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THE MAXIMAL IDEAL SPACE OF $H^{\infty}(\mathbb{D})$ WITH RESPECT TO THE HADAMARD PRODUCT

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Abstract. It is shown that the space of all regular maximal ideals in the Banach algebra $H^{\infty}(\mathbb{D})$ with respect to the Hadamard product is isomorphic to $\mathbb{N}_0$. The multiplicative functionals are exactly the evaluations at the n-th Taylor coefficient. It is a consequence that for a given function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H^{\infty}(\mathbb{D})$ and for a function $F(z)$ holomorphic in a neighborhood U of 0 with $F(0) = 0$ and $a_n \in U$ for all $n \in \mathbb{N}_0$ the function $g(z) = \sum_{n=0}^{\infty} F(a_n) z^n$ is in $H^{\infty}(\mathbb{D})$.

Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be power series on $\mathbb{D}$. Then the *Hadamard product* of f and g is defined by $f * g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. The Hadamard product on the space $H(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$ is continuous with respect to the topology of compact convergence. In [1] R. Brooks has shown that the space of all maximal ideals in the space $H(\mathbb{D})$ is isomorphic to the Stone-Čech-compactification $\beta\mathbb{N}_0$ of $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and the multiplicative functionals on $H(\mathbb{D})$ are given by the coefficient functionals $\delta_n \colon H(\mathbb{D}) \to \mathbb{C}$ defined by $\delta_n(f) := a_n$ (where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $|z| < 1$ and $n \in \mathbb{N}_0$). In this note we discuss the subalgebra $H^{\infty}(\mathbb{D})$ of all bounded holomorphic functions which has been considered for example in [3]. Our main result states that the non-trivial multiplicative functionals on $H^{\infty}(\mathbb{D})$ are of the form $\delta_n, n \in \mathbb{N}_0$ (as in the case of $H(\mathbb{D})$). In contrast to the algebra $H(\mathbb{D})$ the space $H^{\infty}(\mathbb{D})$ is even a Banach algebra with respect to the supremum norm which is denoted by $||f||_{\infty}$ for $f \in H^{\infty}(\mathbb{D})$. It follows that the maximal modular ideals of $H^{\infty}(\mathbb{D})$ are the kernels of the multiplicative functionals and therefore the space of all maximal modular ideals of $H^{\infty}(\mathbb{D})$ is isomorphic to $\mathbb{N}_0$. Note that $H(\mathbb{D})$ possesses a unit element $\gamma(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ which is not in the subalgebra $H^{\infty}(\mathbb{D})$.

The results

Let B be the space of all $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=0}^{\infty} |a_n| < \infty$; clearly, $\|f\|_{\infty} \leq \sum_{n=0}^{\infty} |a_n|$, so $B \subset H^{\infty}(\mathbb{D})$. We note that if $f = \sum_{n=0}^{\infty} a_n z^n \in H^{\infty}(\mathbb{D})$, then $\sum_{n=0}^{\infty} |a_n|^2 = \|f\|_2^2 \leq \|f\|_{\infty}^2 < \infty$, where $||f||_2 := \sqrt{\sum_{n=0}^{\infty} |a_n|^2}$. Hence for any $f, g \in H^{\infty}(\mathbb{D})$, we have $f * g \in B$, since $\sum_{n=0}^{\infty} |a_n b_n| \leq \|f\|_2 \|g\|_2$ by the

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Cauchy- Schwarz inequality; this also shows that $H^\infty(\mathbb{D})$ is a Banach algebra under Hadamard multiplication.

Proposition 1. *Let A be the Banach algebra obtained by adjoining a unit to $H^\infty(\mathbb{D})$. If $f = \sum_{n=0}^\infty a_n z^n \in B$, then $\sigma_A(f) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.*

Proof. We must show that if $\lambda \notin \{a_n : n \in \mathbb{N}_0\}$ and $\lambda \neq 0$, then $\lambda - f$ is invertible in A (the other inclusion is easy). Let $g(z) = \sum_{n=0}^\infty \frac{a_n}{\lambda - a_n} z^n$; since $|a_n| < |\lambda|/2$ for sufficiently large n, we have $|a_n/(\lambda - a_n)| \leq (2/|\lambda|)|a_n|$ for sufficiently large n, so $g \in B \subset H^\infty(\mathbb{D})$. Since

$$\lambda g - f = \sum_{n=0}^\infty \frac{\lambda a_n - \lambda a_n + a_n^2}{\lambda - a_n} z^n = f * g,$$

we see that $(\lambda - f) * (1 + g) = \lambda$, so $\lambda - f$ is invertible in A. □

The next result is the main step of our proof. Although we need it only for Banach algebras, it is valid for the larger class of all Fréchet algebras, cf. [4] for definition. We denote by Δ_A the set of all continuous multiplicative non-trivial functionals.

Theorem 2. *If A is a unital Frechet algebra, and S is a countable subset of Δ_A with the property that $\sigma_A(f^2) = \{\varphi(f^2) : \varphi \in S\}$ for all $f \in A$, then $S = \Delta_A$.*

Proof. Let $S = \{\varphi_n : n \in \mathbb{N}\}$. Suppose that there exists $\varphi \in \Delta_A \setminus S$. As $\varphi \neq \varphi_n$ for all $n \in \mathbb{N}$, the sets $A_n := \ker(\varphi_n - \varphi)$ and $B_n := \ker(\varphi_n + \varphi)$ are closed hyperplanes, in particular they are nowhere dense. By the Baire category theorem there exists $f \in A$ such that $f \notin A_n$ and $f \notin B_n$ for all $n \in \mathbb{N}$. This means that $\varphi_n(f) \neq \varphi(f)$ and $\varphi_n(f) \neq -\varphi(f)$ for all $n \in \mathbb{N}$. On the other hand we know that $\lambda := \varphi(f) \in \sigma_A(f)$ since φ is multiplicative. Hence $\lambda^2 \in \sigma_A(f^2)$. By assumption there exists $n \in \mathbb{N}$ with $\lambda^2 = \varphi_n(f^2) = (\varphi_n(f))^2$. Hence $\lambda = \varphi_n(f)$ or $\lambda = -\varphi_n(f)$, a contradiction. □

Theorem 3. *The non-trivial multiplicative functionals on $H^\infty(\mathbb{D})$ are of the form $\delta_n, n \in \mathbb{N}_0$.*

Proof. By the above, $f^2 \in B$ for all $f \in H^\infty(\mathbb{D})$. Now apply Proposition 1 and Theorem 2. □

Theorem 4. *Let U be an open neighborhood of zero and $F\colon U \to \mathbb{C}$ holomorphic with $F(0) = 0$. If $f(z) = \sum_{n=0}^\infty a_n z^n$ is in $H^\infty(\mathbb{D})$ and $a_n \in U$ for all $n \in \mathbb{N}_0$, then $F(f)(z) := \sum_{n=0}^\infty F(a_n) z^n$ is in $H^\infty(\mathbb{D})$.*

Proof. This is just the functional calculus for Banach algebras (without unit element) using the fact that $\sigma_A(f) \subset U$ by Theorem 3. □

Remark 5. There is no continuous functional calculus on $H^\infty(\mathbb{D})$. Consider for example $F(x) = |x|$. Let $g(z) = (1-z)^{-i} = \sum_{n=0}^\infty b_n z^n$. Then $F(g) = \sum_{n=0}^\infty |b_n| z^n$ is not bounded since $|b_n| \geq \frac{1}{n}$ and $\sum_{n=0}^\infty |b_n|$ is divergent; cf. [5, p. 68].

One should observe that analyticity plays no role, other than in the proof that $H^\infty(\mathbb{D})$ is a Banach algebra under Hadamard multiplication; since $H^\infty(\mathbb{D})$ is isometrically imbedded in $L^\infty(\mathbb{T})$, and the Hadamard product is just convolution, one can just as easily state and prove the corresponding theorem for $L^\infty(\mathbb{T})$, or any of its subspaces having the form $E = \{f \in L^\infty(\mathbb{T}) : \hat{f}_n = 0 \text{ for all } n \notin S\}$, where $\hat{f}_n$ is the nth Fourier coefficient of f, and S is any subset of $\mathbb{Z}$. Of course, $H^\infty(\mathbb{D})$ is the special case of $S = \mathbb{N}_0$. Each such E is a Banach algebra under convolution, and every nontrivial homomorphism of E to $\mathbb{C}$ has the form $f \mapsto \hat{f}_n$ for some $n \in S$.

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