THE MAXIMAL IDEAL SPACE OF $H^\infty(\mathbb{D})$
WITH RESPECT TO THE HADAMARD PRODUCT

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Abstract. It is shown that the space of all regular maximal ideals in the Banach algebra $H^\infty(\mathbb{D})$ with respect to the Hadamard product is isomorphic to $\mathbb{N}_0$. The multiplicative functionals are exactly the evaluations at the $n$-th Taylor coefficient. It is a consequence that for a given function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $H^\infty(\mathbb{D})$ and for a function $F(z)$ holomorphic in a neighborhood $U$ of 0 with $F(0) = 0$ and $a_n \in U$ for all $n \in \mathbb{N}_0$ the function $g(z) = \sum_{n=0}^{\infty} F(a_n) z^n$ is in $H^\infty(\mathbb{D})$.

Introduction

Let $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ be the open unit disk and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be power series on $\mathbb{D}$. Then the Hadamard product of $f$ and $g$ is defined by $f \cdot g(z) = \sum_{n=0}^{\infty} a_n b_n z^n$. The Hadamard product on the space $H(\mathbb{D})$ of all holomorphic functions on $\mathbb{D}$ is continuous with respect to the topology of compact convergence. In [1] R. Brooks has shown that the space of all maximal ideals in the space $H(\mathbb{D})$ is isomorphic to the Stone-Čech-compactification $\beta\mathbb{N}_0$ of $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$ and the multiplicative functionals on $H(\mathbb{D})$ are given by the coefficient functionals $\delta_n : H(\mathbb{D}) \to \mathbb{C}$ defined by $\delta_n(f) := a_n$ (where $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $|z| < 1$ and $n \in \mathbb{N}_0$). In this note we discuss the subalgebra $H^\infty(\mathbb{D})$ of all bounded holomorphic functions which has been considered for example in [3]. Our main result states that the non-trivial multiplicative functionals on $H^\infty(\mathbb{D})$ are of the form $\delta_n$, $n \in \mathbb{N}_0$ (as in the case of $H(\mathbb{D})$). In contrast to the algebra $H(\mathbb{D})$ the space $H^\infty(\mathbb{D})$ is even a Banach algebra with respect to the supremum norm which is denoted by $||f||_\infty$ for $f \in H^\infty(\mathbb{D})$. It follows that the maximal modular ideals of $H^\infty(\mathbb{D})$ are the kernels of the multiplicative functionals and therefore the space of all maximal modular ideals of $H^\infty(\mathbb{D})$ is isomorphic to $\mathbb{N}_0$. Note that $H(\mathbb{D})$ possesses a unit element $\gamma(z) = \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$ which is not in the subalgebra $H^\infty(\mathbb{D})$.

The Results

Let $B$ be the space of all $f(z) = \sum_{n=0}^{\infty} a_n z^n$ such that $\sum_{n=0}^{\infty} |a_n| < \infty$; clearly, $||f||_\infty \leq \sum_{n=0}^{\infty} |a_n|$, so $B \subset H^\infty(\mathbb{D})$. We note that if $f = \sum_{n=0}^{\infty} a_n z^n \in H^\infty(\mathbb{D})$, then $\sum_{n=0}^{\infty} |a_n|^2 = ||f||_2^2 \leq ||f||_\infty^2 < \infty$, where $||f||_2 := \sqrt{\sum_{n=0}^{\infty} |a_n|^2}$. Hence for any $f, g \in H^\infty(\mathbb{D})$, we have $f \cdot g \in B$, since $\sum_{n=0}^{\infty} |a_n b_n| \leq ||f||_2 ||g||_2$ by the

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Cauchy-Schwarz inequality; this also shows that $H^\infty(\mathbb{D})$ is a Banach algebra under Hadamard multiplication.

**Proposition 1.** Let $A$ be the Banach algebra obtained by adjoining a unit to $H^\infty(\mathbb{D})$. If $f = \sum_{n=0}^{\infty} a_n z^n \in B$, then $\sigma_A(f) = \{a_n : n \in \mathbb{N}_0\} \cup \{0\}$.

**Proof.** We must show that if $\lambda \notin \{a_n : n \in \mathbb{N}_0\}$ and $\lambda \neq 0$, then $\lambda - f$ is invertible in $A$ (the other inclusion is easy). Let $g(z) = \sum_{n=0}^{\infty} \frac{a_n}{\lambda - a_n} z^n$, since $|a_n| < |\lambda|/2$ for sufficiently large $n$, we have $|a_n/(\lambda - a_n)| \leq (2/|\lambda|)|a_n|$ for sufficiently large $n$, so $g \in B \subset H^\infty(\mathbb{D})$. Since

$$\lambda - f = \sum_{n=0}^{\infty} \frac{\lambda a_n - \lambda an + a_n^2}{\lambda - a_n} z^n = f * g,$$

we see that $(\lambda - f) * (1 + g) = \lambda$, so $\lambda - f$ is invertible in $A$. \qed

The next result is the main step of our proof. Although we need it only for Banach algebras, it is valid for the larger class of all Fréchet algebras, cf. [4] for definition. We denote by $\Delta_A$ the set of all continuous multiplicative non-trivial functionals.

**Theorem 2.** If $A$ is a unital Fréchet algebra, and $S$ is a countable subset of $\Delta_A$ with the property that $\sigma_A(f^2) = \{\varphi(f^2) : \varphi \in S\}$ for all $f \in A$, then $S = \Delta_A$.

**Proof.** Let $S = \{\varphi_n : n \in \mathbb{N}\}$. Suppose that there exists $\varphi \in \Delta_A \setminus S$. As $\varphi \neq \varphi_n$ for all $n \in \mathbb{N}$, the sets $A_n := \ker(\varphi_n - \varphi)$ and $B_n := \ker(\varphi_n + \varphi)$ are closed hyperplanes, in particular they are nowhere dense. By the Baire category theorem there exists $f \in A$ such that $f \notin A_n$ and $f \notin B_n$ for all $n \in \mathbb{N}$. This means that $\varphi_n(f) \neq \varphi(f)$ and $\varphi_n(f) \neq -\varphi(f)$ for all $n \in \mathbb{N}$. On the other hand we know that $\lambda := \varphi(f) \in \sigma_A(f)$ since $\varphi$ is multiplicative. Hence $\lambda^2 \in \sigma_A(f^2)$. By assumption there exists $n \in \mathbb{N}$ with $\lambda^2 = \varphi_n(f^2) = (\varphi_n(f))^2$. Hence $\lambda = \varphi_n(f)$ or $\lambda = -\varphi_n(f)$, a contradiction. \qed

**Theorem 3.** The non-trivial multiplicative functionals on $H^\infty(\mathbb{D})$ are of the form $\delta_n, n \in \mathbb{N}_0$.

**Proof.** By the above, $f^2 \in B$ for all $f \in H^\infty(\mathbb{D})$. Now apply Proposition 1 and Theorem 2. \qed

**Theorem 4.** Let $U$ be an open neighborhood of zero and $F : U \rightarrow \mathbb{C}$ holomorphic with $F(0) = 0$. If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is in $H^\infty(\mathbb{D})$ and $a_n \in U$ for all $n \in \mathbb{N}_0$, then $F(f)(z) := \sum_{n=0}^{\infty} F(a_n) z^n$ is in $H^\infty(\mathbb{D})$.

**Proof.** This is just the functional calculus for Banach algebras (without unit element) using the fact that $\sigma_A(f) \subset U$ by Theorem 3. \qed

**Remark 5.** There is no continuous functional calculus on $H^\infty(\mathbb{D})$. Consider for example $F(x) = |x|$. Let $g(z) = (1-z)^{-1} = \sum_{n=0}^{\infty} b_n z^n$. Then $F(g) = \sum_{n=0}^{\infty} |b_n| z^n$ is not bounded since $|b_n| \geq \frac{1}{n}$ and $\sum_{n=0}^{\infty} |b_n|$ is divergent; cf. [5, p. 68].

One should observe that analyticity plays no role, other than in the proof that $H^\infty(\mathbb{D})$ is a Banach algebra under Hadamard multiplication; since $H^\infty(\mathbb{D})$ is isometrically imbedded in $L^\infty(\mathbb{T})$, and the Hadamard product is just convolution, one can just as easily state and prove the corresponding theorem for $L^\infty(\mathbb{T})$, or any of
its subspaces having the form \( E = \{ f \in L^\infty(\mathbb{T}) : \hat{f}_n = 0 \text{ for all } n \notin S \} \), where \( \hat{f}_n \) is the \( n \)th Fourier coefficient of \( f \), and \( S \) is any subset of \( \mathbb{Z} \). Of course, \( H^\infty(\mathbb{D}) \) is the special case of \( S = \mathbb{N}_0 \). Each such \( E \) is a Banach algebra under convolution, and every nontrivial homomorphism of \( E \) to \( \mathbb{C} \) has the form \( f \mapsto \hat{f}_n \) for some \( n \in S \).

References


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