}$, will be chosen simply to satisfy the triangle inequality.

We now complete the construction. Let $Q^0 = \{q^0_j : j \geq 1\}$ satisfy

$$\begin{aligned} \operatorname{dist}(q^0_j, q^0_k) &= d_1/2, \\ \operatorname{dist}(q^0_j, y) &= (1 + 2^{-j})d_1 \text{ for each } y \text{ in } Y, \\ \operatorname{dist}(q^0_j, z) &= (1 + 2^{-j})d_1 + d_k \text{ for each } z \text{ in } Z^k. \end{aligned}$$

For each $(i_1, \dots, i_k)$ we also add a sequence $Q^{i_1,\cdots,i_k} = \{q_j^{i_1,\cdots,i_k} : j \geq 1\}$ such that for $p \geq 1$, $r \neq l$ and $(i_1, \dots, i_k) \neq (j_1, \dots, j_k)$,

$$\begin{aligned}
\operatorname{dist}\left(q_r^{i_1,\cdots,i_k}, q_l^{i_1,\cdots,i_k}\right) &= d_{k+1}/2, \\
\operatorname{dist}\left(q_r^{i_1,\cdots,i_k}, q_p^{j_1,\cdots,j_k}\right) &= \operatorname{dist}(Y_{i_1,\dots,i_k}, Y_{j_1,\dots,j_k}), \\
\operatorname{dist}\left(q_r^{i_1,\cdots,i_k}, q_p^{0}\right) &= d_1 - d_{k+1},
\end{aligned}$$

and for $k \neq s$,

$$\operatorname{dist}\left(q_r^{i_1,\cdots,i_k}, q_p^{j_1,\cdots,j_s}\right) = |d_{k+1} - d_{s+1}| + \operatorname{dist}(Y_{i_1,\dots,i_k}, Y_{j_1,\dots,j_s}).$$

For each $y \in Y_{i_1,\dots,i_k}$, $z \in Z^l_{i_1,\dots,i_k}$ and $(i_1, \dots, i_k) \neq (j_1, \dots, j_k)$, we also set

$$\begin{aligned}
\operatorname{dist}\left(q_r^{i_1,\cdots,i_k}, y\right) &= (1 + 2^{-r})d_{k+1}, \\
\operatorname{dist}\left(q_r^{j_1,\cdots,j_k}, y\right) &= (1 + 2^{-r})d_{k+1} + \operatorname{dist}(Y_{i_1,\dots,i_k}, Y_{j_1,\dots,j_k}), \\
\operatorname{dist}\left(q_r^{i_1,\cdots,i_k}, z\right) &= (1 + 2^{-r})d_{k+1} + d_l, \\
\operatorname{dist}\left(q_r^{j_1,\cdots,j_k}, z\right) &= (1 + 2^{-r})d_{k+1} + \operatorname{dist}(Y_{i_1,\dots,i_k}, Y_{j_1,\dots,j_k}) + d_l.
\end{aligned}$$

Our final separable metric space X will then be

$$X = Y \cup \bigcup_{i=1}^{\infty} Z^k \cup Q^0 \cup \bigcup_{k \geq 1} \{Q^{i_1,\cdots,i_k} : 1 \leq i_j \leq n_j \text{ for each } 1 \leq j \leq k\}.$$

Lemma 2.2. *If $V \subseteq Z$ satisfies $\mu(V) > 0$, then*

$$P_0^1(V) = \infty.$$

Proof. Let $\mu(V) = a$. Let η_k be the number of cylinder sets of the Cantor middle third set C of diameter 3^{-M_k} needed to cover $f(V)$ (there is a total of $N_k = 2^{M_k}$ such intervals). Then for k sufficiently large (recalling the definition of μ and that $s = \log 2/\log 3$),

$$\begin{aligned}
\eta_k \left(3^{-M_k}\right)^s &\geq a/2, \\
\eta_k 2^{-M_k} &\geq a/2, \\
\eta_k &\geq a N_k/2.
\end{aligned}$$

For each k,

$$\eta_k = \#\left\{(i_1, \dots, i_k) : Y_{i_1,\dots,i_k} \cap V \neq \emptyset,\ 1 \leq i_j \leq n_j,\ 1 \leq j \leq k\right\}.$$

Thus we may be assured that for k large enough, there is a fixed proportion of the N_k cylinder sets $Y_{i_1,\dots,i_k}$ in which we may centre a ball of a packing of V. This ensures that in packing V we may put disjoint balls of radius d_k centred in each of η_k distinct cylinder sets $Y_{i_1,\dots,i_k}$. We now check that these balls are indeed disjoint.

Let $(i_1, i_2, \dots)$ and $(j_1, j_2, \dots)$ be points of V with $i_k \neq j_k$. Then the two balls $B((i_1, i_2, \dots), d_k)$ and $B((j_1, j_2, \dots), d_k)$ cannot intersect in Y, since

$$\operatorname{dist}(Y_{i_1,\dots,i_k}, Y_{j_1,\dots,j_k}) \geq 2d_k.$$

Nor can these two balls intersect in $\bigcup Z^l$, since $B((i_1, i_2, \dots), d_k) \cap \bigcup Z^l \subset \bigcup_l Z^l_{i_1,\dots,i_k}$ and $\operatorname{dist}(Z^l_{i_1,\dots,i_k}, Z^l_{j_1,\dots,j_k}) \geq 2d_k$ for each $l \geq 1$. Finally, they cannot intersect in any set $Q^{l_1,\cdots,l_p}$ since if $q_j^{l_1,\cdots,l_p} \in B((i_1, i_2, \dots), d_k)$, then

$$\operatorname{dist}(Y_{j_1,\dots,j_k}, q_j^{l_1,\cdots,l_p}) \geq 2d_k.$$

Suppose $B(x, d_k)$ is one of these balls. Since $x \in V \subset Z$, $B(x, d_k)$ contains the point (x, d_k) of $Z^k_{i_1,\dots,i_k}$, and hence has diameter at least d_k (and, of course, no more than $2d_k$). Therefore

$$P^1_{2d_k}(V) \geq \eta_k d_k \geq aN_k d_k/2.$$

Since $N_k d_k \to \infty$ as $k \to \infty$ by choice of N_k and d_k, recall (2), the result follows. □

Lemma 2.3. *If B is a Borel subset of X such that $B \supset S$, then $\mathcal{P}^1(B) = \infty$.*

Proof. Now, $B \cap Y$ is a Borel subset of Y containing S. Therefore, by Lemma 2.1, if $B \subseteq \bigcup_{i=1}^\infty B_i$, then there is an i such that $\mu((B_i \cap Y) \setminus S) > 0$. Then, by Lemma 2.2, $P^1_0((B_i \cap Y) \setminus S) = \infty$. Since $P^1_0(B_i) \geq P^1_0((B_i \cap Y) \setminus S)$, we have $\mathcal{P}^1(B) = \infty$. □

Lemma 2.4. $\mathcal{P}^1(S) = 0$.

Proof. Let $\{B(x_i, r_i)\}$ be a packing of S in X. Fix i, let k be such that $d_k < r_i \leq d_{k-1}$, and suppose that $x_i \in S \cap Y_{j_1,\dots,j_k}$. We now establish an upper bound on the diameter of $B(x_i, r_i)$ in terms of d_k rather than d_{k-1}.

Now, since $\operatorname{dist}(Y_{j_1,\dots,j_{k-1}}, Y \setminus Y_{j_1,\dots,j_{k-1}}) = 2d_{k-1}$, we may be sure that $B(x_i, r_i)$ cannot intersect $Y \setminus Y_{j_1,\dots,j_{k-1}}$. Similarly $B(x_i, r_i)$ cannot intersect $\left(\bigcup_{p\geq 1} Z^p\right) \setminus \left(\bigcup_{p\geq k} Z^p_{j_1,\dots,j_{k-1}}\right)$. Finally, if $p < k-1$, or $p \geq k-1$ and $(j_1, \dots, j_{k-1}) \neq (l_1, \dots, l_{k-1})$, then $\operatorname{dist}(q_r^{l_1,\dots,l_p}, Y_{j_1,\dots,j_k}) > d_{k-1}$. Therefore

$$\begin{aligned} B(x_i, r_i) \subseteq Y_{j_1,\dots,j_{k-1}} &\cup \bigcup_{p\geq k} Z^l_{j_1,\dots,j_{k-1}} \\ &\cup \bigcup_{p\geq 0} \{Q^{j_1,\dots,j_{k-1},J_k,\dots,J_{k+p}} : 1 \leq J_{k+l} \leq n_{k+l} \text{ each } 0 \leq l \leq p\}. \end{aligned}$$

Since the diameters of these three sets are bounded above by $2d_k$, $3d_k$ and $2d_k$ respectively, we see that diam $B(x_i, r_i) \leq 7d_k$.

We now find an upper bound on the number of balls of the packing such that $d_k < r_i \leq d_{k-1}$. Note that there can be no ball of the packing besides $B(x_i, r_i)$ with radius greater than d_k and centre in $S \cap Y_{j_1,\dots,j_{k-1}}$. This is because we may find $P \geq 1$ such that $q_p^{j_1,\dots,j_{k-1}} \in B(x_i, r_i)$ for all $p \geq P$. Any other ball centred in $S \cap Y_{j_1,\dots,j_{k-1}}$ with radius greater than d_k must also contain some of these points. (This was the purpose of introducing the sequences $Q^{j_1,\dots,j_{k-1}}$.) Therefore the total number of balls in the packing with radius in the range $(d_k, d_{k-1}]$ is no more than N_{k-1} (one for each set $Y_{j_1,\dots,j_{k-1}}$) and these balls have diameter no more than $7d_k$. So, using (1),

$$P^1_0(S) \leq \lim_{j\to\infty} 7 \sum_{k=j}^\infty N_{k-1} d_k = 0.$$

The result follows. □

Lemmas 2.3 and 2.4 and the fact that $\mathcal{P}^1(S) \leq P^1_0(S)$ immediately give us our final result:

Theorem 2.5. *$\mathcal{P}^1$ is not Borel regular on X.*

Note. The work above relies completely on the fact that we pack by closed balls. As noted in the introduction, this follows recent practice, however in Taylor and Tricot's original definition of packing measure in [11] (see also [12]), open balls were used. If the Hausdorff function h is left-continuous, then packing measure using open balls is Borel regular, since with this definition it is easy to prove that the pre-packing measure P_0^h is the same for a set as for its closure, and therefore that each set is contained in an $F_{\sigma\delta}$ set of the same measure. Thus the work of this paper is in the style of previous papers (see [6, 8]), in which seemingly minor differences between alternative definitions of packing measure are shown to lead to significantly different measure-theoretic properties.

We also note that the choice of Hausdorff function $h(r) = r$ above was in no way crucial to the result. This choice was made partly to simplify the presentation, but mostly to emphasise the fact that the existence of a space on which diameter-type h-packing measure is not Borel regular in no way depends on some delicate choice of h. Clearly a similar space could be constructed for any Hausdorff function h by modifying the choice of the distances d_k in a suitable manner.

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