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AN ELEMENTARY PROOF OF THE PRINCIPLE OF LOCAL REFLEXIVITY

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Abstract. We give an elementary proof of the principle of local reflexivity.

We use only elementary functional analysis to give a simple and short proof of the version of the "principle of local reflexivity" proved in [2], which is an improvement of the original version given in [3]. Other short proofs can be found in [1] and [4]. We use standard notation for Banach spaces. By X, X^* and X^{**}, we denote a real or complex Banach space, its first dual and its second dual respectively; we identify X with the canonical copy of X contained in X^{**}; given a Banach space Y, we write $B_Y := \{y : \|y\| \leq 1\}$ and $S_Y := \{y : \|y\| = 1\}$; given a subset A of X, $\overline{A}^{\sigma(X^{**},X^*)}$ stands for the *weak**-closure of A in X^{**} and $intA$ is the norm interior of A; an operator is a continuous linear function; given $\varepsilon > 0$, an ε-isometry $T : E \longrightarrow Y$ is an operator for which $1 - \varepsilon \leq \|Tx\| \leq 1 + \varepsilon$ for all $x \in S_E$.

We only require the following Lemma 1. We omit its proof, which is an easy exercise based on the separation Hahn-Banach theorem.

Lemma 1. *Let $T : X \longrightarrow Y$ be an operator, $z \in intB_{X^{**}}$ and $y \in Y$ such that $\|T^{**}z - y\| < \varepsilon$. Then we have that $z \in \overline{L}^{\sigma(X^{**},X^*)}$, where $L := \{x \in B_X : \|Tx - y\| < \varepsilon\}$.*

Theorem 2 (Principle of local reflexivity)**.** *Let $E \subset X^{**}$ and $F \subset X^*$ be finite dimensional subspaces. Given $\varepsilon > 0$ there exists an ε-isometry $T : E \longrightarrow X$ such that $T|_{E\cap X} = id|_{E\cap X}$, and $f(Te) = e(f)$ for all $f \in F$ and all $e \in E$.*

Proof. Let $\dim E = n$ and $\dim E \cap X = n - k$. Let $(y_j, h_j)_{j=1}^n$ be a biorthogonal system in $E \times E^*$ such that $\|y_j\| = 1 - \varepsilon$ and $\operatorname{span}\{y_j\}_{j=k+1}^n = E \cap X$. The identity $id : E \longrightarrow X^{**}$ can be given as $id(e) = \sum_{j=1}^n h_j(e)y_j$. We shall find $v_1, \dots, v_k$ in X so that the operator $T : E \longrightarrow X$ defined by $T(e) := \sum_{j=1}^k h_j(e)v_j + \sum_{j=k+1}^n h_j(e)y_j$ is an ε-isometry. Hence, the condition $T|_{E\cap X} = id|_{E\cap X}$ will be satisfied automatically.

Let $W := X^k$ endowed with the norm $\|(x_j)_{j=1}^k\| = \sup_j \|x_j\|$, and select $0 < \alpha < \min\{2/5, (1-\varepsilon)^{-1} - 1, \varepsilon(\sum_{j=1}^n \|h_j\|)^{-1}\}$. Fix $\{f_j\}_{j=1}^M$ a basis in F, $\{e_j\}_{j=1}^N$

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an $\alpha/4$-net in $intB_E$, and $\{u_j\}_{j=1}^N$ in B_{X^*} so that $\|e\| \leq (1+\alpha)\sup_{1\leq j\leq N}|e(u_j)|$ for all $e \in E$. We have that

$$e_j = \sum_{r=1}^{n} \lambda_r^j y_r, j = 1, \ldots, N.$$

Let us write $P := \max_{1\leq j\leq N} \sum_{r=1}^k |\lambda_r^j|$ and define the set

$$C := \left\{ (x_s)_{s=1}^k \in B_W : \left\| \sum_{s=1}^k \lambda_s^j x_s + \sum_{s=k+1}^n \lambda_s^j y_s \right\| < 1, j = 1, \ldots, N \right\}.$$

By the above lemma, we have that $(y_j)_{j=1}^k \in \overline{C}^{\sigma(W^{**},W^*)}$. Now we set the operator $S : W \longrightarrow \mathbf{R}^{M\cdot k + N\cdot k}$ (or into $\mathbf{C}^{M\cdot k + N\cdot k}$) given by $S((x_s)_{s=1}^k) := (f_i(x_r), u_j(x_s))$ for $1 \leq i \leq M$, $1 \leq r \leq k$, $1 \leq j \leq N$, $1 \leq s \leq k$.

Thus $S^{**}((y_j)_{j=1}^k) \in \overline{S(C)}$. Now, since $\overline{W}^{\sigma(W^{**},W^*)} = W^{**}$ and $R(S)$ is closed, we have that $R(S) = R(S^{**})$, and then, for $0 < \beta < \min\{1, \varepsilon(2P)^{-1}\}$, we can find $(c_j)_{j=1}^k \in C$ and $(b_j)_{j=1}^k \in \beta B_W$ so that

$$S^{**}((y_j)_{j=1}^k) = S((c_j)_{j=1}^k) + S((b_j)_{j=1}^k).$$

We take $v_j := c_j + b_j$ for $j = 1, \ldots, k$ in the definition of T. Thus, we already have the condition $f(Te) = e(f)$ for all $f \in F$ and all $e \in E$. Now, since $\|Te_j\| \leq 1 + \|\sum_{r=1}^k \lambda_r^j b_r\| \leq 1 + \beta P$ for $j = 1, \ldots, N$, it is completely straightforward to check that T is an ε-isometry.

□

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