DEGENERATIONS FOR MODULES
OVER REPRESENTATION-FINITE ALGEBRAS

GRZEGORZ ZWARA

(Communicated by Ken Goodearl)

Abstract. Let $A$ be a representation-finite algebra. We show that a finite dimensional $A$-module $M$ degenerates to another $A$-module $N$ if and only if the inequalities $\dim_K \text{Hom}_A(M, X) \leq \dim_K \text{Hom}_A(N, X)$ hold for all $A$-modules $X$. We prove also that if $\text{Ext}^1_A(X, X) = 0$ for any indecomposable $A$-module $X$, then any degeneration of $A$-modules is given by a chain of short exact sequences.

1. Introduction and main results

Let $A$ be a finite dimensional associative $K$-algebra with an identity over an algebraically closed field $K$. If $a_1 = 1, \ldots, a_n$ is a basis of $A$ over $K$, we have the structure constants $a_{ijk}$ defined by $a_i a_j = \sum a_{ijk} a_k$. The affine variety $\text{mod}_A(d)$ of $d$-dimensional unital left $A$-modules consists of $n$-tuples $m = (m_1, \ldots, m_n)$ of $d \times d$-matrices with coefficients in $K$ such that $m_1$ is the identity matrix and $m_i m_j = \sum a_{ijk} m_k$ holds for all indices $i$ and $j$. The general linear group $\text{Gl}_d(K)$ acts on $\text{mod}_A(d)$ by conjugation, and the orbits correspond to the isomorphism classes of $d$-dimensional modules (see [11]). We shall agree to identify a $d$-dimensional $A$-module $M$ with the point of $\text{mod}_A(d)$ corresponding to it. We denote by $O(M)$ the $\text{Gl}_d(K)$-orbit of a module $M$ in $\text{mod}_A(d)$. Then one says that a module $N$ in $\text{mod}_A(d)$ is a degeneration of a module $M$ in $\text{mod}_A(d)$ if $N$ belongs to the Zariski closure $\overline{O(M)}$ of $O(M)$ in $\text{mod}_A(d)$, and we denote this fact by $M \leq_{\text{deg}} N$. Thus $\leq_{\text{deg}}$ is a partial order on the set of isomorphism classes of $A$-modules of a given dimension. It is not clear how to characterize $\leq_{\text{deg}}$ in terms of representation theory.

There has been a work by S. Abeasis and A. del Fra [1], K. Bongartz [6], [9], [8], Ch. Riedtmann [13], and A. Skowroński and the author [15], [16], [17] and [18] connecting $\leq_{\text{deg}}$ with other partial orders $\leq_{\text{ext}}$ and $\leq$ on the isomorphism classes in $\text{mod}_A(d)$. They are defined in terms of representation theory as follows:

- $M \leq_{\text{ext}} N$: $\Leftrightarrow$ there are modules $M_i, U_i, V_i$ and short exact sequences $0 \rightarrow U_i \rightarrow M_i \rightarrow V_i \rightarrow 0$ in $\text{mod}_A$ such that $M = M_1$, $M_{i+1} = U_i \oplus V_i$, $1 \leq i \leq s$, and $N = M_{s+1}$ for some natural number $s$.
- $M \leq N$: $\Leftrightarrow [M, X] \leq [N, X]$ holds for all modules $X$.

Received by the editors May 6, 1997 and, in revised form, August 28, 1997.

1991 Mathematics Subject Classification. Primary 14L30, 16G60, 16G70.

©1999 American Mathematical Society

1313
Here and later on we abbreviate \( \dim_K \Hom_A(X,Y) \) by \([X,Y]\). Then for modules \( M \) and \( N \) in \( \mod_A(d) \), the following implications hold:

\[
M \leq \text{ext} N \implies M \leq \text{deg} N \implies M \leq N
\]

(see [9], [13]). Unfortunately, the reverse implications are not true in general, and it would be interesting to find out when they are. K. Bongartz proved in [9] that it is the case for all representations of Dynkin quivers and the double arrow. Moreover, in [8] K. Bongartz proved that \( \leq \text{deg} \) and \( \leq \text{ext} \) coincide for all modules over tame concealed algebras. Recently, the author proved in [17] that \( \leq \text{ext} \) and \( \leq \text{deg} \) coincide for all modules over representation-finite blocks of group algebras, and in [18] that \( \leq \text{ext} \) and \( \leq \text{deg} \) coincide for all modules over tame concealed algebras. The main aim of this paper is to prove the following theorem.

**Theorem 1.** Let \( A \) be a representation-finite algebra and \( M, N \) two modules with \( M \leq N \). Then there are \( A \)-modules \( Z, Z' \), and two exact sequences

\[
0 \to N \to M \oplus Z \to Z \to 0 \quad \text{and} \quad 0 \to Z' \to M \oplus Z' \to N \to 0.
\]

In [13] Riedtmann proved that each of the exact sequences \( 0 \to N \to M \oplus Z \to Z \to 0 \) and \( 0 \to Z' \to M \oplus Z' \to N \to 0 \) implies that \( M \leq \text{deg} N \). Hence we get the following fact which solves a long standing problem (see [13]).

**Corollary.** The partial orders \( \leq \) and \( \leq \text{deg} \) coincide for all modules over representation-finite algebras.

We note that for a representation-finite algebra \( A \) we may deduce the dimension of the spaces \( \Hom_A(M,N) \) from the Auslander-Reiten quiver of \( A \) (see [10]), and hence it is rather easy to decide when \( M \leq N \) for any \( A \)-modules \( M \) and \( N \).

There are many examples of representation-finite algebras for which the orders \( \leq \text{deg} \) and \( \leq \text{ext} \) are not equivalent (see [17]). Our second aim in this paper is to prove the following theorem.

**Theorem 2.** Let \( B \) be an algebra and assume that \( \text{Ext}_B^1(X,X) = 0 \) for any indecomposable \( B \)-module \( X \). Then the partial orders \( \leq, \leq \text{deg} \) and \( \leq \text{ext} \) coincide for all \( B \)-modules.

It is well-known that every representation-directed algebra [14] satisfies the above condition. Hence, Theorem 2 extends the corresponding result by Bongartz proved in [9].

The paper is organized as follows. In Section 2 we fix the notation, recall the relevant definitions and facts, and prove some preliminary results on modules which we apply in our investigations. Section 3 is devoted to the proofs of Theorems 1 and 2.

For basic background on the topics considered here we refer to [5], [9], [8], [11] and [14]. The results presented in this paper form a part of the author’s doctoral dissertation written under the supervision of Professor A. Skowroński. The author gratefully acknowledges support from the Polish Scientific Grant KBN No. 2 PO3A 020 08.

### 2. Preliminary results

**2.1.** Throughout the paper \( A \) denotes a fixed finite dimensional associative \( K \)-algebra with an identity over an algebraically closed field \( K \). We denote by \( \mod A \) the category of finite dimensional left \( A \)-modules and by \( \rad(\mod A) \) the Jacobson
radical of mod $A$. By an $A$-module we mean an object from mod $A$. Further, we denote by $\Gamma_A$ the Auslander-Reiten quiver of $A$, and by $\tau = \tau_A$ and $\tau^- = \tau^-_A$ the Auslander-Reiten translations $D\tau$ and $\tau D$, respectively. We shall agree to identify the vertices of $\Gamma_A$ with the corresponding indecomposable modules. For a module $M$ we denote by $[M]$ the image of $M$ in the Grothendieck group $K_0(A)$ of $A$. Thus $[M] = [N]$ if and only if $M$ and $N$ have the same simple composition factors including the multiplicities.

2.2. Following [13], for $M, N$ from mod $A$, we set $M \leq N$ if and only if $[M, X] \leq [N, X]$ for all $A$-modules $X$. The fact that $\leq$ is a partial order on the isomorphism classes of $A$-modules follows from a result by M. Auslander [3] (see also [6]). Observe that, if $M$ and $N$ have the same dimension and $M \leq N$, then $[M] = [N]$. Moreover, M. Auslander and I. Reiten have shown in [4] that, if $M$ and $N$ are $A$-modules with $[M] = [N]$, then for all nonprojective indecomposable $A$-modules $X$ and all noninjective indecomposable modules $Y$ the following formulas hold (see [12]):

$$[X, M] - [M, \tau X] = [X, N] - [N, \tau X],$$

Hence, if $[M] = [N]$, then $M \leq N$ if and only if $[X, M] \leq [X, N]$ for all $A$-modules $X$.

2.3. Let $M$ and $N$ be $A$-modules with $[M] = [N]$ and

$$\Sigma: 0 \to D \to E \to F \to 0$$

an exact sequence in mod $A$. Following [13] we define the additive functions $\delta_{M,N}$, $\delta'_{M,N}$ and $\delta_\Sigma$ on $A$-modules $X$ as follows:

$$\delta_{M,N}(X) = [N, X] - [M, X],$$
$$\delta'_{M,N}(X) = [X, N] - [X, M],$$
$$\delta_\Sigma(X) = \delta_{E,D\oplus F}(X) = [D \oplus F, X] - [E, X],$$
$$\delta'_\Sigma(X) = \delta'_{E,D\oplus F}(X) = [X, D \oplus F] - [X, E].$$

From the Auslander-Reiten formulas (2.2) we get the following very useful equalities:

$$\delta_{M,N}(X) = \delta'_{M,N}(\tau^- X), \quad \delta_{M,N}(\tau X) = \delta'_{M,N}(X)$$

for all $A$-modules $X$. Observe also that $\delta_{M,N}(I) = 0$ for any injective $A$-module $I$, and $\delta'_{M,N}(P) = 0$ for any projective $A$-module $P$. In particular, the following conditions are equivalent:

1. $M \leq N$,
2. $\delta_{M,N}(X) \geq 0$ for all $X \in \Gamma_A$,
3. $\delta'_{M,N}(X) \geq 0$ for all $X \in \Gamma_A$.

2.4. For an $A$-module $M$ and an indecomposable $A$-module $Z$, we denote by $\mu(M, Z)$ the multiplicity of $Z$ as a direct summand of $M$. For a noninjektive indecomposable $A$-module $U$, we denote by $\Sigma(U)$ an Auslander-Reiten sequence

$$\Sigma(U): 0 \to U \to E(U) \to \tau^- U \to 0.$$
We shall need the following lemma.

**Lemma 2.5.** Let $M$, $N$ be $A$-modules with $[M] = [N]$ and $U$ an indecomposable $A$-module. Then:

(i) If $U$ is noninjective, then $\delta_{\Sigma(U)}(M) = \mu(M,U)$.
(ii) If $M \leq N$, then $\mu(N,U) - \mu(M,U) \leq \delta_{M,N}(U) + \delta_{M,N}(U)$.

**Proof.** If $U$ is noninjective, then the Auslander-Reiten sequence $\Sigma(U)$ induces an exact sequence

$$0 \to \text{Hom}_A(\tau^{-}U, M) \to \text{Hom}_A(E(U), M) \to \text{rad}(U, M) \to 0,$$

and hence we get

$$[U \oplus \tau^{-}U, M] - [E(U), M] = [U, M] - \dim_K \text{rad}(U, M) = \mu(M, U).$$

This implies (i). Similarly, we have

$$[U \oplus \tau^{-}U, N] - [E(U), N] = \mu(N, U).$$

Then we obtain

$$\mu(N, U) - \mu(M, U) = ([U \oplus \tau^{-}U, N] - [U \oplus \tau^{-}U, M]) - ([E(U), N] - [E(U), M])$$

$$= \delta'_{M,N}(U) + \delta'_{M,N}(\tau^{-}U) - \delta'_{M,N}(E(U))$$

$$\leq \delta'_{M,N}(U) + \delta'_{M,N}(\tau^{-}U) = \delta_{M,N}(U) + \delta_{M,N}(U).$$

Assume now that $U$ is injective. Then $\text{Hom}_A(U/\text{soc}(U), M) \simeq \text{rad}(U, M)$, and so

$$[U, M] - [U/\text{soc}(U), M] = \mu(M, U).$$

Similarly, we have

$$[U, N] - [U/\text{soc}(U), N] = \mu(N, U).$$

Therefore, we get

$$\mu(N, U) - \mu(M, U) = ([U, N] - [U, M]) - ([U/\text{soc}(U), N] - [U/\text{soc}(U), M])$$

$$= \delta_{M,N}(U) - \delta_{M,N}(U/\text{soc}(U)) \leq \delta_{M,N}(U)$$

$$= \delta'_{M,N}(U) + \delta_{M,N}(U).$$

Hence, (ii) also holds. \qed

We shall need also the following Lemma (3 + 3 + 2) from [2, (2.1)] and its direct consequence.

**Lemma 2.6.** Let

$$\Sigma_1 : 0 \to M_1 \xrightarrow{[v_1]} M_2 \oplus N_1 \xrightarrow{[f_2, u_2]} N_2 \to 0,$$

$$\Sigma_2 : 0 \to M_2 \xrightarrow{[v_2]} M_3 \oplus N_2 \xrightarrow{[f_3, v_2]} N_3 \to 0$$

be short exact sequences in $\text{mod} \ A$. Then the sequence

$$\Sigma_3 : 0 \to M_1 \xrightarrow{[v_1 v_1]} M_3 \oplus N_1 \xrightarrow{[f_3, -v_2 u_2]} N_3 \to 0$$

is exact. Moreover, we have $\delta_{\Sigma_3} = \delta_{\Sigma_1} + \delta_{\Sigma_2}$. 

2.7. A short exact sequence

\[ 0 \rightarrow U \xrightarrow{f} W \xrightarrow{g} V \rightarrow 0 \]

is said to be without isomorphism provided \( f \in \text{rad}(U, W) \) and \( g \in \text{rad}(W, V) \). Let \( \Sigma : 0 \rightarrow U \xrightarrow{f} W \xrightarrow{g} V \rightarrow 0 \) be any exact sequence. It is easy to see that if \( f \in \text{rad}(U, W) \), then there is an exact sequence without isomorphism \( 0 \rightarrow U \rightarrow W' \rightarrow V' \rightarrow 0 \) such that \( W = W' \oplus Y \) and \( V = V' \oplus Y \) for some \( A \)-modules \( W' \), \( V' \) and \( Y \). Dually, if \( g \in \text{rad}(W, V) \), then there is an exact sequence without isomorphism \( 0 \rightarrow U' \rightarrow W' \rightarrow V' \rightarrow 0 \) such that \( U = U' \oplus Z \) and \( W = W' \oplus Z \) for some \( A \)-modules \( U' \), \( W' \) and \( Z \). Moreover, if \( \Sigma \) is nonsplittable, then there is a nonsplittable exact sequence without isomorphism \( 0 \rightarrow U' \rightarrow W' \rightarrow V' \rightarrow 0 \) such that \( U = U' \oplus Y \), \( W = W' \oplus Y \oplus Z \) and \( V = V' \oplus Z \) for some \( A \)-modules \( U' \), \( W' \), \( V' \), \( Y \) and \( Z \).

**Lemma 2.8.** Let \( \Sigma : 0 \rightarrow U \xrightarrow{f} W \xrightarrow{g} V \rightarrow 0 \) be an exact sequence without isomorphism. Then:

(i) For any nonzero direct summand \( U' \) of \( U \), \( \delta_{\Sigma}(U') > 0 \) holds.

(ii) For any nonzero direct summand \( V' \) of \( V \), \( \delta_{\Sigma}(V') > 0 \) holds.

**Proof.** (i) Let \( U' \) be a nonzero direct summand of \( U \). The sequence \( \Sigma \) induces an exact sequence

\[ 0 \rightarrow \text{Hom}(V, U') \rightarrow \text{Hom}(W, U') \rightarrow \text{Hom}(U, U'). \]

Assume that \( f^* \) is an epimorphism. Then there is a homomorphism of \( A \)-modules \( h : W \rightarrow U' \) such that \( f^*(h) = hf : U \rightarrow U' \) is a projection. But then \( f \notin \text{rad}(U, W) \), which yields a contradiction. Hence, \( [V, U'] - [W, U'] + [U, U'] > 0 \), and consequently \( \delta_{\Sigma}(U') > 0 \).

The proof of (ii) is dual. \( \square \)

As a consequence of the above lemma, we get the following fact.

**Lemma 2.9.** Let \( \Sigma : 0 \rightarrow U \rightarrow W \rightarrow V \rightarrow 0 \) be a nonsplittable exact sequence. Then \( \delta_{\Sigma}(U) > 0 \) and \( \delta_{\Sigma}(V) > 0 \).

**Lemma 2.10.** Let \( X \) be an \( A \)-module and \( \Sigma : 0 \rightarrow U \rightarrow W \rightarrow V \rightarrow 0 \) a nonsplittable short exact sequence of \( A \)-modules.

(i) If \( \delta_{\Sigma}(X) > 0 \), then there exists a nonsplittable exact sequence of \( A \)-modules

\[ \Phi : 0 \rightarrow X \rightarrow Y \rightarrow V \rightarrow 0, \]

such that \( \delta_{\Phi} \leq \delta_{\Sigma} \).

(ii) If \( \delta'_{\Sigma}(X) > 0 \), then there exists a nonsplittable exact sequence of \( A \)-modules

\[ \Phi : 0 \rightarrow U \rightarrow Z \rightarrow X \rightarrow 0, \]

such that \( \delta_{\Phi} \leq \delta_{\Sigma} \).

**Proof.** (i) The first part of the proof is due to U. Markolf (see the proof of Theorem 4 in [7]). Let \( X \) be an \( A \)-module such that \( \delta_{\Sigma}(X) > 0 \). Then the last map in the following exact sequence

\[ 0 \rightarrow \text{Hom}(V, X) \rightarrow \text{Hom}(W, X) \rightarrow \text{Hom}(U, X) \rightarrow \text{Ext}_{A}^{1}(V, X) \rightarrow \text{Ext}_{A}^{1}(W, X) \]

is not a monomorphism. Therefore, we find a nonsplittable exact sequence of \( A \)-modules \( \Phi : 0 \rightarrow X \rightarrow Y \rightarrow V \rightarrow 0 \), whose pullback under \( W \rightarrow V \) is a splittable.
sequence. Thus we get the following commutative diagram with exact rows and columns:

\[
\begin{array}{ccccccccc}
& 0 & 0 \\
& \downarrow & \downarrow \\
U & = & U \\
& \downarrow & \downarrow \\
0 & \rightarrow & X & \rightarrow & X \oplus W & \rightarrow & W & \rightarrow & 0 \\
& \| & \downarrow & \downarrow \\
0 & \rightarrow & X & \rightarrow & Y & \rightarrow & V & \rightarrow & 0 \\
& & & \downarrow & \downarrow \\
0 & & 0 & & 0 & & 0 & & 0
\end{array}
\]

So, we have an exact sequence \( \Theta : 0 \rightarrow U \rightarrow X \oplus W \rightarrow Y \rightarrow 0 \). Observe that \( \delta_\Sigma = \delta_\Phi + \delta_\Theta \). This implies that \( \delta_\Phi \leq \delta_\Sigma \).

The proof of (ii) is dual. \( \square \)

**Lemma 2.11.** If \( M <_{\text{deg}} N \), then \( \delta_{M,N}(N) > 0 \) and \( \delta'_{M,N}(N) > 0 \).

**Proof.** Suppose that \( \delta'_{M,N}(N) = 0 \). By Theorem 2.4 in [9], we know that if a module \( U \) embeds into \( N \) and \([U, N] = [U, M]\), then \( U \) also embeds into \( M \). Applying this fact for \( U = N \), we obtain that \( N \) embeds into \( M \). But the modules \( M \) and \( N \) have the same dimension. This implies that \( M \) is isomorphic to \( N \), which gives a contradiction. Hence, \( \delta'_{M,N}(N) > 0 \) and \( \delta_{M,N}(N) > 0 \) by duality. \( \square \)

3. Proof of Theorems 1 and 2

Throughout this section \( A \) denotes a representation-finite algebra.

**Lemma 3.1.** Let \( M \) and \( N \) be two \( A \)-modules with \( M < N \), and let

\[
\Sigma : 0 \rightarrow U \rightarrow W \rightarrow V \rightarrow 0
\]

be a short exact sequence without isomorphism in \( \text{mod } A \) such that \( \delta_\Sigma \leq \delta_{M,N} \).

Then there exists a short exact sequence without isomorphism in \( \text{mod } A \)

\[
\Phi : 0 \rightarrow U \rightarrow Y \rightarrow Z \rightarrow 0
\]

such that \( \delta_\Sigma \leq \delta_\Phi \leq \delta_{M,N} \) and \( \delta_\Phi(Y) = \delta_{M,N}(Y) \).

**Proof.** Let

\[
\Sigma : 0 \rightarrow U \rightarrow W \rightarrow V \rightarrow 0
\]

be a short exact sequence of \( A \)-modules without isomorphism such that \( \delta_\Sigma \leq \delta_{M,N} \).

Take a short exact sequence without isomorphism in \( \text{mod } A \),

\[
\Phi : 0 \rightarrow U \rightarrow Y \rightarrow Z \rightarrow 0
\]

such that \( \delta_\Sigma \leq \delta_\Phi \leq \delta_{M,N} \), and which is maximal in the following sense. For any short exact sequence without isomorphism \( \Phi' \) in \( \text{mod } A \) starting at \( U \) and satisfying inequalities \( \delta_\Phi \leq \delta_\Phi' \leq \delta_{M,N} \), we have \( \delta_\Phi = \delta_\Phi' \). Since \( \sum_{X \in \Gamma_A} \delta_{M,N}(X) \) is finite, such a sequence \( \Phi \) exists. Assume now that \( Y = Y_1 \oplus Y_2 \), where \( Y_1 \) is indecomposable and \( \delta_\Phi(Y_1) < \delta_{M,N}(Y_1) \). Then \( Y_1 \) is noninjective and we have an Auslander-Reiten sequence

\[
\Sigma(Y_1) : 0 \rightarrow Y_1 \xrightarrow{h} E \rightarrow \tau^- Y_1 \rightarrow 0,
\]
and of course
\[ \Phi : 0 \to U \to Y_1 \oplus Y_2 \xrightarrow{f_1,f_2} Z \to 0. \]
Since \( f_1 \in \text{rad}(Y_1,Z) \), the push out of the Auslander-Reiten sequence \( \Sigma(Y_1) \) is a splittable sequence, so we obtain the following commutative diagram with exact rows:
\[
\begin{array}{ccc}
0 & \to & Y_1 \\
\downarrow f_1 & & \downarrow \quad \downarrow \\
0 & \to & \tau^{-}Y_1 \oplus Z \\
\end{array}
\]
This implies that there exists a nonsplittable exact sequence
\[ \Psi : 0 \to Y_1 \xrightarrow{(h,f_1)} E \oplus Z \to \tau^{-}Y_1 \oplus Z \to 0. \]
Applying Lemma 2.6 for \( \Phi \) and \( \Psi \), we get a new exact sequence
\[ 0 \to U \xrightarrow{\tau} Y_2 \oplus E \to Z \oplus \tau^{-}Y_1 \to 0. \]
Since \( \Phi \) is a sequence without isomorphism, we have \( i \in \text{rad}(U,Y_2 \oplus E) \). Hence, there is a sequence without isomorphism in mod \( A \)
\[ \Theta : 0 \to U \to Y \to Z \to 0, \]
with \( Y_2 \oplus E = Y \oplus W \) and \( Z \oplus \tau^{-}Y_1 = Z \oplus W \) for some \( A \)-module \( W \). Thus, by Lemmas 2.6 and 2.5(i), for any \( A \)-module \( X \) we have
\[ \delta_{\Theta}(X) = \delta_{\Phi}(X) + \delta_{\Psi}(X) = \delta_{\Phi}(X) + \delta_{\Sigma(Y_1)}(X) = \delta_{\Phi}(X) + \mu(X,Y_1). \]
Since \( \delta_{\Phi} \leq \delta_{M,N} \) and \( \delta_{\Phi}(Y_1) \leq \delta_{M,N}(Y_1) - 1 \), we get \( \delta_{\Sigma} \leq \delta_{\Theta} \leq \delta_{M,N} \). This gives a contradiction with our choice of the sequence \( \Phi \). Hence, \( \delta_{\Phi}(Y) = \delta_{M,N}(Y) \), and this finishes the proof. \( \square \)

**Lemma 3.2.** If \( M < N \), then \( \delta_{M,N}(N) > 0 \) and \( \delta'_{M,N}(N) > 0 \).

**Proof.** We proceed by induction on \( \sum_{X \in \Gamma_A} \delta_{M,N}(X) > 0 \). Applying equalities (2.3), we obtain \( \sum_{X \in \Gamma_A} \delta_{M,N}(X) = \sum_{X \in \Gamma_A} \delta'_{M,N}(X) \). Assume \( M < N \) and that \( \delta_{M,N}(N) = 0 \) or \( \delta'_{M,N}(N) = 0 \). By duality, we may assume that \( \delta_{M,N}(N) = 0 \) and moreover, the modules \( M \) and \( N \) have no nonzero common direct summand. Let \( \mathcal{F} \) be the set of all modules in \( \Gamma_A \) which are a direct summands of \( N \). Take \( Y \in \mathcal{F} \). By Lemma 2.5(ii), we get
\[ \mu(N,Y) = \mu(N,Y) - \mu(M,Y) \leq \delta_{M,N}(Y) + \delta'_{M,N}(Y) = \delta_{M,N}(Y). \]
So, the module \( Y \) is noninjective and there is an Auslander-Reiten sequence \( \Sigma(Y) \).

We define a new exact sequence without isomorphism
\[ \Sigma : 0 \to N \to E(N) \to \tau^{-}N \to 0, \]
where \( E(N) = \bigoplus_{Y \in \mathcal{F}} E(Y) \mu(N,Y) \) and \( \tau^{-}N = \bigoplus_{Y \in \mathcal{F}} (\tau^{-}Y) \mu(N,Y) \). Applying Lemma 2.5(i), we obtain
\[ \delta_{\Sigma}(Y) = \mu(N,Y) \leq \delta_{M,N}(Y), \]
for any \( Y \in \Gamma_A \). Consequently \( \delta_{\Sigma} \leq \delta_{M,N} \) and, from Lemma 3.1, there is an exact sequence without isomorphism
\[ \Phi : 0 \to N \to W \to V \to 0 \]
with $\delta_\Phi \leq \delta_{M,N}$ and $\delta_\Phi(W) = \delta_{M,N}(W)$. Then $M \oplus V \leq W$ and $\delta_{M \oplus V,W}(W) = 0$. Observe that $\delta_{M,N} - \delta_{M \oplus V,W} = \delta_\Phi$ and, from Lemma 2.9, $\delta_\Phi(N) > 0$. This leads to

$$\sum_{X \in \Gamma_A} \delta_{M \oplus V,W}(X) < \sum_{X \in \Gamma_A} \delta_{M,N}(X).$$

It follows from our inductive assumption that the modules $M \oplus V$ and $W$ are isomorphic. Then the sequence $\Phi$ has the form

$$0 \to N \to V \oplus M \to V \to 0,$$

and this implies that $M <_{\text{deg}} N$, by Proposition 3.4 in [13]. Applying Lemma 2.11, we get $\delta_{M,N}(N) > 0$, and hence a contradiction. This finishes the proof. \hfill \square

3.3. Proof of Theorem 1. Let $M$ and $N$ be $A$-modules with $M \leq N$. We may assume that $M < N$. Let $r(X) = \min\{\delta_{M,N}(X), \mu(N,X)\}$, for any $X \in \Gamma_A$, and let $\mathcal{F}$ be the set of all vertices of $\Gamma_A$ with $r(X) > 0$. The set $\mathcal{F}$ does not contain injective $A$-modules and is nonempty, by Lemma 3.2. Let $N' = \bigoplus_{X \in \mathcal{F}} X^r(X) = \bigoplus_{X \in \Gamma_A} X^r(X)$ and $N'' = \bigoplus_{X \in \Gamma_A} X^{\mu(N,X)-r(X)}$. Then $N = N' \oplus N''$. We define a new exact sequence without isomorphism

$$\Sigma : 0 \to \bigoplus_{X \in \mathcal{F}} X^r(X) \to \bigoplus_{X \in \mathcal{F}} E(X)^r(X) \to \bigoplus_{X \in \mathcal{F}} (\tau^- X)^r(X) \to 0.$$

Applying Lemma 2.5(i), we obtain $\delta_\Sigma(X) = r(X) \leq \delta_{M,N}(X)$, for any $X \in \Gamma_A$. Consequently, $\delta_\Sigma \leq \delta_{M,N}$ and, by Lemma 3.1, there is an exact sequence without isomorphism

$$\Phi : 0 \to N' \to W \to Z \to 0$$

with $\delta_\Sigma \leq \delta_\Phi \leq \delta_{M,N}$ and $\delta_\Phi(W) = \delta_{M,N}(W)$. Then $M \oplus Z \leq N'' \oplus W$ and $\delta_{M \oplus Z,N'' \oplus W}(W) = 0$. Let $N_1$ be any indecomposable direct summand of $N''$. Then $r(N_1) < \mu(N,N_1)$, and this leads to $\delta_\Sigma(N_1) = r(N_1) = \delta_{M,N}(N_1)$. Hence,

$$\delta_{M \oplus Z,N'' \oplus W}(N_1) = \delta_{M,N}(N_1) - \delta_\Phi(N_1) = \delta_\Sigma(N_1) - \delta_\Phi(N_1) \leq 0.$$

So, $\delta_{M \oplus Z,N'' \oplus W}(N_1) = 0$. This implies that $\delta_{M \oplus Z,N'' \oplus W}(N'') = 0$, and furthermore $\delta_{M \oplus Z,N'' \oplus W}(N' \oplus W) = 0$. Hence, $M \oplus Z \cong N'' \oplus W$, by Lemma 3.2. Finally, the sequence $\Phi$ induces an exact sequence $0 \to N' \oplus N'' \to N'' \oplus W \to Z \to 0$, which has the form $0 \to N \to M \oplus Z \to Z \to 0$. In a similar way we obtain an exact sequence $0 \to Z' \to M \oplus Z' \to N \to 0$. \hfill \square

Lemma 3.4. Let $M$, $N$ and $X$ be $A$-modules such that $M < N$ and $X \in \Gamma_A$. Then we have:

(i) If $\delta_{M,N}(X) > 0$, then there exist an indecomposable direct summand $N_1$ of $N$ and a nonsplittable exact sequence $\Phi : 0 \to N_1 \to Y \to X \to 0$ without isomorphism such that $\delta_\Phi \leq \delta_{M,N}$.

(ii) If $\delta_{M,N}(X) > 0$, then there exist an indecomposable direct summand $N_1$ of $N$ and a nonsplittable exact sequence $\Phi : 0 \to X \to Y \to N_1 \to 0$ without isomorphism such that $\delta_\Phi \leq \delta_{M,N}$.

Proof. (i) Assume that $\delta_{M,N}(X) > 0$. Applying Theorem 1 we get the exact sequence $\Sigma : 0 \to N \to M \oplus Z \to Z \to 0$, in mod $A$. Further, applying Lemma 2.10(ii), we obtain a nonsplittable exact sequence $\Psi : 0 \to N \to W \to Y \to 0$ with $\delta_\Psi \leq \delta_\Sigma = \delta_{M,N}$. Then, by Lemma 2.9, there is an indecomposable direct summand
$N_1$ of $N$ with $\delta_{B}(N_1) > 0$. Finally, by Lemma 2.10(i), we obtain a nonsplittable exact sequence $\Phi : 0 \to N_1 \to Y \to X \to 0$ with $\delta_{B} \leq \delta_{\Phi} \leq \delta_{M,N}$.

We obtain (ii) by duality. \hfill $\square$

3.5. **Proof of Theorem 2.** Let $B$ be an algebra and assume that $\text{Ext}_B^1(X,X) = 0$ for any indecomposable $B$-module $X$. It is well-known that then $B$ is representation-finite. Let $M$ and $N$ be two $B$-modules with $M \leq N$. We shall show that $M \leq_{\text{ext}} N$. We proceed by induction on $[N,N] - [M,M] \geq 0$. If $[N,N] - [M,M] = 0$, then by Lemma 1.2 in [9], $M$ is isomorphic to $N$. Hence, we may assume that $M < N$, and that $M$ and $N$ have no common nonzero direct summand. Take any indecomposable direct summand $N_1$ of $N$. Applying Lemma 2.5(ii), we obtain that $\delta_{M,N}(N_1) + \delta'_{M,N}(N_1) > 0$. Without loss of generality, we may assume that $\delta_{M,N}(N_1) > 0$. Now applying Lemma 3.4, we get a nonsplittable exact sequence

$$\Sigma : 0 \to N_1 \to Y \to N_2 \to 0$$

with $\delta_{\Sigma} \leq \delta_{M,N}$, for some $A$-module $Y$ and some indecomposable direct summand $N_2$ of $N$. Since $\text{Ext}_B^1(N_1,N_1) = 0$, the modules $N_1$ and $N_2$ are not isomorphic. Thus, $N = N_1 \oplus N_2 \oplus N_3$, for some $A$-module $N_3$. Moreover, $M \leq Y \oplus N_3 <_{\text{ext}} N$. This implies that $[Y \oplus N_3, Y \oplus N_3] < [N,N]$, by Lemma 1.2 in [9]. Then

$$[Y \oplus N_3, Y \oplus N_3] - [M,M] < [N,N] - [M,M]$$

and $M \leq_{\text{ext}} Y \oplus N_3$, by our inductive assumption. Finally, we obtain $M <_{\text{ext}} N$, and this finishes the proof. \hfill $\square$

**References**


Faculty of Mathematics and Informatics, Nicholas Copernicus University, Chopina 12/18, 87-100 Toruń, Poland

E-mail address: gzwara@mat.uni.torun.pl