INTEGRABILITY OF SUPERHARMONIC FUNCTIONS, UNIFORM DOMAINS, AND HÖLDER DOMAINS

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Abstract. Let $S^+(D)$ denote the space of all positive superharmonic functions on a domain $D \subset \mathbb{R}^n$. Lindqvist showed that $\log S^+(D)$ is a bounded subset of $BMO(D)$. Using this, we give a characterization of finitely connected 2-dimensional uniform domains and remarks on Hölder domains.

1. Notation and Main Result

Let $S^+(D)$ and $H^+(D)$ denote the spaces of all positive superharmonic and positive harmonic functions on a domain $D \subset \mathbb{R}^n$, $n \geq 2$, respectively. The quasihyperbolic metric $k_D$ on $D$ is defined by

$$ k_D(x, y) = \inf_{\gamma} \int_\gamma \frac{ds}{d(\cdot, \partial D)}, $$

where $d$ denotes the Euclidean distance, and the infimum is taken over all rectifiable curves $\gamma \subset D$ joining $x$ to $y$ (cf. [2]). We say that $D$ is a Hölder domain if

$$ k_D(x, x_0) \leq \frac{1}{\alpha} \log \left( 2 + \frac{1}{d(x, \partial D)} \right) + C, \quad x \in D, $$

for some $\alpha, C > 0$. Note that $D$ is a Hölder domain if $\sup_{x \in D} \int_{B_x} e^{pk_D(\cdot, x_0)} dm < \infty$ for some $p > 0$, where $B_x$ denotes the ball with center $x$ and radius $d(x, \partial D)/2$, and $m$ denotes the $n$-dimensional Lebesgue measure. Smith-Stegenga showed the following remarkable characterization of Hölder domains, which asserts that the local exponential integrability of the quasihyperbolic metric implies the global one:

**Proposition 1.1** ([9]). If $D$ is a Hölder domain in $\mathbb{R}^n$, then $\int_D e^{pk_D(\cdot, x_0)} dm < \infty$ for some $p > 0$.

Using this or a similar $BMO$ argument, Smith-Stegenga, Masumoto, and Stegenga-Ullrich investigated the $L^p$ integrability of $S^+(D)$ functions:

**Proposition 1.2** ([10], [7], [11], cf. [12]). If $D$ is a Hölder domain in $\mathbb{R}^n$, then $S^+(D) \subset L^p(D)$ for some $p > 0$. Conversely, if $D$ is a finitely connected subdomain of $\mathbb{R}^2$ and $S^+(D) \subset L^p(D)$ for some $p > 0$, then $D$ is a Hölder domain.

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Lindqvist clarified the argument of Stegenga-Ullrich by showing the following. Let $BMO(D)$ be the space of all locally integrable functions $f$ on $D$ satisfying
\[ \|f\|_{s,D} = \sup |B|^{-1} \int_B |f - f_B|dm < \infty, \]
where the supremum is taken over all balls $B$ in $D$, $|B| = m(B)$, and $f_B$ denotes the integral mean of $f$ over $B$.

**Proposition 1.3** ([4]). For an arbitrary subdomain $D$ of $\mathbb{R}^n$, we have
\[ \|\log u\|_{s,D} \leq C(n), \quad u \in S^+(D). \]

In our former paper [3], we obtained various estimations for the integrability of $BMO(D)$ functions. So, Lindqvist’s theorem immediately provides the corresponding results for $S^+(D)$. Now we state one of them.

We say that a proper subdomain $D$ of $\mathbb{R}^n$ is a uniform domain if
\[ k_D(x,y) \leq C \log \left( 2 + \frac{d(x, \partial D) + d(y, \partial D) + |x-y|}{\min\{d(x, \partial D), d(y, \partial D)\}} \right), \quad x, y \in D, \]
for some $C > 0$. Each bounded uniform domain is Hölder. For $p > 0$ and a measurable subset $E$ of $\mathbb{R}^n$, we set
\[ N_p(E) = |E|^{-1} \inf_{y \in \mathbb{R}^n} \left( \int_E |x - y|^p dm(x) \right)^{\frac{1}{p}}. \]
$N_p(E)$ is invariant under similarities of $\mathbb{R}^n$, and a kind of distance between $E$ and balls. Then from [3], Theorem 5.3, we have

**Theorem 1.1.** If $D$ is a uniform domain in $\mathbb{R}^n$, then there exist constants $p_0$, $p$, $C > 0$ such that for each $u \in S^+(D)$ and each measurable subset $E$ of $D$, we have
\[ \left( |E|^{-1} \int_E u^p dm \right) \left( |E|^{-1} \int_E u^{-p} dm \right) \leq CN_{p_0}(E)^2. \]

Our main aim of the present paper is to show that the converse holds if $D$ is a finitely connected subdomain of $\mathbb{R}^2$.

**Theorem 1.2** (Main Theorem). Let $D$ be a finitely connected proper subdomain of $\mathbb{R}^2$. Assume that there exist constants $p_0$, $p$, $C > 0$ such that for each $u \in S^+(D)$ and each measurable subset $E$ of $D$, we have
\[ \left( |E|^{-1} \int_E u^p dm \right) \left( |E|^{-1} \int_E u^{-p} dm \right) \leq CN_{p_0}(E)^2. \]
Then $D$ is a uniform domain.

The proof of the Main Theorem is given in §2. In §3, we list some other immediate consequences of Lindqvist’s theorem and of the author [3]. Using these results, we investigate the integrability of $S^+(D)$ functions on Hölder domains (§4) and the boundedness of domains with some integrability condition for $S^+(D)$ (§5).

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2. Proof of the Main Theorem

To prove the Main Theorem, we need a lemma, which also plays a fundamental role in §§4 and 5. From the Harnack inequality, \(|\log u(x) - \log u(y)| \leq Ck_D(x, y)|, x, y \in D\), holds for each \(u \in H^+(D)\), where \(C = C(n) > 0\). Conversely,

**Lemma 2.1.** Let \(D\) be a finitely connected proper subdomain of \(\mathbb{R}^2\). Assume that each boundary component contains more than two points. Then there exist constants \(C_1, C_2 > 0\) such that for each pair of points \(x_1, x_2\) on \(D\), we can find a pair of an arc \(\gamma\) joining \(x_1\) to \(x_2\) and a \(H^+(D)\) function \(u\) satisfying

\[
\int_{\gamma_{yz}} \frac{ds}{d(\cdot, \partial D)} \leq C_1(\log u(y) - \log u(z)) + C_2, \quad y, z \in \gamma,
\]

where \(\gamma_{yz}\) denotes the portion of \(\gamma\) joining \(y\) to \(z\) and \(y\) is between \(z\) and \(x_2\).

**Proof.** Since \(k_D\) is conformally invariant modulo constant factors, we may assume that \(D\) is a bounded domain surrounded by a finite number of circles, and that \(x_1\) and \(x_2\) are sufficiently close to some boundary components \(F_1\) and \(F_2\), respectively. We may assume \(F_1 \neq F_2\). A similar argument holds when \(F_1 = F_2\). We may also assume that \(F_1 = \{|x| = 1\}\) the outer boundary of \(D\), and \(F_2 = \{|x| = a\}\), \(0 < a < 1\). Take \(b > 0\) so that \(\{a < |x| < a + 2b\} \subset D\), \(\{1 - 2b < |x| < 1\} \subset D\). Let \(x_j = r_je^{i\theta_j}\), \(j = 1, 2\). Let \(\gamma_1\) and \(\gamma_2\) be the segments joining \(x_1\) to \(x'_1 = (1 - b)e^{i\theta_1}\) and \(x''_2 = (a + b)e^{i\theta_2}\) to \(x_2\), respectively. We can take an arc \(\gamma' \subset D\) joining \(x'_1\) to \(x''_2\) so that \(\gamma'\) is uniform. Thus \(\int_{\gamma'} d(\cdot, \partial D)^{-1}ds \leq C\). Let \(u\) be the Martin kernel function for \(x'_1\) to \(x''_2\) in \(\partial D\), i.e.

\[
u(x) = \lim_{D \ni y \to x_2} (g_D(x, y)/g_D(x_0, y)),
\]

where \(g_D\) is the Green function on \(D\) and \(x_0\) is a fixed point on \(D\). Then it is easy to check that \(u(x) \equiv (|x| - a)^{-1}\), \(x \in \gamma_2\), \(u(x) \approx 1 - |x|\), \(x \in \gamma_1\), and \(u(x) \approx 1\), \(x \in \gamma'\). So \(\gamma = \gamma_1 \cup \gamma' \cup \gamma_2\) and \(u\) satisfy the required condition. \(\square\)

**Proof of the Main Theorem.** Assume that \(D\) satisfies the condition of the Main Theorem. In general, if \(D\) is uniform, then \(D \setminus \{x\}, x \in D\), is also uniform. Thus we may assume that \(D\) has no punctures. Let \(x, y \in D\) and set \(l = |x - y|\). We may assume that \(d(x, \partial D) \leq d(y, \partial D) \leq l\). Let \(r = d(x, \partial D)/2\) and \(B_x\) (resp. \(B_y\)) denote the ball with center \(x\) (\(y\)) and radius \(r\). Let \(E = B_x \cup B_y\). From Lemma 2.1, there exists a \(H^+(D)\) function \(u\), \(u(y) = 1\), satisfying

\[
k_D(x, y) \leq C_1 \log u(x) + C.
\]

Then

\[
\int_E u^{-p}dm \geq C_{r^n} e^{pC_{r^n}^{-1}k_D(x, y)}, \quad \int_E u^{-p}dm \geq C_{r^n}, \quad |E|N_{p_0}(E) \leq C(l^{p_0r^n})^{\frac{p_0}{p_0 + n}}.
\]

Hence

\[
e^{pC_{r^n}^{-1}k_D(x, y)} \leq C \left( \frac{1}{r} \right)^{\frac{2p_0}{p_0 + n}},
\]

and so \(D\) is a uniform domain. \(\square\)
3. Some other direct consequences of Lindqvist’s theorem

In the present section, we list some other consequences of Lindqvist’s theorem and of the author [3]. For a weight \( w \) on a ball \( B \), we set

\[
M_p(w, B) = \begin{cases} 
\text{ess sup}_B w, & p = \infty, \\
\left( |B|^{-1} \int_B w^p \, dm \right)^{\frac{1}{p}}, & p \neq 0, p \neq \pm \infty, \\
\exp\left( |B|^{-1} \int_B \log w \, dm \right), & p = 0, \\
\text{ess inf}_B w, & p = -\infty.
\end{cases}
\]

\( M_p(w, B) \) is a non-decreasing function of \( p \). Let \( A(D) \) denote the space of all balls \( B \) on \( D \) satisfying \( d(B, \partial D) \geq \text{rad}(B) \), where \( \text{rad}(B) \) denotes the radius of \( B \). We say that a weight \( w \) on a domain \( D \) satisfies the \( A_\infty \) condition locally on \( D \) \((w \in A_{\infty}^{\text{loc}}(D))\) if \( 0 < M_1(w, B) \leq K M_0(w, B) < \infty \), \( B \in A(D) \), for some \( K > 0 \). The typical example of \( A_{\infty}^{\text{loc}}(D) \) weights are given by

\[
w = d(\cdot, \partial D)^\alpha (k_D(\cdot, x_0) + 1)^\beta u^\gamma, \quad \alpha, \beta \in \mathbb{R}, \quad -\infty < \gamma < -\frac{n}{n-2}, \quad u \in S^+(D).
\]

**Lemma 3.1.** Let \( u \in S^+(D) \), \( -\infty \leq p < \frac{n}{n-2} \) \((\frac{n}{n-2} = \infty \) if \( n = 2 \), and \( B \in A(D) \). Then \( 0 < M_p(u, B) \leq C M_{-\infty}(u, B) < \infty \), where \( C = C(n, p) > 0 \).

**Proof.** Let \( g_D(\cdot, y) \) be the Green function on \( D \) with pole \( y \). We may assume \( 1 \leq p < \frac{n}{n-2} \). From the Harnack inequality, it is easy to check that \( M_p(g_D(\cdot, y), B) \leq C M_-\infty(g_D(\cdot, y), B), B \in A(D) \). Since each \( S^+(D) \) function can be approximated by an increasing sequence of Green potentials, we may assume that \( u \) is a Green potential of a positive measure \( \nu \) on \( D \). Then

\[
M_p(u, B) \leq \int_D M_p(g_D(\cdot, y), B) \, d\nu(y) \leq C \int_D M_-\infty(g_D(\cdot, y), B) \, d\nu(y) \leq C M_-\infty(u, B).
\]

\( \blacksquare \)

If \( w \in A_{\infty}^{\text{loc}}(D) \), then \( \log w \in BMO(D) \). So Lemma 3.1 gives another proof of Lindqvist’s theorem. Let \( \phi \) be a non-negative, non-decreasing, continuous function on \([0, \infty)\) such that \( \phi(t) > 0, t > 0 \). We say that \( \phi \) is tame if \( \phi(t+1) \leq C \phi(t), t \geq 1 \). Let \( x_0 \in D \), and let \( B_0 \) be the ball in \( D \) with center \( x_0 \) and radius \( d(x_0, \partial D)/2 \). Then from [3], Theorem 2.3, we have

**Theorem 3.1.** Let \( \phi \) be tame, \( D \) a proper subdomain of \( \mathbb{R}^n \), \( w \) a \( A_{\infty}^{\text{loc}}(D) \) weight with constant factor \( K \), and \( E \) a measurable subset of \( D \). Assume that

\[
\int_E \phi(p_0 k_D(\cdot, x_0)) \, w dm < \infty
\]

for some \( p_0 > 0 \). Then for each \( p, 0 < p < C_1 \min\{p_0, 1\} \), and each \( u \in S^+(D) \), we have

\[
\int_E \phi(p) \log u - (\log u)_{B_0} \, w dm \leq C_2 \left( \int_{B_0} w dm + \int_E \phi(p_0 k_D(\cdot, x_0)) \, w dm \right),
\]

where \( C_1 = C_1(n, K, \phi) > 0 \) and \( C_2 = C_2(n, K, \phi, p_0) > 0 \).
This result has an advantage in that it gives an estimation of $S^+(D)$ functions not only from above but also from below. It is to be noted that from Lemma 3.1, we may replace the constant $(\log u)_{B_0} (= \log M_0(u, B_0))$ with $\log M_p(u, B_0)$, $-\infty \leq p < \frac{n}{n-2}$, in particular, with $\log(\min_{p_0} u)$ or $\log(u_{B_0})$. As to $H^+(D)$ functions, Theorem 3.1 is rather trivial, because the pointwise version $|\log u(x) - \log u(x_0)| \leq Ck_D(x, x_0)$ holds by the Harnack inequality. In the case of $\phi(t) = e^t$, $t^p$, we have

**Corollary 3.1.** Let $D$, $w$ and $E$ be as above. Assume that $\int_E e^{p_0 k_D(\cdot, x_0)} wdm < \infty$ for some $p_0 > 0$. Then for each $p$, $0 < p < C_1 \min\{p_0, 1\}$, we have

$$\int_D u^{\pm p} wdm \leq C_2 (u_{B_0})^{\pm p} \left( \int_{B_0} wdm + \int_E e^{p_0 k_D(\cdot, x_0)} wdm \right), \quad u \in S^+(D),$$

where $C_1 = C_1(n, K) > 0$ and $C_2 = C_2(n, K, p_0) > 0$.

**Corollary 3.2.** Let $D$, $w$ and $E$ be as above. Let $0 < p < \infty$. Assume that $\int_E k_D^p(\cdot, x_0) wdm < \infty$. Then

$$\int_D |\log u|^p wdm \leq C \left( \int_{B_0} wdm + \int_E k_D(\cdot, x_0)^p wdm \right), \quad u \in S^+(D), \ u_{B_0} = 1,$$

where $C = C(n, K, p) > 0$.

As to the non-weighted case, from [3], Corollary 5.2, we have

**Theorem 3.2.** Let $D$ be a proper subdomain of $\mathbb{R}^n$ and $E$ a measurable subset of $D$. Assume that $\int_E e^{p_0 k_D(\cdot, x_0)} dm < \infty$ for some $p_0 > 0$. Then $N_{p_0}(E) < \infty$ and for each $p$, $0 < p < C_1 \min\{p_0, 1\}$, and each $u \in S^+(D)$, we have

$$\left( \int_E u^p dm \right) \left( \int_E u^{-p} dm \right) \leq C_2 \left( \left| E \right| N_{p_0}(E)^2 + \left( \inf_{x_0 \in \partial D} \int_E e^{p_0 k_D(\cdot, x_0)} dm \right)^2 \right),$$

where $C_1 = C_1(n) > 0$, and $C_2 = C_2(n, p_0) > 0$.

Note that, in general, we can omit neither the first term nor the second term on the right side of the inequality (cf. the Main Theorem).

### 4. Remarks on the Hölder Domain

In the present section, we give some characterizations of finitely connected Hölder domains in $\mathbb{R}^2$. First, we show the following analogy of Proposition 1.2. Recall that $B_0$ is the disk with center $x_0$ and radius $d(x_0, \partial D)/2$.

**Theorem 4.1.** If $D$ is a Hölder domain in $\mathbb{R}^n$, then $\int_D u^{-p} dm \leq C(u_{B_0})^{-p}$, $u \in S^+(D)$, for some $p, C > 0$. Conversely, if $D$ is a finitely connected subdomain of $\mathbb{R}^2$ and $\int_D u^{-p} dm \leq C(u_{B_0})^{-p}$, $u \in S^+(D)$, for some $p, C > 0$, then $D$ is a Hölder domain.

**Proof.** Assume that $D \subset \mathbb{R}^2$ is finitely connected and that $\int_D u^{-p} dm \leq C(u_{B_0})^{-p}$, $u \in S^+(D)$, for some $p, C > 0$. Since positive constants are in $S^+(D)$, $D$ must have finite area. In general, if $D$ is Hölder, then $D \setminus \{x\}, x \in D$, is also Hölder. So we may assume that $D$ has no punctures. Let $x \in D$. From Lemma 2.1, we can take $u \in H^+(D), u(x_0) = 1$, so that $k_D(x, x_0) \leq -C_1 \log u(x) + C$. Then

$$d(x, \partial D)^2 \exp(pC_1^{-1}k_D(x, x_0)) \leq C \int_{B_0} u^{-p} dm \leq C,$$
so $D$ is Hölder. The remaining implication follows from Corollary 3.1 and Proposition 1.1.

Next, we consider another class of harmonic functions. Let $QLH(D)$ be the space of all harmonic, Lipschitz continuous functions $h$ on $D$ with respect to the quasihyperbolic metric endowed with the norm

$$\|h\|_L = \sup_{x \in D} |\nabla h(x)|d(x, \partial D).$$

Then $QLH(D)$ agrees with the space of all harmonic $BMO(D)$ functions, and the norms $\| \cdot \|_{s, D}$ and $\| \cdot \|_L$ are comparable with constant factors depending only on $n$. Smith-Stegenga [9] showed that a domain $D$ is a Hölder domain if and only on $D$, $\|f\|_{s, D} \leq 1$. We show that we may replace $BMO(D)$ with its subspace $QLH(D)$ under some additive condition:

**Theorem 4.2.** Let $D$ be a proper subdomain of $\mathbb{R}^2$ which is conformally equivalent to some Hölder domain. Assume that there exist constants $p, C > 0$ such that

$$\int_D e^{p|h-h(x_0)|}dm \leq C, \quad h \in QLH(D), \; \|h\|_L \leq 1.$$  

Then $D$ is a Hölder domain.

**Lemma 4.1.** Let $D$ be as above. Then there exist constants $C_1, C_2 > 0$ depending only on $D$ and $x_0$ such that for each $x \in D$ we can find a real $QLH(D)$ function $h$, $\|h\|_L \leq 1$, $h(x_0) = 0$, so that $k_D(x, x_0) \leq C_1h(x) + C_2$.

**Proof.** Since $k_D$ (and so $QLH(D)$) is conformally invariant, we may assume that $D$ is a Hölder domain from the beginning. Let $x \in D$. Take $x' \in \partial D$ so that $d(x, \partial D) = |x - x'|$. Let $h(y) = \log |y - x'|$. Then $\|h\|_L \leq C$ and

$$|h(x) - h(x_0)| \geq \log \frac{1}{d(x, \partial D)} - C \geq Ck_D(x, x_0) - C.$$

**Proof of Theorem 4.2.** Assume that $D$ satisfies the condition of Theorem 4.2 with $p = p_0$. Let $x \in D$. Take a real $QLH(D)$ function $h$ satisfying the condition of the lemma above. Let $B = \{y \mid |y - x| \leq d(x, \partial D)\}$. Then $k_D(x, x_0) \leq C_1h(y) + C$, $y \in B$, and so for $p = p_0C_1^{-1}$, we have

$$d(x, \partial D)^2e^{p k_D(x, x_0)} \leq C \int_B e^{ph}dm \leq C.$$

Thus $D$ is a Hölder domain.

Combining Proposition 1.2, and Theorems 4.1, 4.2, we have

**Corollary 4.1.** For a finitely connected proper subdomain $D$ of $\mathbb{R}^2$, the following conditions are equivalent:

1. $D$ is a Hölder domain;
2. $\int_D u^p dm \leq Cu(x_0)^p$, $u \in S^+(D)$, holds for some $p, C > 0$;
3. $\int_D u^{-p} dm \leq C u(x_0)^{-p}$, $u \in S^+(D)$, holds for some $p, C > 0$;
4. $\int_D e^{p|h-h(x_0)|/\|h\|_L} dm \leq C$, $h \in QLH(D)$, holds for some $p, C > 0$.  

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Note that since \( \log |x - x_1|, x_1 \in \partial D \), belongs to \( QLH(D) \), if \( D \) satisfies the condition (4), then \( D \) has finite area, and so is conformally equivalent to some Hölder domain.

Finally in the present section, we give a remark on Lipschitz domains. We say that a bounded domain \( D \) in \( \mathbb{R}^n \) is \( k \)-Lipschitz \((k > 0)\) if \( D \) and \( \partial D \) are given locally by a Lipschitz function whose Lipschitz constant is at most \( k \). Each Lipschitz domain is a uniform domain. For various estimations of \( S^+(D) \) functions on Lipschitz domains, see Maeda-Suzuki [5], Masumoto [6], and Aikawa [1]. Let \( \alpha = \alpha_n(\tan^{-1}(1/k)) \), where \( \alpha_n \) denotes the maximal order of barriers (cf. [5]). Let \( D \) be a \( k \)-Lipschitz domain. Then it is known that if

\[
0 < p < \min\{n/(n + \alpha - 2), 1/(\alpha - 1)\},
\]

then \( \int_D u^p dm \leq C u(x_0)^p, u \in S^+(D) \), holds ([6], [1]). Moreover, from the estimation \( u(x) \geq C d(x, \partial D)^{\alpha} u(x_0), u \in H^+(D) \), it is easy to see that if \( 0 < p < 1/\alpha \), then \( \int_D u^{-p} dm \leq C u(x_0)^{-p}, u \in H^+(D) \), holds.

5. Boundedness of domains with some integrability condition

In the present section, we give another application of Lindqvist’s theorem and [3]. We investigate integrability conditions for \( H^+(D) \) which ensure the boundedness of \( D \). Proposition 1.2 shows that if \( D \) is a finitely connected proper subdomain of \( \mathbb{R}^2 \), and at least one boundary component contain more than two points. Assume

\[
\int_D \phi(p \log^+ u) dm \leq C_1, \quad u \in H^+(D), \quad u(x_0) = 1,
\]

holds for some \( p, C_1 > 0 \). Then for each \( x \in D \) we can take an arc \( \gamma \) on \( D \) joining \( x_0 \) to \( x \) so that \( |\gamma| \leq C_2 \), where \( C_2 = C_2(D, x_0, \phi, p) > 0 \). In particular, \( D \) is bounded.

(1) In (1), we may replace \( \log^+ u \) with \( \log^+ \frac{1}{u} \).

(2) Conversely, let \( \int_1^{\infty} \phi(t)^{\frac{1}{pt^k}} dt = \infty \). Then there exists an unbounded proper subdomain \( D \) of \( \mathbb{R}^n \) which is homeomorphic to an open ball satisfying

\[
\int_D \phi(p |\log u|) dm \leq C, \quad u \in S^+(D), \quad u_{B_0} = 1,
\]

for some \( p, C > 0 \).

Proof of Theorem 5.1. (2) follows from Theorem 3.1, Lindqvist’s theorem, and [3], Theorem 6.1. Next, assume that \( \phi \) and \( D \) satisfy the condition in (1) with \( p = p_0 \). We may assume that \( D \) has no punctures. Let \( x \in D \). From Lemma 2.1, we can take a pair of an arc \( \gamma : x = x(t), 0 \leq t \leq a \), joining \( x_0 \) to \( x \) and a \( H^+(D) \) function \( u, u(x_0) = 1 \), satisfying \( \int_{\gamma_0} \frac{d}{(\partial D)} \leq C \log u(y) + C, y \in \gamma \). Let \( t_0 = 0 \) and set \( t_1 = \max\{t > t_0 \mid |x(t) - x(t_0)| \leq d(x(t_0), \partial D)/2\} \). If \( t_1 < a \), then set \( t_2 = \max\{t > t_1 \mid |x(t) - x(t_1)| \leq d(x(t_1), \partial D)/2\} \). Repeating this process, we...
obtain a sequence \(0 = t_0 < t_1 < \cdots < t_{k-1} < t_k = a\). We may assume \(k \geq 4\). Set \(x_j = x(t_j)\). Since \(\int_{t_j}^{t_{j+1}} d(x(t), \partial D)^{-1} \left| dx(t) \right| \geq C, 0 \leq j \leq k - 2\), we have
\[
\int_{t_j}^{t_{j+1}} \left| dx(t) \right| = C_1 \log u(x_j) + C, \quad 0 \leq j \leq k - 1.
\]
Let \(B_j, j \geq 2\), be the ball with center \(x_j\) and radius \(r_j = d(x_j, \partial D)/10\). Then \(B_j, 0 \leq j \leq k - 1\), are disjoint. Thus, for \(p = p_0 C^{-1} 1\), we have
\[
\sum_{j=2}^{k-1} r_j \leq \left( \sum_{j=2}^{k-1} \phi(p_j)^{-1} \right)^{1/2} \left( \sum_{j=2}^{k-1} \phi(p_j)^{r_j^2} \right)^{1/2}
\]
\[
\leq C \left( \int_{D} \phi(t)^{-1} dt \right)^{1/2} \left( \int_{D} \phi(p_0 \log^+ u) dm \right)^{1/2}.
\]
Let \(\gamma'\) be the associated polygon joining \(x_0, x_1, \ldots, x_k\). Then \(|\gamma'| \leq C \sum_{j=2}^{k-1} r_j\), hence (1) follows. Finally, if we take \(u \in H^+(D), u(x_0) = 1\), so that \(\int_{\gamma'} \frac{ds}{u^\alpha(\partial D)} \leq -C \log u(\gamma) + C, y \in \gamma\), and repeating the argument above, we get (1)'.

**Corollary 5.1.**

(1) Let \(1 < p < \infty\). Let \(D\) be a finitely connected proper subdomain of \(\mathbb{R}^2\) and at least one boundary component contain more than two points. Assume that
\[
\int_{D} (\log^+ u)^p dm \leq C, \quad u \in H^+(D), \quad u_{B_0} = 1,
\]
for some \(C > 0\). Then \(D\) is bounded.

(1)' In (1), we may replace \(\log^+ u\) with \(\log^+ \frac{1}{u}\).

(2) Let \(0 \leq p \leq n - 1\). Then, there exists an unbounded proper subdomain \(D\) of \(\mathbb{R}^n\) which is homeomorphic to an open ball satisfying
\[
\int_{D} |\log u|^p dm \leq C, \quad u \in S^+(D), \quad u_{B_0} = 1,
\]
for some \(C > 0\).

**References**


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