A SIMPLE PROOF OF A CURIOUS CONGRUENCE BY SUN

ZUN SHAN AND EDWARD T. H. WANG

(Communicated by David Rohrlich)

Abstract. In this note, we give a simple and elementary proof of the following curious congruence which was established by Zhi-Wei Sun:

\[
\frac{(p-1)/2}{2} \sum_{k=1}^{[3p/4]} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{[3p/4]} \frac{(-1)^{k-1}}{k} \quad (\text{mod } p).
\]

In [4], the following curious congruence for odd prime \( p \) was established by Zhi-Wei Sun:

\[
\frac{(p-1)/2}{2} \sum_{k=1}^{[3p/4]} \frac{1}{k \cdot 2^k} \equiv \sum_{k=1}^{[3p/4]} \frac{(-1)^{k-1}}{k} \quad (\text{mod } p).
\]

The author’s proof, using Pell sequences, is fairly complicated. In fact, a recent article [3] on congruence modulo \( p \) ends in the remark that “It seems unlikely that (1) can be proved with the simple approach that we have used here.” In the present note, we give a simple and elementary proof of (1). Throughout, \( p \) denotes an odd prime.

First of all, it is well known (e.g. [1], [2]) that for \( k = 0, 1, 2, \ldots, p - 1 \),

\[
\left( \frac{p-1}{k} \right) \equiv (-1)^k \quad (\text{mod } p).
\]

From (2) we get

\[
\frac{2^{p-1} - 1}{2} = \frac{(1 + 1)^p - 2}{2p} = \frac{1}{2p} \sum_{k=1}^{p-1} \left( \begin{array}{c} p \\ k \end{array} \right) = \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{k} \left( \begin{array}{c} p \\ k \end{array} \right) \left( p - 1 \right) \equiv \frac{1}{2} \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} \quad (\text{mod } p).
\]
Let $\varepsilon = e^{\pi i/4}$. Then

$$(1 + \varepsilon)^p + (1 - \varepsilon)^p = 2 + 2 \sum_{1 \leq k \leq p \atop k \text{ even}} \left(\begin{array}{c} p \\ k \end{array} \right) \varepsilon^k$$

$$= 2 + 2p \sum_{1 \leq k \leq p \atop k \text{ even}} \frac{1}{k} \left(\begin{array}{c} p - 1 \\ k - 1 \end{array} \right) \varepsilon^k$$

$$\equiv 2 - 2p \sum_{1 \leq k \leq p \atop k \text{ even}} \frac{\varepsilon^k}{k} \pmod{p^2}$$

(4)

$$= 2 - 2p \left( \sum_{k=1}^{[\frac{n+1}{2}]} (-1)^k \frac{1}{4k} + i \sum_{k=1}^{[\frac{n+1}{2}]} \frac{(-1)^{k-1}}{4k-2} \right)$$

$$= 2 - \frac{p}{2} \sum_{k=1}^{[\frac{n+1}{2}]} (-1)^k \frac{1}{k} + ip \sum_{k=1}^{[\frac{n+1}{2}]} \frac{(-1)^k}{2k-1}$$

$$= 2 - \frac{p}{2} A + ipB$$

where

$$A = \sum_{k=1}^{[\frac{n+1}{2}]} \frac{(-1)^k}{k} \quad \text{and} \quad B = \sum_{k=1}^{[\frac{n+1}{2}]} \frac{(-1)^k}{2k-1}.$$ 

Since $\overline{\varepsilon} = \varepsilon^{-1}$, taking modulus of both sides of (4) yields

$$4 - 2pA \equiv \left(2 - \frac{p}{2} A\right)^2 + p^2 B^2$$

$$\equiv 4 - 2pA \equiv ((1 + \varepsilon)^p + (1 - \varepsilon)^p)((1 + \varepsilon^{-1})^p + (1 - \varepsilon^{-1})^p)$$

$$= (2 + \varepsilon + \varepsilon^{-1})^p + (2 - \varepsilon - \varepsilon^{-1})^p$$

$$= (2 + \sqrt{2})^p + (2 - \sqrt{2})^p$$

$$= 2^{p+1} + 2 \sum_{1 \leq k \leq p \atop k \text{ even}} \left(\begin{array}{c} p \\ k \end{array} \right) 2^{p-k} \sqrt{2}^k$$

(5)

$$= 2^{p+1} + 2^{p+1} \sum_{k=1}^{(p-1)/2} \left(\begin{array}{c} p \\ 2k \end{array} \right) \frac{1}{2^{2k}}$$

$$= 2^{p+1} + 2^{p+1} \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^{2k}} \left(\begin{array}{c} p - 1 \\ 2k - 1 \end{array} \right)$$

$$\equiv 2^{p+1} - 2^p \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \pmod{p^2}.$$
From (5) and (3) we obtain, since \(2^{p-1} \equiv 1 \mod p\),
\[
A \equiv -\frac{2^p - 2}{p} + 2^{p-1} \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k}
\equiv \sum_{k=1}^{p-1} \frac{(-1)^k}{k} + \sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \pmod{p}
\]
and so
\[
\sum_{k=1}^{(p-1)/2} \frac{1}{k \cdot 2^k} \equiv -\sum_{k=1}^{p-1} \frac{(-1)^k}{k} + A = \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} + \sum_{k=1}^{[\frac{p}{2}]} \frac{(-1)^{k-1}}{k}
\equiv \sum_{k=1}^{p-1} \frac{(-1)^{k-1}}{k} - \sum_{k=p-[\frac{p-1}{2}]}^{p-1} \frac{(-1)^{k-1}}{k} \pmod{p}
\equiv \sum_{k=1}^{[\frac{p}{2}]} \frac{(-1)^{k-1}}{k}
\]
and (1) is proved.

Acknowledgement

This paper was written when the first author was visiting the Mathematics Department of Wilfrid Laurier University, December 1996–August 1997. The hospitality of Wilfrid Laurier University is greatly appreciated.

References