PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 4, April 1999, Pages 1079–1087
S 0002-9939(99)05195-3

A COUNTEREXAMPLE TO THE FREDHOLM ALTERNATIVE FOR THE p-LAPLACIAN

PAVEL DRÁBEK AND PETER TAKÁČ

(Communicated by Jeffrey Rauch)

Abstract. The following nonhomogeneous Dirichlet boundary value problem for the one-dimensional p-Laplacian with $1 < p < \infty$ is considered:
(*)
$$-(|u'|^{p-2}u')' - \lambda|u|^{p-2}u = f(x) \text{ for } 0 < x < T; \quad u(0) = u(T) = 0,$$
where $f \equiv 1 + h$ with $h \in L^\infty(0,T)$ small enough. Solvability properties of Problem (*) with respect to the spectral parameter $\lambda \in \mathbb{R}$ are investigated. We focus our attention on some fundamental differences between the cases $p \neq 2$ and $p = 2$. For $p \neq 2$ we give a counterexample to the classical Fredholm alternative (which is valid for the linear case $p = 2$).

1. Introduction and main results

We consider the following elliptic boundary value problem:
$$-(\psi_p(u'))' - \lambda\psi_p(u) = f(x) \text{ for } 0 < x < T; \quad u(0) = u(T) = 0. \tag{1.1}$$
Here, $\psi_p(v) \stackrel{\text{def}}{=} |v|^{p-2}v$ for $p \in (1,\infty)$ and $v \in \mathbb{R}$. Hence, $\Delta_p u = (\psi_p(u'))'$ is the one-dimensional p-Laplacian. We assume that $0 < T < \infty$, $f \in L^\infty(0,T)$, and $\lambda \in \mathbb{R}$ is a spectral parameter. Finally, under a solution of Problem (1.1) we understand a (real-valued) function $u \in C^1[0,T]$ such that $u(0) = u(T) = 0$, the function $\psi_p(u')$ is absolutely continuous, and Eq. (1.1) holds a.e. in $(0,T)$.

For the case $p = 2$, Problem (1.1) reduces to the well-known Dirichlet problem for the Laplace operator. The solvability of this linear boundary value problem is fully described by the classical Fredholm alternative. In the past, several attempts have been made to extend the Fredholm alternative to problems involving nonlinear operators. It has been shown by various authors (see e.g. del Pino and Manásevich [11]) that the set of all eigenvalues of the homogeneous Problem (1.1) (that is, for $f \equiv 0$ in $(0,T)$) forms a sequence
$$\lambda_k = (k\pi_p/T)^p \quad \text{for } k = 1, 2, \ldots, \tag{1.2}$$

Received by the editors July 16, 1997.
1991 *Mathematics Subject Classification.* Primary 34B15; Secondary 34C10.
Key words and phrases. Nonuniqueness and multiplicity of solutions, resonance for the p-Laplacian, nonlinear Fredholm alternative.
The first author's research was supported in part by the Grant Agency of the Czech Republic, Project 201/97/0395, by the Ministry of Education of the Czech Republic, Project Nr. VR 97156, and by the University of Rostock, Germany.
The second author's research was supported in part by Deutsche Forschungsgemeinschaft (Germany).

where

$$\pi_p = 2(p-1)^{1/p} \int_0^1 (1-s^p)^{-1/p}\, ds. \tag{1.3}$$

The case when $\lambda \notin \{\lambda_k : k = 1, 2, \dots\}$ is extensively treated in Fučík et al. [8]. Then, for any $f \in L^\infty(0,T)$, the existence of at least one solution to Problem (1.1) follows from a general result in [8, Chapt. II, Theorem 3.2]. Unlike in the linear case $p = 2$, for $p \neq 2$ the uniqueness holds only if $\lambda \leq 0$. If $0 < \lambda < \lambda_1$, then there exists some $f \in L^\infty(0,T)$ for which Problem (1.1) has at least two distinct solutions; cf. Fleckinger et al. [7, Example 2] for $1 < p < 2$ and del Pino, Elgueta and Manásevich [10, Eq. (5.26), p. 12] for $2 < p < \infty$. However, if $0 < \lambda < \lambda_1$ and $0 \leq f \in L^\infty(0,T)$, then uniqueness is preserved for all $p \in (1,\infty)$; see Díaz and Saa [6] or Fleckinger et al. [7, Theorem 1]. Making the special choice $f \equiv 1$ in $(0,T)$, del Pino and Manásevich [11] have demonstrated nonuniqueness for certain values of the parameter $\lambda > \lambda_1$, that is, for all $\lambda \in \bigcup_{k=1}^\infty \mathcal{J}_k$ where $\mathcal{J}_k$ is an open bounded interval with one of the end-points equal to λ_{2k}. The length of $\mathcal{J}_k$ tends to infinity as $k \to \infty$, and some of the intervals overlap for k large enough. The number of solutions increases to infinity as $\lambda \to \infty$. Unfortunately, from their work it is not clear if, for $p \neq 2$, the lack of uniqueness persists for all $\lambda \geq \lambda_1$.

The case when $\lambda = \lambda_k$, for some $k \in \mathbb{N}$, is much more delicate. In the linear case $p = 2$ the classical Fredholm alternative provides a very transparent necessary and sufficient condition for the solvability of Problem (1.1), namely,

$$\int_0^T f(x) \sin(x/T)\, dx = 0. \tag{1.4}$$

It is shown in Binding, Drábek and Huang [2, Theorem D] that for $p \neq 2$ the condition

$$\int_0^T f(x) \sin_p(x/T)\, dx = 0 \tag{1.5}$$

is not necessary for the solvability of Problem (1.1) with $\lambda = \lambda_1$, where $\sin_p(x/T)$ denotes the eigenfunction associated with the first eigenvalue λ_1. This result easily extends to every eigenvalue $\lambda = \lambda_k$, $k \in \mathbb{N}$, with the corresponding eigenfunction $\sin_p(kx/T)$ in place of $\sin_p(x/T)$ in Eq. (1.5). The first attempt to show that the set of all functions f on the right-hand side of Problem (1.1), for which this problem has at least one solution, need not be a manifold was made in Binding, Drábek and Huang [3]. More precisely, considering Eq. (1.1) subject to the boundary conditions $u(0) = 0$ and $u(T) = u'(T)$, the authors of [3] employed the method of linearization and the general implicit function theorem (see e.g. Deimling [5, Theorem 15.1]) to show the following result: *given any $\lambda \geq 0$, there exists an open set $M \subset L^2(0,T)$, $0 \notin M$, such that for any $f \in M$ the problem*

$$\begin{cases} -(\psi_p(u'))' - \lambda\psi_p(u) = f(x) & \textit{for } 0 < x < T; \\ u(0) = 0, \ u(T) = u'(T), \end{cases} \tag{1.6}$$

has at least one solution. Obviously, this result holds for all eigenvalues of the corresponding homogeneous problem; the first one is equal to zero. Unfortunately, the same approach does not work for the homogeneous Dirichlet boundary conditions which appear to be more complicated.

The purpose of the present paper is twofold. First, when $p \neq 2$, we obtain that the nonuniqueness of the solution to Problem (1.1) persists for all $\lambda > 0$ regardless if λ is an eigenvalue (the resonant case) or not (the nonresonant case). Second, we show that for every $\lambda = \lambda_{2k}$ with $k \geq 1$, and for every $\lambda = \lambda_{2k+1}$ with $k > k(p) > 0$ ($k(p) \to \infty$ as $p \to 2$), the resonant Problem (1.1) has at least one solution for $f \equiv 1 + h$, where h belongs to an open neighborhood of zero in $L^\infty(0,T)$. We prove the following two theorems.

Theorem 1.1 (Nonuniqueness)**.** *Let $1 < p < \infty$, $p \neq 2$, and $\lambda > 0$. Then there exists $f \in C([0,T])$ such that Problem* (1.1) *has at least two distinct solutions.*

Theorem 1.2 (Resonance)**.** *Let $1 < p < \infty$, $p \neq 2$, and set*

$$k(p) = \begin{cases} \frac{p-1}{2-p} & \text{if } 1 < p < 2; \\ \frac{1}{p-2} & \text{if } 2 \leq p < \infty. \end{cases} \tag{1.7}$$

Let either $\lambda = \lambda_{2k}$ with $k \geq 1$, or else $\lambda = \lambda_{2k+1}$ with $k > k(p)$, where k is an integer. Then there exists a number δ, $0 < \delta < 1$, such that Problem (1.1) *has at least one solution for any $f(x) = 1 + h(x)$ with $\|h\|_{L^\infty(0,T)} < \delta$.*

Remark 1.3. Notice that $0 < k(p) < 1$ whenever $1 < p < 3/2$ or $3 < p < \infty$, in which case the conlusion of Theorem 1.2 holds true for every $\lambda = \lambda_{2k+1}$ with $k \geq 1$. For $2 \neq p \in [3/2, 3]$ we have $1 \leq k(p) < \infty$ with $k(p) \to \infty$ as $p \to 2$.

We point out that both these results, Theorems 1.1 and 1.2, provide counterexamples to the classical Fredholm alternative and that there are still many open questions left. It is not clear how the number of solutions in the nonresonant case relates to the number of eigenvalues lying below λ. It is also not clear if a conclusion similar to that in our Theorem 1.2 holds true for all eigenvalues of the homogeneous problem (1.1). Also the problem of sufficiency of the condition

$$\int_0^T f(x) \sin_p(kx/T)\, dx = 0 \tag{1.8}$$

remains open.

In this article we use exclusively common techniques from the theory of ordinary differential equations. The one-dimensional problem is easier to treat than the corresponding multi-dimensional one for the following two reasons. First, the set of all eigenvalues of the p-Laplacian is fully described only in dimension one, and second, it is less technical to provide counterexamples. Besides, one can hardly expect "better" behavior of the boundary value problem in the case of a partial differential equation than in the case of an ordinary differential equation.

This article is organized as follows. In Section 2 we establish some standard properties of the initial value problem associated with Eq. (1.1). We concentrate on the existence, uniqueness, and continuous dependence of the solution upon the parameter λ, the function f, and the initial condition. In Section 3 we give the proofs of Theorems 1.1 and 1.2.

2. The initial value problem

Let us consider the initial value problem corresponding to Eq. (1.1),

$$(\psi_p(u'))' = -\lambda\psi_p(u) - f(x) \quad \text{for } x \in \mathcal{J}; \quad u(x_0) = u_0,\ u'(x_0) = u'_0. \tag{2.1}$$

Of course, $\mathcal{J}$ denotes a nondegenerate compact interval of the form

$$\mathcal{J} = \{x \in [0,T] : |x - x_0| \leq \eta\}$$

for some $x_0 \in [0, T]$ and $\eta > 0$ small enough depending upon the initial conditions u_0 and u_0'. We investigate the existence, uniqueness, and continuous dependence of the (local) solution $u : \mathcal{J} \to \mathbb{R}$ to Problem (2.1) upon the parameter λ, the function f, and the initial conditions u_0 and u_0'. We write $\Lambda \equiv (\lambda, f, u_0, u_0')$ in $\mathbb{R} \times L^\infty(0, T) \times \mathbb{R}^2$ and $u(x) \equiv u(x; \Lambda)$ for $x \in \mathcal{J}$ to stress the dependence of $u(x)$ upon all the data listed. Under a solution of Problem (2.1) in $\mathcal{J}$ we understand a (real-valued) function $u \in C^1(\mathcal{J})$ such that $u(x_0) = u_0$ and $u'(x_0) = u_0'$, the function $\psi_p(u')$ is absolutely continuous, and Eq. (2.1) holds a.e. in $\mathcal{J}$.

It is well-known that existence can be proved by standard arguments using Schauder's fixed point theorem. We state the existence result rigorously proved in Reichel and Walter [13, Theorem 1, p. 49].

Proposition 2.1 ([13]). *Let $1 < p < \infty$, $\lambda \in \mathbb{R}$, $f \in L^\infty(0, T)$, $x_0 \in [0, T]$, and $u_0, u_0' \in \mathbb{R}$. Then the initial value problem* (2.1) *has a solution u in an interval $\mathcal{J}$, for some number $\eta > 0$ small enough. Moreover, if B is any (nonempty) bounded set in the Banach space $\mathbb{R} \times L^\infty(0, T) \times \mathbb{R}^2$, then the number $\eta \equiv \eta(B) > 0$ may be chosen independent from a particular choice of the data $\Lambda = (\lambda, f, u_0, u_0') \in B$. Finally, η may be chosen so small that all possible solutions $u(\bullet; \Lambda)$ of Problem* (2.1) *in $\mathcal{J}$ for any data $\Lambda \in B$ belong to the same compact set K in $C^1(\mathcal{J})$.*

Notice that K is a compact set in the Banach space $C^1(\mathcal{J})$ if and only if both sets K and $K' = \{dv/dx : v \in K\}$ are compact in $C^0(\mathcal{J})$.

It is also well-known that uniqueness implies continuous dependence of the solution upon various data entering the initial value problem; cf. Coddington and Levinson [4, Chapt. II, §4]. This is a consequence of compactness from the proof of Proposition 2.1 in [13]. The continuous dependence can be stated as follows.

Proposition 2.2. *Let $1 < p < \infty$, $\lambda \in \mathbb{R}$, $f \in L^\infty(0, T)$, $x_0 \in [0, T]$, and $u_0, u_0' \in \mathbb{R}$. Assume that the data $\Lambda = (\lambda, f, u_0, u_0')$ from $\mathbb{R} \times L^\infty(0, T) \times \mathbb{R}^2$ are given such that the solution $u \equiv u(\bullet; \Lambda)$ of Problem* (2.1) *in $\mathcal{J}$ is unique. Let $\Lambda^{(n)} = (\lambda^{(n)}, f^{(n)}, u_0^{(n)}, {u_0'}^{(n)})$, $n = 1, 2, \ldots$, be any sequence converging to Λ in $\mathbb{R} \times L^\infty(0, T) \times \mathbb{R}^2$, and let $u^{(n)}$ be any (possibly nonunique) solution of Problem* (2.1) *in $\mathcal{J}$ with the data $\Lambda^{(n)}$ in place of Λ, for each $n = 1, 2, \ldots$. Then $u^{(n)} \to u$ in $C^1(\mathcal{J})$ as $n \to \infty$, where $\eta > 0$ is small enough.*

Proof. We omit the proof because it requires only few minor changes in Coddington and Levinson [4, Chapt. II, Theorem 4.3]. ☐

The uniqueness of the solution is essentially proved in McKenna, Reichel and Walter [9, Appendix], del Pino, Manásevich and Murúa [12, Appendix], and Reichel and Walter [13, Theorem 4, p. 57].

Proposition 2.3 ([9], [13]). *Let $1 < p < \infty$, $0 < \lambda < \infty$, $0 \le f \in L^\infty(0, T)$, $x_0 \in [0, T]$, and $u_0, u_0' \in \mathbb{R}$. In addition, if $u_0' = 0$, $u_0 < 0$ and $p > 2$, then assume*

$$\operatorname{ess\,sup}_{|x - x_0| \le \eta_0} f(x) < \lambda \psi_p(-u_0) \quad \textit{for some } \eta_0 > 0, \tag{2.2}$$

and if $u_0' = u_0 = 0$, then assume $1 < p \le 2$ and also

$$\operatorname{ess\,inf}_{|x - x_0| \le \eta_0} f(x) > 0 \quad \textit{for some } \eta_0 > 0. \tag{2.3}$$

Then the initial value problem (2.1) *has a unique solution u in an interval $\mathcal{J}$, for some η small enough with $0 < \eta \le \eta_0$.*

Proof. We will distinguish among the following possibilities.

Case $u_0' \neq 0$. This is Part (α)(ii) for $2 \le p < \infty$ and Part (α)(iii) for $1 < p \le 2$ of Theorem 4 in [13, p. 57].

Case $u_0' = 0$ and $u_0 \neq 0$. Then Problem (2.1) is equivalent to the fixed point equation

$$u(x) = u_0 - \int_{x_0}^{x} \psi_{p'} \left(\int_{x_0}^{s} [\lambda \psi_p(u(t)) + f(t)]\, dt \right) ds \quad \text{for } x \in \mathcal{J}. \tag{2.4}$$

Notice that $\psi_{p'}$ is the inverse function of $\psi_p(v) = |v|^{p-2}v$ where $p' = \frac{p}{p-1}$ denotes the conjugate exponent to p. Since $u_0 \neq 0$, the function ψ_p is locally Lipschitz continuous near u_0.

For $1 < p \le 2$ the function $\psi_{p'}$ is locally Lipschitz continuous. Therefore, the desired uniqueness follows by a fixed point argument using the contraction mapping principle; cf. Part (β)(iii) of Theorem 4 in [13, p. 57] or [9, Appendix, p. 1224] or [12, Appendix, p. 92].

For $2 \le p < \infty$ we need to distinguish between the cases $u_0 > 0$ and $u_0 < 0$. If $u_0 > 0$, then the uniqueness for Problem (2.1) follows from Part (β)(vi) of Theorem 4 in [13, p. 57] which is actually proved in [9, Appendix, p. 1224]. If $u_0 < 0$, then we need to apply Part (β)(v) of Theorem 4 in [13, p. 57] which requires the inequality (2.2). Again, the proofs employ fixed point arguments using the contraction mapping principle.

Case $u_0' = u_0 = 0$. Since also $1 < p \le 2$ and the inequality (2.3) is satisfied, we may apply Part (α)(iv) of Theorem 4 in [13, p. 57]. Proposition 2.3 is proved. ☐

Remark 2.4. Strictly speaking, in all the references [9, Appendix, p. 1224], [12, Appendix, p. 92] and [13, Theorem 4, p. 57] the right-hand side of Eq. (2.1), that is $g(x,u) \equiv -\lambda\psi_p(u) - f(x)$, is assumed to be continuous in both variables x and u. Nevertheless, from the proofs of the existence and uniqueness results quoted above it is clear that they hold true also for $f \in L^\infty(0,T)$. Only in the proof of Part (β)(v) of Theorem 4 in [13, p. 57] few minor technical adjustments are needed.

For $f \equiv 1 + h$, Ineq. (2.3) holds, provided $\|h\|_{L^\infty(0,T)} < 1$, whereas Ineq. (2.2) is satisfied if $\|h\|_{L^\infty(0,T)} < \frac{p-1}{p+1}$ and u is a "global" solution of Problem (2.1) in $[0,T]$ satisfying $u(x_1) = 0$ for some $x_1 \in [0,T]$, and $(x_1 - x_0)u'(x) \ge 0$ for every x between x_0 and x_1. More precisely, we have the following result:

Lemma 2.5. *Let $1 < p < \infty$, $\lambda \in \mathbb{R}$, $f \in L^\infty(0,T)$, $x_0 \in [0,T]$, and $u_0 < 0 = u_0'$. Assume that u is a solution of Problem* (2.1) *in some nondegenerate compact interval $\mathcal{J}$ containing x_0, $\mathcal{J} \subset [0,T]$, and there exists a point $x_1 \in \mathcal{J}$ such that $u(x_1) = 0$ and $(x_1 - x_0)u'(x) \ge 0$ for all $x \in [a,b]$, where $a = \min\{x_0, x_1\}$ and $b = \max\{x_0, x_1\}$. Then we have*

$$\begin{aligned} & f(x) - 1 \ge -\delta \ (f(x) - 1 \le \delta, \ \textit{respectively}) \ \ \textit{for a.e. } x \in \mathcal{J} \\ & \Longrightarrow \quad \lambda\psi_p(u_0) + 1 - (p-1)\left(\frac{|u'(x_1)|^p}{u_0} - 1 \right) \le p\delta \ (\ge -p\delta) \end{aligned} \tag{2.5}$$

for every number $\delta \ge 0$. In particular, if

$$\operatorname*{ess\,sup}_{x \in \mathcal{J}} |f(x) - 1| \le \delta < \frac{p-1}{p+1},$$

then

$$\lambda\psi_p(u_0) + \operatorname*{ess\,sup}_{x\in\mathcal{J}} f(x) \le \lambda\psi_p(u_0) + 1 + \delta < 0. \tag{2.6}$$

Proof. Following [13, p. 60] we first multiply Eq. (2.1) by u' and then integrate the product over the interval $[a,b]$, thus arriving at

$$\frac{p-1}{p}|u'(x_1)|^p + \int_{x_0}^{x_1} f(t)\, u'(t)\, dt = \frac{\lambda}{p}|u_0|^p. \tag{2.7}$$

From now on, let us assume $f-1 \ge -\delta$ a.e. in $\mathcal{J}$; the other case $f-1\le\delta$ a.e. in $\mathcal{J}$ is analogous. Since $u_0 < 0 = u(x_1)$ and u' does not change sign in $[a,b]$, we get

$$-\int_{x_0}^{x_1}(f(t)-1)u'(t)\, dt = -\int_a^b (f(t)-1)|u'(t)|\, dt \le \delta \int_a^b |u'(t)|\, dt = -\delta u_0.$$

We add this inequality to Eq. (2.7) to obtain

$$\frac{p-1}{p}(|u'(x_1)|^p - u_0) - \frac{1}{p}(\lambda|u_0|^{p-2}u_0 + 1)u_0 \le -\delta u_0.$$

Multiplying the last inequality by $-p/u_0$ we arrive at (2.5). Finally, from (2.5) with $\operatorname{ess\,sup}_{\mathcal{J}} |f-1| \le \delta < \frac{p-1}{p+1}$ we deduce

$$\lambda\psi_p(u_0) + 1 + \delta \le (p-1)\left(\frac{|u'(x_1)|^p}{u_0} - 1\right) + (p+1)\delta \le -(p-1) + (p+1)\delta$$

which yields (2.6). Lemma 2.5 is proved. □

3. Proofs of the theorems

3.1. **Proof of Theorem 1.1.** The case $0<\lambda<\lambda_1$ is treated in Fleckinger et al. [7, Example 2] for $1<p<2$ and in del Pino, Elgueta and Manásevich [10, Eq. (5.26), p. 12] for $2<p<\infty$. Therefore, from now on, we restrict ourselves to the case $0<\lambda<\lambda_k$ for $k\ge 2$.

In this proof we first decompose the interval $[0,T]$ into k subintervals $\mathcal{J}_j = [(j-1)T/k, jT/k]$ for $j=1,2,\dots,k$. We then "paste together" f and u from the available examples [7, Example 2] and [10, Eq. (5.26), p. 12]. This pasting together is done by extending both functions f and u, rescaled to the subinterval $[0,T/k]$, as odd functions first to $[T/k, 2T/k]$, then to $[2T/k, 3T/k]$ and so on. Unfortunately, in neither of these two examples is it guaranteed that $f(0)=f(T)=0$, and consequently, only a piecewise continuous L^∞-function f with $f\in C(\mathcal{J}_j)$ can thus be obtained. However, the particular behavior of f and u near $x=0$ or $x=T$ in [7, Example 2] and [10, Eq. (5.26), p. 12] plays no essential role as long as the solution u of Problem (1.1) satisfies $u\in C^2([0,T])$, $\psi_p(u')\in C^1([0,T])$, together with $u(0)=u(T)=0$, $u'(0)=-u'(T)>0$ and $u''(0)=u''(T)=0$. It is easy to modify the two examples in this way. Then also $f(0)=f(T)=0$. As we wish to construct a continuous function f over $[0,T]$ for which Problem (1.1) has at least two distinct solutions, it is necessary to carry out this modification of f and u. Notice that for every $k=1,2,\dots$, $\lambda_{1;k}\equiv\lambda_k$ is the first eigenvalue of the homogeneous problem

$$\begin{cases} -(\psi_p(u'))' - \lambda\psi_p(u) = 0 & \text{for } \frac{jT}{k} < x < \frac{(j+1)T}{k}; \\ u\left(\frac{jT}{k}\right) = u\left(\frac{(j+1)T}{k}\right) = 0, \end{cases} \tag{3.1}$$

for each $j = 0, 1, \dots, k-1$. Now fix λ such that $0 < \lambda < \lambda_k$ where $k \geq 2$ is an integer. We need to distinguish between the cases $1 < p < 2$ and $2 < p < \infty$.

(i) **Case** $p > 2$**.** Given any ε, $0 < \varepsilon < \frac{T}{2k}$, consider a function $v_0 \in C^2\left([0, T/k]\right)$ such that

$$v_0(x) = \begin{cases} 0 & \text{for } x \in \left[0, \frac{\varepsilon}{2}\right] \cup \left[\frac{T}{k} - \frac{\varepsilon}{2}, \frac{T}{k}\right]; \\ 1 & \text{for } x \in \left[\varepsilon, \frac{T}{k} - \varepsilon\right], \end{cases}$$

and define $h_0 \stackrel{\text{def}}{=} -(\psi_p(v_0'))' - \lambda\psi_p(v_0)$ in $[0, T/k]$, cf. [10, Eq. (5.26), p. 12] or [7, Example 1]. Obviously, we have $h_0 \in C\left([0, T/k]\right)$ and $h_0(x) = 0$ for $x \in \left[0, \frac{\varepsilon}{2}\right] \cup \left[\frac{T}{k} - \frac{\varepsilon}{2}, \frac{T}{k}\right]$. We extend the functions v_0 and h_0 to the interval $[T/k, 2T/k]$ by $v_1(x) = -v_0\left(\frac{2T}{k} - x\right)$ and $h_1(x) = -h_0\left(\frac{2T}{k} - x\right)$ for $T/k \leq x \leq 2T/k$. It follows that

$$v_1(x) = \begin{cases} 0 & \text{for } x \in \left[\frac{T}{k}, \frac{T}{k} + \frac{\varepsilon}{2}\right] \cup \left[\frac{2T}{k} - \frac{\varepsilon}{2}, \frac{2T}{k}\right]; \\ -1 & \text{for } x \in \left[\frac{T}{k} + \varepsilon, \frac{2T}{k} - \varepsilon\right], \end{cases}$$

and $h_1 = -(\psi_p(v_1'))' - \lambda\psi_p(v_1)$ in $[T/k, 2T/k]$. Let now $j \in \{0, 1, \dots, k-1\}$ and $jT/k \leq x \leq (j+1)T/k$. We define $v_j(x) = v_0\left(x - \frac{jT}{k}\right)$ and $h_j(x) = h_0\left(x - \frac{jT}{k}\right)$ if j is even, and $v_j(x) = v_1\left(x - \frac{jT}{k}\right)$ and $h_j(x) = h_1\left(x - \frac{jT}{k}\right)$ if j is odd. Finally, setting $u_1(x) \stackrel{\text{def}}{=} v_j(x)$ and $f(x) \stackrel{\text{def}}{=} h_j(x)$ for $jT/k \leq x \leq (j+1)T/k$ and $j = 0, 1, \dots, k-1$, we have constructed functions $u_1 \in C^2([0, T])$ and $f \in C([0, T])$ satisfying

$$-(\psi_p(u_1'))' - \lambda\psi_p(u_1) = f(x) \quad \text{for } 0 < x < T; \quad u_1(0) = u_1(T) = 0.$$

Furthermore, repeating the argument from [10, p. 12] or [7, Example 1] applied to Eq. (3.1), we can construct a C^1-function w_0 in the interval $[0, T/k]$ such that $w_0 \not\equiv v_0$, $-(\psi_p(w_0'))' - \lambda\psi_p(w_0) = h_0$, and $w_0(0) = w_0\left(T/k\right) = 0$. More precisely, the function w_0 is a global energy minimizer for the nonhomogeneous Dirichlet boundary value problem corresponding to Eq. (3.1). Analogously as we did for v_0 and v_j above, we extend w_0 to a function w_j defined in $jT/k \leq x \leq (j+1)T/k$ for each $j = 0, 1, \dots, k-1$. Thus, setting $u_2(x) \stackrel{\text{def}}{=} w_j(x)$ for $jT/k \leq x \leq (j+1)T/k$ and $j = 0, 1, \dots, k-1$, we have constructed another function $u_2 \in C^1([0, T])$, $u_2 \not\equiv u_1$, satisfying $\psi_p(u_2') \in C^1([0, T])$ and

$$-(\psi_p(u_2'))' - \lambda\psi_p(u_2) = f(x) \quad \text{for } 0 < x < T; \quad u_2(0) = u_2(T) = 0.$$

We conclude that u_1 and u_2 are two distinct solutions of Problem (1.1).

(ii) **Case** $1 < p < 2$**.** Here we modify an example from [7, Example 2]. Let m be a number satisfying $\max\left\{\frac{p}{p-1}, \frac{1}{2-p}\right\} \leq m < \infty$. Given any ε, $0 < \varepsilon < \frac{T}{8k}$, take a function $v_0 \in C^2\left([0, T/k]\right)$ such that

$$\begin{aligned} v_0(x) &= \left|x - \tfrac{T}{2k}\right|^m && \text{for } \left|x - \tfrac{T}{2k}\right| \leq \varepsilon; \\ v_0(x) &= \left(\tfrac{T}{k}\right)^m - \left|x - \tfrac{jT}{4k}\right|^m && \text{for } \left|x - \tfrac{jT}{4k}\right| \leq \varepsilon,\ j = 1, 3; \\ v_0'(x) &\neq 0 && \text{for all } x \neq \tfrac{jT}{4k},\ j = 1, 2, 3, \end{aligned}$$

together with $v_0(0) = v_0\left(T/k\right) = 0$ and $v_0''(0) = v_0''\left(T/k\right) = 0$. Clearly, we have $\psi_p(v_0') \in C^1\left([0, T/k]\right)$ with $(\psi_p(v_0'))'(x) = 0$ for $x = 0$ and $x = T/k$. Let us define v_j and h_j for $j = 0, 1, \dots, k-1$, and u_1 and f as in Case (i) above. Then the existence of another solution u_2 to Problem (1.1), which is different from u_1, now follows by the same arguments as in [7, Example 2]. Again, u_2 is constructed from a global energy minimizer w_0 for the nonhomogeneous Dirichlet boundary value problem corresponding to Eq. (3.1). Thus, we have shown nonuniqueness also for $1 < p < 2$.

The proof of Theorem 1.1 is finished.

3.2. **Proof of Theorem 1.2.** Let us define the sequence $\{\mu_k\}_{k=1}^{\infty}$ by $\mu_k = (kp'\pi_p/T)^p$ for $k = 1, 2, \dots$, where π_p is defined by Eq. (1.3). The proof of our assertion follows from the proof of Theorem 2.1 in del Pino and Manásevich [11] combined with our Proposition 2.2. Indeed, let us investigate the following two cases.

(i) **Case** $\lambda = \lambda_{2k}$ with some $k \geq 1$. Then $\lambda_{2k} \in (\lambda_{2k-1}, \mu_k)$ if $1 < p < 2$, and $\lambda_{2k} \in (\mu_k, \lambda_{2k+1})$ if $p > 2$. In either case, the Dirichlet problem

$$-(\psi_p(u'))' - \lambda_{2k}\psi_p(u) = 1 \;\text{ for } 0 < x < T; \quad u(0) = u(T) = 0, \tag{3.2}$$

has at least one solution, by Theorem 2.1(c) and (d) in [11]. But inspecting the proof of Theorem 2.1(c,d) in [11, p. 137] and combining it with the continuous dependence stated in our Proposition 2.2, we obtain immediately that the Dirichlet problem

$$\begin{cases} -(\psi_p(u'))' - \lambda_{2k}\psi_p(u) = 1 + h(x) & \text{for } \; 0 < x < T; \\ u(0) = u(T) = 0, \end{cases} \tag{3.3}$$

has at least one solution for any $h \in L^\infty(0,T)$ such that $\|h\|_{L^\infty(0,T)} < \delta$ with $\delta > 0$ small enough. The continuous dependence of the solution u to the initial value problem corresponding to Eq. (3.3) is used essentially in the shooting method employed in [11]. The details of the proof are the same as there.

(ii) **Case** $\lambda = \lambda_{2k+1}$ with $k > k(p)$ where $k(p)$ is defined in (1.7). Then $\lambda_{2k+1} \in (\lambda_{2k}, \mu_k)$ if $1 < p < 2$, and $\lambda_{2k+1} \in (\mu_{k+1}, \lambda_{2(k+1)})$ if $p > 2$. In either case, the Dirichlet problem

$$-(\psi_p(u'))' - \lambda_{2k+1}\psi_p(u) = 1 \;\text{ for } 0 < x < T; \quad u(0) = u(T) = 0, \tag{3.4}$$

has at least one solution, by Theorem 2.1(f) in [11]. Again, inspecting the proof of Theorem 2.1(f) in [11, p. 137] and combining it with the continuous dependence stated in our Proposition 2.2, we obtain analogously as before that the Dirichlet problem

$$\begin{cases} -(\psi_p(u'))' - \lambda_{2k+1}\psi_p(u) = 1 + h(x) & \text{for } \; 0 < x < T; \\ u(0) = u(T) = 0, \end{cases} \tag{3.5}$$

has at least one solution for any $h \in L^\infty(0,T)$ such that $\|h\|_{L^\infty(0,T)} < \delta$ with $\delta > 0$ small enough.

We have finished our proof of Theorem 1.2.

Remark 3.1. Notice that for $\delta > 0$ small enough, the solutions of the nonautonomous problems (3.3) and (3.5) have the same number of interior zeros in $(0, T)$ as the corresponding solutions of the autonomous problems (3.2) and (3.4), respectively. Of course, the corresponding solution of (3.3) (or (3.5)) has to lie sufficiently close to that of (3.2) (or (3.4), respectively) with respect to the $C^1([0,T])$-norm. This is a direct consequence of the proof of Theorem 2.1(c,d) and (f) in [11] combined with the continuous dependence from our Proposition 2.2. Hence, Eq. (3.3) has a solution with $2k-2$ zeros in $(0,T)$ if $1<p<2$, and with $2k$ zeros in $(0,T)$ if $p>2$. Similarly, Eq. (3.5) has a solution with $2k-1$ zeros in $(0,T)$ if $1<p<2$, and with $2k+1$ zeros in $(0,T)$ if $p>2$.

References

1. A. Anane, *Simplicité et isolation de la première valeur propre du p-laplacien avec poids*, Comptes Rendus Acad. Sc. Paris, Série I, **305** (1987), 725–728. MR **89e:**35124
2. P. A. Binding, P. Drábek, Y. X. Huang, *On the Fredholm alternative for the p-Laplacian*, Proc. Amer. Math. Soc. **125** (1997), 3555–3559. MR **98b:**35058
3. P. A. Binding, P. Drábek, Y. X. Huang, *On the range of the p-Laplacian*, Appl. Math. Letters **10** (1997), 77–82. MR **98g:**34035
4. E. A. Coddington and N. Levinson, *"Theory of Ordinary Differential Equations"*, McGraw-Hill, Inc., New York, 1955. MR **16:**1022b
5. K. Deimling, *"Nonlinear Functional Analysis"*, Springer-Verlag, Berlin–Heidelberg–New York, 1985. MR **86j:**47001
6. J. I. Díaz and J. E. Saa, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, Comptes Rendus Acad. Sc. Paris, Série I, **305** (1987), 521–524. MR **89e:**35051
7. J. Fleckinger, J. Hernández, P. Takáč and F. de Thélin, *Uniqueness and positivity for solutions of equations with the p-Laplacian*, Proceedings of the Conference on Reaction-Diffusion Equations, Trieste, Italy, October 1995. Marcel Dekker, Inc., New York–Basel, 1997. MR **98h:**35074
8. S. Fučík, J. Nečas, J. Souček and V. Souček, *"Spectral Analysis of Nonlinear Operators"*, Lecture Notes in Mathematics, Vol. **346**, Springer-Verlag, New York–Berlin–Heidelberg, 1973. MR **57:**7280
9. P. J. McKenna, W. Reichel and W. Walter, *Symmetry and multiplicity for nonlinear elliptic differential equations with boundary blow-up*, Nonlinear Anal. **28**(7) (1997), 1213–1225. MR **97h:**35049
10. M. A. del Pino, M. Elgueta and R. F. Manásevich, *A homotopic deformation along p of a Leray-Schauder degree result and existence for* $(|u'|^{p-2}u')'+f(t,u)=0$, $u(0)=u(T)=0$, $p>1$, J. Differential Equations **80**(1) (1989), 1–13. MR **91i:**34018
11. M. A. del Pino and R. F. Manásevich, *Multiple solutions for the p-Laplacian under global nonresonance*, Proc. Amer. Math. Soc. **112**(1) (1991), 131–138. MR **91h:**34026
12. M. A. del Pino, R. F. Manásevich and A. E. Murúa, *Existence and multiplicity of solutions with prescribed period for a second order quasilinear o.d.e.*, Nonlinear Anal. **18**(1) (1992), 79–92. MR **92m:**34055
13. W. Reichel and W. Walter, *Radial solutions of equations and inequalities involving the p-Laplacian*, J. Inequalities and Appl. **1** (1997), 47–71.

Department of Mathematics, University of West Bohemia, P.O. Box 314, CZ–306 14 Plzeň, Czech Republic

E-mail address: pdrabek@kma.zcu.cz

Fachbereich Mathematik, Universität Rostock, Universitätsplatz 1, D–18055 Rostock, Germany

E-mail address: peter.takac@mathematik.uni-rostock.de