THE LOCATION OF THE ZEROS OF THE HIGHER ORDER
DERIVATIVES OF A POLYNOMIAL

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Abstract. Let $p(z)$ be a complex polynomial of degree $n$ having $k$ zeros in a disk $D$. We deal with the problem of finding the smallest concentric disk containing $k-l$ zeros of $p^{(l)}(z)$. We obtain some estimates on the radius of this disk in general as well as in the special case, where $k$ zeros in $D$ are isolated from the other zeros of $p(z)$. We indicate an application to the root-finding algorithms.

1. Introduction

Let us consider the following problem: If $k$ zeros of a polynomial $p(z)$ of degree $n$ $(2 \leq k \leq n)$ lie in a disk $D$ of radius $r$, what is the smallest concentric disk that contains $k-l$ zeros of $p^{(l)}(z)$ $(1 \leq l \leq k-1)$? Since the problem is scaling and translation invariant, we can assume that the disk $D$ is the closed unit disk $\Delta := \{ z \in \mathbb{C} : |z| \leq 1 \}$. Let $P_{n,k}$ denote the class of complex polynomials of degree $n$ having exactly $k$ zeros in $\Delta$. We define the function $\rho(n, k, l)$, $n \geq k > l$, as follows:

$$(1.1) \quad \rho(n, k, l) = \sup_{p \in P_{n,k}} \min \left\{ R > 0 : D(0, R) \text{ contains at least } k-l \text{ zeros of } p^{(l)}(z) \right\}.$$  

Because of scaling and translation invariance we can conclude that if $D(c, r)$ contains $k$ zeros of the polynomial $p(z)$, then $D(c, r\rho(n, k, l))$ contains $k-l$ zeros of $p^{(l)}(z)$.

The problem of estimating $\rho(n, k, l)$ has a long history in the case $l = 1$. The results listed below can be found in Marden’s book ([2]). The Gauss-Lucas Theorem states that $\rho(n, n, 1) = 1$. Result $\rho(n, 2, 1) = \cot(\pi/n)$ is due to Alexander, Kakeya and Szegő. Biernacki proved that

$$(1.2) \quad \rho(n, n-1, 1) \leq (1 + 1/n)^{1/2} \quad \text{and} \quad \rho(n, k, 1) \leq \prod_{i=1}^{n-k} \frac{((n+i)/(n-i)},$$

and Marden showed that

$$(1.3) \quad \rho(n, k, 1) \leq \csc \frac{\pi}{2(n-k+1)}.$$
More recently, Coppersmith and Neff ([1]) proved that, under the condition that \( k \) zeros in \( \Delta \) are centered at 0,

\[
\rho(n, k, l) \leq 10.477 \frac{(n - k + 1)(k - l)}{\sqrt{k}}
\]

and

\[
\rho(n, k, k - 1) \leq C \max \left\{ (n - k + 1)^{1/2}k^{-1/4}, (n - k + 1)^{-2/3} \right\}.
\]

One can also impose an additional condition on \( p \in P_{n,k} \) to have zeros in \( \Delta \) isolated from the other zeros. Following Pan ([3]), we define the isolation ratio of a polynomial \( p \) with respect to a disk \( D \), \( I(p, D) \), as:

\[
I(p, D) := \sup \{ \mu > 0 : D(c, \mu r) \text{ contains exactly the same zeros of } p \text{ as } D(c, r) \}.
\]

We say that a disk \( D \) is \( f \)-isolated if \( I(p, D) \geq f \). The isolation ratio is also scaling and translation invariant. We define the isolation ratio of \( p \) as \( I(p) := I(p, \Delta) \).

Renegar ([5]) proved the following result:

\[
I(p, D(c, r)) := \sup \{ \mu > 0 : D(c, \mu r) \text{ contains exactly the same zeros of } p \text{ as } D(c, r) \}.
\]

We say that a disk \( D \) is \( f \)-isolated if \( I(p, D) \geq f \). The isolation ratio is also scaling and translation invariant. We define the isolation ratio of \( p \) as \( I(p) := I(p, \Delta) \).

Renegar ([5]) proved the following result:

\[
I(p) \geq 15n^3, \quad \text{then } \rho(n, k, k - 1) \leq \frac{3n}{2} \text{ and } I \left( p^{(k-1)}, \overline{D}(0, 3n/2) \right) \geq \frac{I(p)}{10n^2}.
\]

In other words, if \( k \) zeros of a polynomial \( p \) are well-isolated in some disk, then \( p^{(k-1)} \) has an isolated single zero in a larger concentric disk. Using Walsh’s Coincidence Theorem ([2]) we improve the bound in (1.7). We also generalize Biernacki’s proof of (1.2) and obtain an upper bound for \( \rho(n, k, l) \) which is smaller than (1.4) for some special choices of \( n, k \) and \( l \).

Results of this type have found applications in constructing low complexity algorithms for finding zeros of polynomials. Smale and Renegar ([7], [5]) derived quantitative criteria for a point to be in the domain of quadratic convergence of Newton’s algorithm. The basic idea is that if a zero is isolated from other zeros, then we need only crude approximation to obtain fast convergence. However, if we have an isolated cluster of \( k \) zeros, then \( p^{(k-1)} \) should have a single zero nearby. In this case, the result like (1.7) allows us to apply Newton’s algorithm to \( p^{(k-1)} \) with quadratic convergence. To establish initial isolation of clusters of zeros, various global search algorithms are used ([4], [5]).

2. Main results

First we prove the following proposition:

**Proposition 2.1.**

\[
(2.1) \quad \text{If } I(p) \geq 1 + 2 \frac{l(n-k)}{k-l+1}, \text{ then } \rho(n, k, l) \leq 1 \text{ and } \frac{I(p^{(l)})}{k-l+1+l(n-k)} \geq \frac{(k-l+1)I(p) - l(n-k)}{k-l+1+l(n-k)}.
\]

**Proof.** Let \( I(p) = R \) and let \( p(z) = f(z)g(z) \) where \( f(z) \) is a polynomial of degree \( k \) whose zeros lie in \( \Delta \) and \( g(z) \) is a polynomial of degree \( n - k \) whose zeros lie in \( \Delta \). For any disk \( D \) containing \( \Delta \), we have

\[
I(p^{(l)}) = \sup \{ \mu > 0 : D(c, \mu r) \text{ contains exactly the same zeros of } p^{(l)} \text{ as } D(c, r) \}.
\]

We say that a disk \( D \) is \( f \)-isolated if \( I(p^{(l)}, D) \geq f \). The isolation ratio is also scaling and translation invariant. We define the isolation ratio of \( p^{(l)} \) as \( I(p^{(l)}) := I(p^{(l)}, \Delta) \).

Renegar ([5]) proved the following result:

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Renegar ([5]) proved the following result:

\[
I(p^{(l)}) \geq 15n^3, \quad \text{then } \rho(n, k, k - 1) \leq \frac{3n}{2} \text{ and } I \left( p^{(k-1)}, \overline{D}(0, 3n/2) \right) \geq \frac{I(p^{(l)})}{10n^2}.
\]
\{z : |z| > R\}. Let \(m = \min(n-k, l)\). If \(z\) is a solution of the equation

\[
p^{(l)}(z) = \sum_{i=0}^{m} \binom{l}{i} g^{(i)}(z)f^{(l-i)}(z) = 0,
\]
then the Walsh Coincidence Theorem implies that there exist \(x_1, |x_1| \leq 1\) and \(x_2, |x_2| \geq R\) such that

\[
\sum_{i=0}^{m} \binom{l}{i} ((z-x_2)^{n-k})^{(i)} ((z-x_1)^{k})^{(l-i)} = 0,
\]

i.e.

\[
\sum_{i=0}^{m} \binom{l}{i} \frac{(n-k)!}{(n-k-i)!}(z-x_2)^{n-k-i} \frac{k!}{(k-l+i)!}(z-x_1)^{k-l+i} = 0,
\]

i.e.

\[
(z-x_1)^{k-l}(z-x_2)^{n-k-m} \sum_{i=0}^{m} \binom{l}{i} \frac{n-l}{n-k-i} (z-x_2)^{m-i}(z-x_1)^i = 0.
\]

Also \(z\) must lie in the interval \([x_1, x_2]\) and either \(z = x_1\) or \(z = x_2\) or \(z\) is a zero of the polynomial

\[
q(z) = \sum_{i=0}^{m} \binom{l}{i} \frac{n-l}{n-k-i}(z-x_2)^{m-i}(z-x_1)^i.
\]

Let \(w = \frac{z-x_1}{z-x_2}\) and let

\[
h(w) = \sum_{i=0}^{m} \binom{l}{i} \frac{n-l}{n-k-i} w^i = \sum_{i=0}^{m} a_i w^i.
\]

All zeros of \(h(w)\) lie on the negative real axis and by the Eneström-Kakeya Theorem they satisfy the inequality

\[
\min_{0 \leq i \leq m-1} \frac{a_i}{a_{i+1}} \leq |w| \leq \max_{0 \leq i \leq m-1} \frac{a_i}{a_{i+1}}.
\]

We have that

\[
\frac{a_i}{a_{i+1}} = \frac{(i+1)(k-l+i+1)}{(l-i)(n-k)};
\]

therefore

\[
\zeta := \min_{0 \leq i \leq m-1} \frac{a_i}{a_{i+1}} = \frac{k-l+1}{l(n-k)}
\]

and

\[
\xi := \max_{0 \leq i \leq m-1} \frac{a_i}{a_{i+1}} = \frac{m(m+k-l)}{(l-m+1)(n-k-m+1)}.
\]

Since all zeros of \(h(w)\) lie in the interval \([-\xi, -\zeta]\), all zeros of \(q(z)\) lie in the interval \([-\zeta, -\xi]\). By continuity, \(p^{(l)}(z)\) has \(k-l\) zeros in \(\mathbb{R}\) if \(\left|\frac{c \xi_2 + \xi_1}{c+1}\right| \geq 1\). Since

\[ |z_k + z_{k+1}| \leq \frac{cR}{k+1}, \text{ the above inequality holds if } \frac{cR}{k+1} \geq 1, \text{ i.e. if } R \geq \left(1 + \frac{2}{\gamma}\right) = \left(1 + 2\frac{(n-k)}{k-l+1}\right). \text{ These are the only zeros of } p^{(l)}(z) \text{ in the disk} \]

\[ D \left(0, \frac{cR}{\zeta + 1}\right) = D \left(0, \frac{(k-l+1)R - l(n-k)}{k-l+1 + l(n-k)} \right). \]

We use Proposition 2.1 to prove the following result analogous to (1.7).

**Theorem 2.2.**

\[ (2.13) \quad \text{If } I(p) \geq \frac{n^2}{4}, \text{ then } \rho(n, k, k-1) \leq 1 \text{ and } I(p^{(k-1)}) \geq \frac{4I(p)}{n^2}. \]

**Proof.** If \( l = k - 1 \) and \( I(p) \geq 1 + (k-1)(n-k) \), then, by Proposition 2.1, \( \Sigma \) contains a zero of \( p^{(k-1)} \). Since \( 1 + (k-1)(n-k) \leq k(n-k) \leq n^2/4 \), the same is true if \( I(p) \geq n^2/4 \). Also,

\[ (2.14) \quad I(p^{(k-1)}) \geq \frac{2I(p) - (k-1)(n-k)}{2 + (k-1)(n-k)} > \frac{I(p)}{1 + (k-1)(n-k)} \geq \frac{I(p)}{k(n-k)} \geq \frac{4I(p)}{n^2}. \]

**Remark 2.3.** We don’t know of any estimates for \( \rho(n, k, k-1) \) if the isolation ratio of \( p \) is smaller than \( n^2/4 \). In applications to zero-finding we can use the classical Graeffe process of root-squaring to control the isolation ratio of a polynomial ((3],[6]). If \( p_0 \) has zeros \( z_1, \ldots, z_n \) and isolation ratio \( C > 1 \), then the polynomial

\[ (2.15) \quad p_1(z) := p_0(\sqrt{z})p_0(-\sqrt{z}) \]

has zeros \( z_1^2, \ldots, z_n^2 \) and isolation ratio \( C^2 \). By iterating this process we obtain polynomial \( p_m \) with zeros \( z_1^{2^m}, \ldots, z_n^{2^m} \) and isolation ratio \( C^{2^m} \).

We can also apply Proposition 2.1 to obtain an upper bound for \( \rho(n, k, l) \).

**Theorem 2.4.**

\[ (2.16) \quad \rho(n, k, l) \leq \frac{n+l}{n-l} \prod_{i=k}^{n-2} \left(1 + \frac{2l(n-l)}{i-l+1}\right). \]

**Proof.** Let us assume that the zeros of \( p(z) \) are ordered by their moduli, \( |z_1| \leq |z_2| \leq \ldots \leq |z_n| \). If \( k = n-1 \), then \( p(z) = f(z)(z - z_n) \), where \( f(z) \) is a polynomial with zeros \( z_i, |z_i| \leq 1, 1 \leq i \leq n-1 \). We have

\[ (2.17) \quad p^{(l)}(z) = f^{(l)}(z)(z - z_n) + lf^{(l-1)}(z). \]

By the Walsh Coincidence Theorem, if \( z \) is a zero of \( p^{(l)}(z) \) it is also a zero of the equation:

\[ (2.18) \quad (z - \gamma)^{n-1-l}(z - z_n) + l((z - \gamma)^{n-1})(z - z_n) = 0 \]

where \( |\gamma| \leq 1 \), i.e.,

\[ (z - \gamma)^{n-1-l}(z - (n-l)(z - z_n) + l(z - \gamma)) = 0. \]
Therefore \( z = \gamma \) or \( z = ((n - l)z_n + l\gamma)/n \). Also if \( |((n - l)z_n + l\gamma)/n| > 1 \), then there are exactly \( k - 1 - l \) zeros of \( p^{(l)}(z) \) in \( \Delta \). This is the case if \( |z_n| > \frac{n+l}{n-l} \). Otherwise, by the Gauss-Lucas Theorem, all zeros of \( p^{(l)}(z) \) are in the disk \( D \left( 0, \frac{n+l}{n-l} \right) \).

Now let’s fix \( K \) and suppose that (2.16) holds for \( k = n - 1 \ldots K - 1 \). By Proposition 2.1, if \( |z_{K+1}| > 1 + \frac{2(n-K)}{R-1+1} \), then there are exactly \( K - l \) zeros of \( p^{(l)}(z) \) in \( \Delta \). Otherwise there are \( K + 1 \) zeros of \( p(z) \) in the disk \( D \left( 0, 1 + \frac{2(n-K)}{K-1+1} \right) \) and by induction there are at least \( K - l \) zeros of \( p^{(l)}(z) \) in \( D(0,R) \), where

\[
R \leq \frac{n+l}{n-l} \prod_{i=K}^{n-2} \left( 1 + \frac{2(n-i)}{i-l+1} \right). \tag{2.20}
\]

Inequality (2.16) can be better in some cases than (1.4), namely if \( k \) and \( l \) are close to \( n \). In the most interesting case \( l = k - 1 \), Coppersmith and Neff gave a lower bound on \( \rho(n,k,k-1) \) and also conjectured that if \( k > n/2 \), then \( \rho(n,k,k-1) \) is bounded by a constant.

**References**


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