QUADRATIC BASE CHANGE FOR \( p \)-ADIC SL\((2) \) AS A THETA CORRESPONDENCE I: OCCURRENCE

DAVID MANDERSCHEID

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Abstract. The local theta correspondence is considered for reductive dual pairs \((SL_2(F), O(F))\) where \(F\) is a \( p \)-adic field of characteristic zero and \(O\) is the orthogonal group attached to a quaternary quadratic form with coefficients in \(F\) and of Witt rank one over \(F\). It is shown that certain representations of \(SL_2(F)\) occur in the correspondence.

This paper is the first in a series of papers examining in detail Howe’s local theta correspondence [H] for a reductive dual pair \((SL_2(F), O(F))\) where \(F\) is a \( p \)-adic field of characteristic zero and \(O\) is the orthogonal group attached to a quaternary quadratic form with coefficients in \(F\) and of Witt rank one over \(F\). In this paper I show that certain representations of \(SL_2(F)\) occur in the correspondence; see Theorem 2.3 and Corollary 2.4. In future papers, among other results, I will show that these are all the representations of \(SL_2(F)\) that occur in the case \(p\) odd; I will also provide strong evidence that this is also true for \(p\) even.

The idea behind our argument is relatively simple and can be explained as follows. First, recall the general setting of theta correspondences for symplectic and orthogonal groups; see, e.g., [MVW]. For \(i = 1, 2\) let \(V_i\) be a finite-dimensional vector space over \(F\) equipped with a nondegenerate bilinear form \(\langle , \rangle_i\); assume that \(\langle , \rangle_1\) is skew-symmetric while \(\langle , \rangle_2\) is symmetric. Equip \(W = V_1 \otimes V_2\) with the skew-symmetric form \(\langle , \rangle\) coming from tensoring the \(\langle , \rangle_i\). Let \(G_1, G_2\) and \(G\) be the isometry groups of \(\langle , \rangle_1, \langle , \rangle_2\) and \(\langle , \rangle\) respectively and identify \(G_1\) and \(G_2\) with subgroups of \(G\) via their usual actions on \(W\). Then \((G_1, G_2)\) is called a reductive dual pair in \(G\). Let \(\chi\) be a nontrivial additive character of \(F\) and let \(\omega_\chi^\infty\) denote the (smooth) oscillator representation of \(\tilde{G}\) attached to \(\chi\) where \(\tilde{G}\) is the (unique) nontrivial two-fold cover of \(G\). For \(H\) a closed subgroup of \(G\) let \(\tilde{H}\) denote the inverse image of \(H\) in \(G\) and let \(\mathcal{R}_\chi(\tilde{H})\) denote the set of irreducible admissible representations of \(\tilde{H}\) which occur as quotients of \(\omega_\chi^\infty|_{\tilde{H}}\). Then \(G_1\) and \(G_2\) commute and \(\mathcal{R}_\chi(G_1 \tilde{G}_2)\) gives rise to a correspondence between \(\mathcal{R}_\chi(\tilde{G}_1)\) and \(\mathcal{R}_\chi(\tilde{G}_2)\). These correspondences are called theta correspondences. We denote these correspondences by \(\theta: \mathcal{R}_\chi(\tilde{G}_1) \to \mathcal{R}_\chi(\tilde{G}_2)\) and \(\theta: \mathcal{R}_\chi(\tilde{G}_2) \to \mathcal{R}_\chi(\tilde{G}_1)\); the direction of \(\theta\) will be clear from context. Theta correspondences are known in general to be bijections for \(p\) odd [W] and for all \(p\) in the cases considered in this paper.
Further in all cases considered here the space $V_2$ will be even-dimensional and then the $\tilde{G}_1$ and $\tilde{G}_2$ are trivial covers so that we write, in an abuse of notation, $R_\chi(G_1)$ and $R_\chi(G_2)$ instead of $R_\chi(\tilde{G}_1)$ and $R_\chi(\tilde{G}_2)$ and elements of these sets will be considered as representations of $G_1$ and $G_2$ respectively.

Now consider the case where $V_1$ is two-dimensional and $V_2$ is as follows. Let $E/F$ be a quadratic extension and set $V_2 = \{ A \in M_2(E) \mid A^t = A \}$ where $^t$ denotes transpose and $\bar{A}$ denotes the matrix obtained from $A$ by applying $\sigma$ to each entry of $A$ where $\Gamma(E/F) = \langle \sigma \rangle$ is the Galois group of $E/F$. Equip $V_2$ with the quadratic form $q(A) = -\det(A)$ where $\det:M_2(E) \to E$ is the determinant map (note that $\det|_{V_2}$ actually maps to $F$). This form is defined over $F$ and of Witt rank one over $F$. Now write $V_2 = V_0 \oplus H$ where $V_0 = \{ A \in V_2 \mid A \text{ is 0 on diagonal} \}$, $H = \{ A \in V_2 \mid A \text{ is 0 off diagonal} \}$ and the summands are orthogonal. Let $H_1, H_0$ and $J$ be the orthogonal groups attached to $V_2, V_0$ and $H$, respectively, and let $R_\chi(G_1), R_\chi^0(G_1)$ and $R_\chi'(G_1)$ denote those representations of $G_1$ occurring in the theta correspondences attached to $\chi$ and the pairs $(G_1, H_1), (G_1, H_0)$ and $(G_1, J)$ respectively. Then the idea behind our argument, roughly speaking, is that if $\pi$ occurs in $R_\chi(G_1)$, then $\pi$ should be the quotient of $\pi_0 \otimes \pi'$ for some $\pi_0$ in $R_\chi^0(G_1)$ and $\pi'$ in $R_\chi'(G_1)$ by a seesaw duality argument [K], the sets $R_\chi^0(G_1)$ and $R_\chi'(G_1)$ being known [C], [ST].

This paper is organized as follows. In the first section we set notation and briefly recall the parametrization of the admissible dual of $G_1$ in [LL]. We also recall the portions of the parametrizations of the theta correspondences attached to $\chi$ and the pairs $(G_1, H_0)$ and $(G_1, J_1)$ we will need. In the second section we construct, by the method outlined above, representations of $G_1$ which occur in $R_\chi(G_1)$. The only other ingredients necessary are Whittaker models and a result of D. Prasad on trilinear forms for $GL_2(F)$ [P1].

Finally we note that instead of using Prasad’s result one could try to use Cognet’s thesis [Co]. The results of Cognet that would be necessary, however, are much more difficult to prove than the necessary result of [P1]. Further they do not as readily yield results as complete as we obtain here and do not apply to the trivial representation of $G_1$. Cognet’s thesis, however, does indicate the intimate connection that exists between the theta correspondences attached to $(G_1, H_1)$ and quadratic base change from $SL_2(F)$ to $SL_2(E)$, a subject we will turn to in future papers and thus the title of this paper.

1. Some notation and parameters

In this section we establish notation, recall the parametrization of the admissible dual of $G_1 = SL_2(F)$, and recall the theta correspondences attached to $\chi$ and $(G_1, H_0)$ and $(G_1, J_1)$. Since this material is known or easily derived from the literature we will be quite brief in our discussion. For unexplained terminology or notation see [M] or [MVW].

Let $F$ be a nonarchimedean local field of characteristic zero and let $p$ denote the residual characteristic of $F$. Let $O = O_F, P = P_F, \omega = \omega_F, k = k_F, q = q_F, \nu_F$ denote, respectively, the ring of integers, the prime ideal, a uniformizing parameter, the residue field, the order of the residue field, and the absolute value on $F$ normalized so that $|x| = q^{-\nu_F(x)}$ where $\nu = \nu_F$ denotes the order function on $F$. Let $U = U_F = O_F^*$ and $U^n = U_F^n = 1 + P_F^n$ for $n$ a positive integer. Further, for $K/F$ Galois, let $\Gamma(K/F)$ denote the associated Galois group and if, in addition,
Let \( [K : F] < \infty \) let \( N_{K/F} = N \) denote the norm map and \( K^1 = K_F^1 \) the norm-one elements in \( K^\times \). Finally, fix an algebraic closure \( \bar{F} \) of \( F \) and a Weil group \( W_F \); let the associated Weil group be as in [T].

For \( G \) a group and \( \sigma \) a representation of a subgroup \( H \), let \( \text{Ind}(G, H; \sigma) \) denote the representation of \( G \) induced by \( \sigma \) (form of induction, e.g., compact, \( L^2 \), determined by context) and, for \( g \) in \( G \), let \( \sigma^g \) denote the representation of \( H^g = gHg^{-1} \) defined by \( \sigma^g(h) = \sigma(g^{-1}hg) \) for \( h \) in \( H^g \). If \( J \) is a subgroup of \( H \), then let \( \sigma|_J \) denote the restriction of \( \sigma \) to \( J \). Further, if \( J \triangleleft H \) and \( \bar{\sigma} \) is a representation of \( H/J \), then we also view \( \bar{\sigma} \) as a representation of \( H \) via inflation. By a character we mean a one-dimensional representation (not necessarily unitary). If \( \chi \) is a character of \( F^\times \), we also view \( \chi \) as a character of \( W_F \) via local class field theory and as a character of \( \text{GL}_2(F) \) by composition with \( \text{det} \), the determinant map. We say representations \( \pi_1 \) and \( \pi_2 \) of \( \text{GL}_2(F) \) are twist equivalent if there exists a character \( \eta \) of \( F^\times \) such that \( \pi_1 \cong \pi_2 \otimes \eta \). Finally, if \( K/F \) is a finite-dimensional extension of fields and \( \text{Gal}(K/F) \), we also view \( \chi \) as a character of \( \text{Gal}(F) \) via composition with \( N_{K/F} \).

We now briefly recall the parametrization of the admissible dual of \( G_1 = G_1(F) = \text{SL}_2(F) \) in [LL]. To do this we first recall the parametrization of the admissible dual of \( G_1' = G_1'(F) = \text{GL}_2(F) \) in [LL] in a form suitable for our purposes. If \( \mu \) and \( \nu \) are characters of \( F^\times \) such that \( \mu(x)\nu^{-1}(x) \neq |x| \) or \(|x|^{-1} \), let \( \pi(\mu, \nu) \) denote the irreducibly induced (normalized induction) principal series representation of \( G_1' \) attached to \( \mu \) and \( \nu \); note that \( \pi(\mu, \nu) \cong \pi(-\nu, \mu) \). If \( \mu(x)\nu^{-1}(x) = |x| \), write \( \mu = \chi|.|^{\frac{1}{2}} \) and \( \nu = \chi|.|^{-\frac{1}{2}} \) with \( \chi \) a character of \( F^\times \) and let \( \sigma(\mu, \nu) \) denote the special representation corresponding to the unique invariant subspace of the space of the associated induced representation from the Borel subgroup of \( G_1' \) and \( \pi(\mu, \nu) \) denote the corresponding quotient. Similarly, if \( \mu(x)\nu^{-1}(x) = |x|^{-1} \), let \( \sigma(\mu, \nu) \) denote the corresponding special representation (now the quotient) and \( \pi(\mu, \nu) \) the corresponding one-dimensional; note that \( \sigma(\mu, \nu) \cong \sigma(\nu, \mu) \) and \( \pi(\mu, \nu) \cong \pi(\nu, \mu) \). Further, if \( K/F \) is quadratic and \( \theta \) is a character of \( K^\times \), we let \( \pi(\theta) = \pi(\rho) \) denote the corresponding irreducible smooth representation of \( G_1' \) associated to \( \rho = \text{Ind}(W_{K/F}, W_{K/F}; \theta) \); note that \( \pi(\theta) \cong \pi(\theta^{-1}) \) and note also that \( \pi \) is supercuspidal if and only if \( \theta \) does not factor through \( N_{K/F} \) which in turn happens if and only if \( \rho \) is irreducible. We call representations of the form \( \pi(\theta) \) Weil representations. The irreducible smooth representations of \( G_1' \) not of one of the above forms are called exceptional and occur only if \( p = 2 \). These representations are supercuspidal and are parametrized naturally by the local Langlands correspondence in terms of the primitive (i.e., not induced from a proper subgroup) two-dimensional representations of \( W_F [Ku] \); for \( \sigma \) such a representation of \( W_F \), we write \( \pi(\sigma) \) for the corresponding exceptional representation. Finally we note that the representations enjoy no other equivalences with the exception that, if \( \mu \) and \( \nu \) are characters of \( F^\times \) with \( \mu\nu^{-1} \) of order two, then \( \pi(\mu, \nu) \cong \pi(\mu_K) \) where \( K/F \) is the quadratic extension of \( F \) associated to \( \mu\nu^{-1} \) by local class field theory.

Then viewing \( G_1 \) as a subgroup of \( G_1' \), we have

**Theorem 1.1 ([LL]).** Let \( \pi_1 \) be an irreducible smooth representation of \( G_1 \). Then there exists an irreducible smooth representation \( \pi \) of \( G_1' \) unique up to twist equivalence which contains \( \pi_1 \) upon restriction to \( G_1 \). The L-packet of \( \pi_1 \) is of the form \( \{ \pi_1, \ldots, \pi_s \} \) where the \( \pi_i \) are distinct irreducible smooth representations of \( G_1 \) and the restriction of \( \pi \) to \( G \) decomposes as \( \bigoplus_{i=1}^s \pi_i \). Further, given \( 1 \leq i, j \leq s \), there exists \( g \) in \( G' \) such that \( \pi_i^g \cong \pi_j \). Moreover:
(i) If \( \pi \) is not a Weil representation, then \( s = 1 \).

(ii) If \( \pi = \pi(\theta) \) with \( \theta \) a character of \( K^\times \) not of order two, then \( s = 2 \) and
\[\pi_i^\alpha \cong \pi_i \] if and only if \( \det g \) is a norm from \( K^\times \). If \( \pi \) is supercuspidal in this setting then \( \rho \) is singly imprimitive (i.e., can only be induced nontrivially from \( W_{K/K} \)).

(iii) If \( \pi = \pi(\theta) \) with \( \theta \) a character of \( K^\times \) of order two, then \( s = 4 \). In this case, \( \rho \) is triply imprimitive and if \( K_i, i = 1,2,3 \), are the fields such that \( \rho \) may be induced from \( W_{K_i/K_i} \), and \( L \) is their composite, then \( \Gamma(L/F) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} \) and \( \pi_i^\alpha \cong \pi_i \) if and only if \( \det g \) is a norm from \( L^\times \).

(iv) The collection of distinct \( L \)-packets partitions \( G_1^\wedge \). Further another representation \( \pi' \) in \( (G_1^\wedge)^\wedge \) gives rise to the same \( L \)-packet as \( \pi \) if and only if \( \pi \) and \( \pi' \) can be realized as follows:

\[\begin{align*}
\text{a)} & \quad \pi = \pi(\mu, \nu) \quad \text{and} \quad \pi' = \pi(\mu', \nu') \quad \text{with} \quad \mu \nu^{-1} = (\mu')^{-1} (\nu')^{-1}, \\
\text{b)} & \quad \pi = \pi(\mu, \nu) \quad \text{and} \quad \pi' = \pi(\mu', \nu') \quad \text{with} \quad \mu \nu^{-1} = (\nu')^{-1} (\mu')^{-1}, \\
\text{c)} & \quad \pi = \pi(\theta) \quad \text{and} \quad \pi' = \pi(\theta') \quad \text{with} \quad \theta \text{ and } \theta' \text{ on } K^\times \text{ with } \theta \theta^{-1} \mid_{K_1} = 1, \\
\text{d)} & \quad \pi = \pi(\sigma) \quad \text{and} \quad \pi' = \pi(\sigma') \quad \text{with} \quad \sigma \text{ and } \sigma' \text{ primitive projectively equivalent representations}.
\end{align*}\]

In what follows we will distinguish among the \( \pi_i \) by their Whittaker models. In particular recall that if \( \pi \) is an infinite-dimensional irreducible smooth representation of \( G' \) and \( \eta \) is a nontrivial character of \( F \), then \( \pi \) has, up to scaling, a unique Whittaker model with respect to \( \eta \) and, of course, if \( \pi \) is a finite-dimensional smooth representation, then it has no Whittaker models. Thus for infinite-dimensional \( \pi \) we let \( \pi(\mu, \nu, \eta) \) denote the component of \( \pi(\mu, \nu) \) upon restriction to \( G_1 \) with \( \eta \)-Whittaker model and similarly for \( \sigma(\mu, \nu) \), \( \pi(\theta) \) and \( \pi(\sigma) \). The only remaining irreducible admissible representation of \( G_1 \) is the trivial representation which we denote by \( 1 \). Finally for \( a \) in \( F^\times \) and \( \pi \) in \( G_1^\wedge \) set \( \pi^a = \pi^g \) where \( g \) is an element of \( G_1^\wedge \) such that \( \det g = a \). It is easy to check that \( \pi^a \) is well defined. We note that \( \pi(\mu, \nu, \eta) \pi^a \) for all \( a \) in \( F^\times \) and similarly for \( \sigma(\mu, \nu) \), \( \pi(\theta) \) and \( \pi(\sigma) \).

We close this section with two lemmas which briefly recall the theta correspondences attached to \( \chi \) and the pairs \( (G_1, H_0) \) and \( (G_1, J_1) \).

**Lemma 1.2** ([C]). If \( \theta \) is a character of \( E^\times \) and \( a \) is an element of \( N_{E/F}(E^\times) \), then \( \pi(\theta, \chi_a) \) occurs in \( \mathcal{R}_A(G_1) \) and these are the only representations of \( G_1 \) that occur. In particular for any character \( \theta \) of \( E^\times \) precisely half of the representations in the \( L \)-packet of \( G_1 \) attached to \( \pi(\theta) \) occur.

**Remark 1.3.** Although the statements concerning Whittaker models in Lemma 1.2 are not explicitly made in [C], they are easy consequences of the results of [C] and Theorem 1.1.

Now consider \( (G_1, J_1) \). Parametrize \( J_1^\wedge \) as follows. First identify \( F^\times \) with the connected component of the identity, \( J_1^0 \), of \( J_1 \) by letting \( a \) in \( F^\times \) act as follows:

\[a \cdot \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} ax & 0 \\ 0 & a^{-1}y \end{pmatrix}\]

where \( x, y \in F \). Let \( \sigma' \) denote the element of \( J_1 \) defined by

\[\sigma' \cdot \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} y & 0 \\ 0 & x \end{pmatrix} \]
Then \( J_1 = J_1^0 \rtimes \langle \sigma' \rangle \). Then one easily checks that if \( \rho \) is a character of \( F^\times \), then \( \Ind (J_1, J_1^0; \rho) \) is irreducible if and only if \( \rho^2 \neq 1 \). If \( \rho^2 \neq 1 \), then, in a slight abuse of notation, we write \( \rho = \Ind (J_1, J_1^0; \rho) \) and note that \( \rho = \rho^{-1} \) as representations of \( J_1 \). If \( \rho^2 = 1 \), then we let \( \rho^+ \) and \( \rho^- \) denote the two components of \( \Ind (J_1, J_1^0; \rho) \). These representations are the extensions of \( \rho \) specified by requiring that \( \rho^+(\sigma') = 1 \) while \( \rho^-(\sigma') = -1 \). All these representations enjoy the obvious equivalences and exhaust \( J_1^0 \).

**Lemma 1.4 ([ST]).** The theta correspondence attached to \( \chi \) and \((G_1, J_1)\) may be parametrized as follows.

(i) \( \mathcal{R}_\chi (J_1) \) contains all of \( J_1^\wedge \) with the exception of \( 1^- \).

(ii) If \( \rho \) is a character of \( F^\times \) such that \( \rho^2 \neq 1 \), then \( \rho \) as a representation of \( J_1^\wedge \) pairs with the representation \( \pi(\rho, 1) \) of \( G_1^0 \).

(iii) If \( \rho \) is a character of \( F^\times \) of order two, then \( \rho^+ \) and \( \rho^- \) pair with the two components of \( \pi(\rho, 1) \) upon restriction to \( G_1 \).

(iv) The trivial representation of \( J_1 \) pairs with the trivial representation of \( G_1 \).

**Remark 1.5.** The relevant statements in [ST] are 1.1.4 and 1.1.5 on p. 1054. It is stated, however, in 1.1.5 that \( F^\times \) (\( N_1 \) in their notation) has a unique character of order two. Although this is incorrect, the proofs given are correct and yield 1.4 as above.

## 2. Some representations which occur

We need two preliminary results. The first is a statement of seesaw duality which can be stated in general as follows. Suspending earlier notation for the moment, a pair of reductive dual pairs \((G_1, G_2)\) and \((G_1', G_2')\) in a symplectic group \( G \) is called a seesaw pair if \( G_1 \supset G_1' \) and \( G_2' \supset G_2 \). This terminology is suggested by the following diagram:

\[
\begin{array}{ccc}
G_1 & \supset & G_1' \\
\downarrow & & \downarrow \\
G_2 & \supset & G_2'
\end{array}
\]

Now if \((\pi, V)\) is an irreducible smooth representation of \( \tilde{G}_1 \) and \((\omega_{\infty}, S)\) is the oscillator representation of \( \tilde{G}_1 \), we set \( S(\pi) = \bigcap \ker f \) where the intersection is over all \( f \) in \( \text{Hom}_{\tilde{G}_1} (S, V) \). Now set \( S[\pi] = S/S(\pi) \) and view it as a \( \tilde{G}_1 \times \tilde{G}_2 \)-space. Then either \( S[\pi] = 0 \) in which case \( \pi \) does not occur in \( \mathcal{R}_\chi(\tilde{G}_1) \) and we set \( \theta_0(\pi) = 0 \) or \( S[\pi] \) is nonzero and by Lemma III.4 of Chapter II of [MVW] there is a smooth representation \( \theta_0(\pi) \) of \( \tilde{G}_2 \) unique up to isomorphism such that \( S[\pi] \cong \pi \otimes \theta_0(\pi) \). Then the strong form of Howe’s duality conjecture is that either \( \theta_0(\pi) \) is zero or is admissible of finite length with unique irreducible quotient \( \theta(\pi) \) (in particular the theta correspondence is a bijection); however, we only need the following concerning \( \theta_0(\pi) \).

**Lemma 2.1** (see, e.g., [P2]). Let \( \pi \) an irreducible smooth representation of \( \tilde{G}_2 \) and \( \pi' \) an irreducible smooth representation of \( \tilde{G}_1 \), then

\[
\text{Hom}_{\tilde{G}_1} (\theta_0(\pi), \pi') \cong \text{Hom}_{\tilde{G}_2} (\theta_0(\pi'), \pi) .
\]
Finally, we need

**Lemma 2.2 ([P1]).** Let $\pi_i$, $i = 1, 2, 3$, be infinite-dimensional irreducible smooth representations of $GL_2(F)$ such that the product of their central characters is trivial and at least one of the representations is not discrete series. Then there exists a unique (up to scalar) $GL_2(F)$-invariant trilinear form on $\pi_1 \otimes \pi_2 \otimes \pi_3$.

We now revert to our previous notation to state and prove the main result of this paper.

**Theorem 2.3.** If $\pi$ is an irreducible smooth representation of $G_1$ whose $L$-packet is not that of a Weil representation coming from $E$, then $\pi$ is in $R^\chi(G_1)$.

**Proof.** First assume that $\pi$ is infinite-dimensional. Let $\pi'_i$ be an irreducible admissible representation of $G'_1$ whose restriction to $G_1$ decomposes as the $L$-packet of $\pi$. Let $\pi'_2$ be a Weil representation of $G'_1$ attached to $E$ and singly imprimitive. Finally let $\pi'_i$ denote an infinite-dimensional nonsimply discrete series representation of $G'_1$ with associated singleton $L$-packet for $G_1$ and also chosen so that the product of the central characters of the $\pi'_i$ is trivial. Clearly such a choice is possible. Then, by Lemma 2.2, $\pi'_1 \otimes \pi'_2 \otimes \pi'_3$ has a $G'_1$-invariant trilinear form. Thus there exist $\pi_i$, $i = 1, 2, 3$, in the $L$-packet of $G_1$ associated to $\pi'_i$ such that $\pi_1 \otimes \pi_2 \otimes \pi_3$ has a nontrivial $G_1$-invariant trilinear form.

Now since $\pi_1 \otimes \pi_2 \otimes \pi_3$ has a $G_1$-invariant trilinear form so does $\pi_i = \pi'_i \otimes \pi'_2 \otimes \pi'_3$ for all $\chi$ in $F^\times$. Further, by Theorem 1.1, $\pi_i \cong \pi$ for all $\chi$ in $F^\times$ and thus $\pi_1 \otimes \pi_2 \otimes \pi_3$ has a $G_1$-invariant trilinear form for all $\chi$ in $F^\times$. Now considering the effect of conjugation on Whittaker models it follows from Theorem 1.1 and Lemma 1.2 that there exists an $\alpha$ in $F^\times$ such that $\pi_i(b, \chi)$ is in $R^\chi_0(G_1)$; fix such an $\alpha$.

Now suppose $\pi_i(\chi) \cong \pi'$, the contragredient of $\pi$. Then $\pi_1 \otimes \pi_2 \otimes \pi'$ has a $G_1$-invariant trilinear form. Moreover $\pi'_2$ and $\pi_1$ are in $R^\chi_0(G_1)$ and $R'_\chi(G_1)$ respectively, in the latter case by Lemma 1.4. Let $\sigma_0$ and $\sigma_1$ be representations in $H'_0$ and $J_i$ respectively such that $\theta(\sigma_0) = \pi'_2$ and $\theta(\sigma_1) = \pi_1$. Now consider the seesaw pair

\[
\begin{array}{ccc}
G_1 \times G_1 & \longrightarrow & H_1 \\
\downarrow & & \downarrow \\
G_1 & \times & H_0 \times J_1.
\end{array}
\]

Now let $W_1 = V_1 \otimes H$ and $W_2 = V_1 \otimes V_0$ with symplectic forms as in the introduction. Then, as usual (see, e.g., [MVW, p. 37]), the restriction of $\omega_\chi$ on $G$ to $Sp(W_1) \times Sp(W_2)$ is given by the (outer) tensor product of $\omega_{\chi,1}$ and $\omega_{\chi,2}$ the oscillator representations of $Sp(W_1)$ and $Sp(W_2)$, respectively, and thus $\theta(\sigma_0 \times \sigma_1) = \pi_1 \otimes \pi'_2$. Then the existence of the $G_1$-invariant trilinear form implies that $\text{Hom}_{G_1} (\theta(\sigma_0 \otimes \sigma_1), \pi)$ is nonzero whence by Lemma 2.1 $\theta(\pi)$ is nonzero so that $\pi$ is in $R^\chi(G_1)$ as desired.

Now suppose $\pi_i(\chi) \cong \pi'$ does not hold. The existence of a $G_1$-invariant trilinear form on $\pi_1 \otimes \pi'_2 \otimes \pi_3$ implies the existence of such a form on $\pi'_1 \otimes \pi'_2 \otimes \pi_3$ for all $b$ in $F^\times$. Now, if $b$ is in $N_{E/F}(E^\times)$, then Theorem 1.1 implies $\pi'_1(\chi) \cong \pi'_2$ and $\pi'_1 \cong \pi_1$. Further $\pi' = \pi^{-1}$ where $-1$ is viewed as an element of $F^\times$ [MVW, p.
It thus suffices to show that there exists $b$ in $N_{E/F}(E^\times)$ such that $\pi_3^{-ba} \cong \pi$. This, however, follows from Theorem 1.1 and our hypothesis that $\pi$ is not in the $L$-packet of a Weil representation attached to $E$.

The only remaining case is $\pi$ trivial. In this case let $\pi_2$ be the component of $\pi(\omega_{E/F}, 1)$ restricted to $G_1$ that occurs in $\mathcal{R}_\chi^0(G_1)$ where $\omega_{E/F}$ is the character of $F^\times$ associated to $E/F$ by class field theory. Further, let $\pi_1 = \pi_2^r$ which in turn is $\pi_2^{-1}$ and thus is in $\mathcal{R}_\chi^0(G_1)$ by Lemma 1.4. Now another seesaw duality argument, as above, gives the result.

**Corollary 2.4.** Let $\pi$ be an irreducible smooth representation of $G_1$. Then $\pi$ occurs in $\mathcal{R}_\chi(G_1)$ if $\pi$ is trivial or if for some $b$ in $N_{E/F}(E^\times)$, $\pi^b$ has a Whittaker model with respect to $\chi$.

**Proof.** For representations not coming from Weil representations attached to $E$ this follows from the previous theorem. For representations coming from Weil representations attached to $E$ this comes from the usual inclusion $\mathcal{R}_\chi^0(G_1) \subseteq \mathcal{R}_\chi(G_1)$ (i.e., persistence) in the theory of towers (see, e.g., [MW]).

**Corollary 2.5.** Let $\{\pi_i\}_{i=1}^s$ be an L-packet for $G_1$. Then

$$|\mathcal{R}_\chi(G_1) \cap \{\pi_i\}| \geq \frac{s}{2}$$

if the packet comes from a Weil representation attached to $E$ and

$$|\mathcal{R}_\chi(G_1) \cap \{\pi_i\}_{i=1}^s| = s$$

otherwise.

**Proof.** Immediate.

**Remark 2.6.** The quadratic spaces $(V_2, q)$ considered here are not the only Witt-rank-one quaternionic quadratic spaces over $F$. To consider all such possible spaces take $V_2$ as before but set $q_{E,a}(A) = aq(A)$ where $a$ is an element of $F^\times$ and the subscript $E$ denotes the dependence of $q$ on $E$. Then as $E$ and $a$ vary the $(V_2, q_{E,a})$ do exhaust (up to isomorphism) the Witt-rank-one quaternionic quadratic spaces over $F$. Moreover $q_{E,a}$ is equivalent to $q_{E',a'}$ if and only if $E = E'$ (recall that we fixed an algebraic closure) and $(a^{-1}a', a^{-1}a') = \omega_{E'/F}(a^{-1}a')$ where $(, )$ is the Hilbert symbol and $\omega_{E/F}$ is the character of $F^\times$ associated to $E/F$ by local class field theory. This can all be checked by comparing discriminants and Hasse invariants. Thus to get all possible theta correspondences it suffices to consider the effect of changing the parameter $a$. This is a straightforward exercise (see, e.g., [M, Remark 2.4(iii)]). The answer is that in Corollary 2.4 $\pi$ occurs in $\mathcal{R}_\chi(G_1)$ if $\pi$ is trivial or, if for some $b$ in $N_{E/F}(E^\times)$, $\pi^b$ has a Whittaker model with respect to $\chi_a$.

**Remark 2.7.** (i) As we noted in the first paragraph of this paper, we can show that the representations shown here to be in $\mathcal{R}_\chi(G_1)$ do in fact exhaust $\mathcal{R}_\chi(G_1)$ at least for $p$ odd. This strengthens Corollary 2.4 to an “if and only if” statement and thereby the inequality in Corollary 2.5 becomes an equality. These stronger results are not amenable to the methods of this paper, however.

(ii) The results of this paper and the indicated strengthening above are consistent with the general conjectures of Prasad [P3]. They are also consistent with Kudla’s conservation conjecture.
References


[P2] D. Prasad, Trilinear forms for representations of $\mathrm{GL}(2)$ and local $\epsilon$-factors, Compositio Math. 75 (1990), 1–46. MR 91i:22023


Department of Mathematics, The University of Iowa, Iowa City, Iowa 52242

E-mail address: david-manderscheid@uiowa.edu