PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 5, Pages 1331–1335
S 0002-9939(99)05210-7
Article electronically published on January 28, 1999

PERIODIC SOLUTIONS OF A PERIODIC DELAY PREDATOR-PREY SYSTEM

LI YONGKUN

(Communicated by Suncica Canic)

ABSTRACT. The existence of a positive periodic solution for

$$\begin{cases} \frac{\mathrm{d}H(t)}{\mathrm{d}t} = r(t)H(t)\left[1 - \frac{H(t-\tau(t))}{K(t)}\right] - \alpha(t)H(t)P(t), \\ \frac{\mathrm{d}P(t)}{\mathrm{d}t} = -b(t)P(t) + \beta(t)P(t)H(t-\sigma(t)) \end{cases}$$

is established, where r, K, α, b, β are positive periodic continuous functions with period $\omega > 0$, and τ, σ are periodic continuous functions with period ω.

1. INTRODUCTION

As pointed out by Freedman and Wu [1] and Kuang [2], it would be of interest to study the global existence of periodic solutions for systems with periodic delays, representing predator-prey or competition systems. The purpose of this article is to consider the following periodic delay predator-prey model:

$$\begin{cases} \dfrac{\mathrm{d}H(t)}{\mathrm{d}t} = r(t)H(t)\left[1 - \dfrac{H(t-\tau(t))}{K(t)}\right] - \alpha(t)H(t)P(t), \\ \dfrac{\mathrm{d}P(t)}{\mathrm{d}t} = -b(t)P(t) + \beta(t)P(t)H(t-\sigma(t)), \end{cases} \tag{1.1}$$

where r, K, α, b, β are positive periodic continuous functions with period $\omega > 0$, and τ, σ are periodic continuous functions with period $\omega > 0$. The system (1.1) was introduced by May in [3, p. 103].

In Section 2, we will use the continuation theorem of coincidence degree theory, which was proposed in [4] by Gaines and Mawhin, to establish the existence of at least one positive ω-periodic solution of system (1.1).

First, consider an abstract equation in a Banach space X,

$$Lx = \lambda Nx, \qquad \lambda \in (0,1), \tag{1.2}$$

where $L\colon \operatorname{Dom} L \cap X \to X$ is a linear operator and λ is a parameter. Let P and Q denote two projectors,

$$P\colon X \cap \operatorname{Dom} L \to \operatorname{Ker} L \quad \text{and} \quad Q\colon X \to X/\operatorname{Im} L.$$

Received by the editors March 5, 1997.
1991 *Mathematics Subject Classification.* Primary 34K15, 34K20, 92A15.
Key words and phrases. Delay equation, predator-prey system, periodic solution.
The author was partially supported by the ABF of Yunnan Province of China.

For convenience we introduce a continuation theorem [4, p. 40] as follows.

Lemma 1.1. *Let X be a Banach space and L a Fredholm mapping of index zero. Assume that $N\colon \overline{\Omega} \to X$ is L-compact on $\overline{\Omega}$ with Ω open bounded in X. Furthermore assume:*

(a) *for each $\lambda \in (0,1)$, $x \in \partial\Omega \cap \operatorname{Dom} L$,*

$$Lx \neq Nx;$$

(b) *for each $x \in \partial\Omega \cap \operatorname{Ker} L$,*

$$QNx \neq 0$$

and

$$\deg\{QNx,\ \Omega \cap \operatorname{Ker} L,\ 0\} \neq 0.$$

Then $Lx = Nx$ has at least one solution in $\overline{\Omega}$.

2. Main result

In what follows, we use the following notation:

$$\bar{u} = \frac{1}{\omega}\int_0^\omega u(t)\mathrm{d}t, \quad (u)_M = \max_{t\in[0,\omega]} |u(t)|, \quad (u)_m = \min_{t\in[0,\omega]} |u(t)|,$$

where u is a periodic continuous function with period ω.

Now we state our fundamental theorem about the existence of a positive ω-periodic solution of system (1.1).

Theorem 2.1. *Assume the following*:

(i) $(b/\beta)_M e^{2\bar{r}\omega} < (K)_m$;
(ii) $\bar{r} > (\overline{r/K})\bar{b}/\bar{\beta}$.

Then system (1.1) *has at least one positive ω-periodic solution.*

Proof. Consider the system

$$\begin{cases} \dfrac{\mathrm{d}x(t)}{\mathrm{d}t} = r(t)\left[1 - \dfrac{e^{x(t-\tau(t))}}{K(t)}\right] - \alpha(t)e^{y(t)}, \\ \dfrac{\mathrm{d}y(t)}{\mathrm{d}t} = -b(t) + \beta(t)e^{x(t-\sigma(t))}, \end{cases} \tag{2.1}$$

where r, K, α, b, β, τ, σ are the same as those in system (1.1). It is easy to see that if the system (2.1) has an ω-periodic solution $(x^*(t), y^*(t))$, then $(e^{x^*(t)}, e^{y^*(t)})$ is a positive ω-periodic solution of system (1.1). Therefore, for (1.1) to have at least one positive ω-periodic solution it is sufficient that (2.1) has at least one ω-periodic solution. In order to apply Lemma 1.1 to system (2.1), we take

$$X = \{(x(t), y(t))^T \in C(R, R^2) : x(t+\omega) = x(t),\ y(t+\omega) = y(t)\}$$

and

$$\|(x,y)^T\| = \max_{t\in[0,\omega]} |x(t)| + \max_{t\in[0,\omega]} |y(t)|.$$

With this norm, X is a Banach space. Let

$$N\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}r(t)\left[1-\frac{e^{x(t-\tau(t))}}{K(t)}\right]-\alpha(t)e^{y(t)}\\-b(t)+\beta(t)e^{x(t-\sigma(t))}\end{bmatrix},$$

$$L\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\dot{x}\\\dot{y}\end{bmatrix},\quad P\begin{bmatrix}x\\y\end{bmatrix}=Q\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}\frac{1}{\omega}\int_0^\omega x(t)\mathrm{d}t\\\frac{1}{\omega}\int_0^\omega y(t)\mathrm{d}t\end{bmatrix},\qquad\begin{bmatrix}x\\y\end{bmatrix}\in X.$$

Since $\operatorname{Ker} L=R^2$ and $\operatorname{Im} L$ is closed in X, L is a Fredholm mapping of index zero. Furthermore, we have that N is L-compact on $\overline{\Omega}$ [4]; here Ω is any open bounded set in X.

Corresponding to equation (1.2), we have

$$\begin{cases}\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}=\lambda\left\{r(t)\left[1-\dfrac{e^{x(t-\tau(t))}}{K(t)}\right]-\alpha(t)e^{y(t)}\right\},\\\dfrac{\mathrm{d}y(t)}{\mathrm{d}t}=\lambda[-b(t)+\beta(t)e^{x(t-\sigma(t))}].\end{cases}\tag{2.2}$$

Suppose that $(x(t),y(t))^T\in X$ is a solution of system (2.2) for a certain $\lambda\in(0,1)$. By integrating (2.2) over the interval $[0,\omega]$, we obtain

$$\int_0^\omega\left\{r(t)\left[1-\frac{e^{x(t-\tau(t))}}{K(t)}\right]-\alpha(t)e^{y(t)}\right\}\mathrm{d}t=0$$

and

$$\int_0^\omega[-b(t)+\beta(t)e^{x(t-\sigma(t))}]\,\mathrm{d}x=0.$$

Thus

$$\int_0^\omega\left[\frac{r(t)e^{x(t-\sigma(t))}}{K(t)}+\alpha(t)e^{y(t)}\right]\mathrm{d}t=\int_0^\omega r(t)\mathrm{d}t\tag{2.3}$$

and

$$\int_0^\omega\beta(t)e^{x(t-\sigma(t))}\,\mathrm{d}t=\int_0^\omega b(t)\mathrm{d}t.\tag{2.4}$$

From (2.2)–(2.4), it follows that

$$\begin{aligned}\int_0^\omega|\dot{x}(t)|\,\mathrm{d}t&\le\lambda\int_0^\omega\left|r(t)\left[1-\frac{e^{x(t-\sigma(t))}}{K(t)}\right]-\alpha(t)e^{y(t)}\right|\mathrm{d}t\\&<\int_0^\omega r(t)\mathrm{d}t+\int_0^\omega\left[\frac{r(t)e^{x(t-\tau(t))}}{K(t)}+\alpha(t)e^{y(t)}\right]\mathrm{d}t\\&=2\int_0^\omega r(t)\mathrm{d}t=2\bar{r}\omega\end{aligned}$$

and

$$\int_0^\omega|\dot{y}(t)|\,\mathrm{d}t\le\lambda\int_0^\omega|-b(t)+\beta(t)e^{x(t-\sigma(t))}|\,\mathrm{d}t<2\bar{b}\omega.$$

That is,

$$\int_0^\omega|\dot{x}(t)|\,\mathrm{d}t<2\bar{r}\omega\tag{2.5}$$

and

$$\int_0^\omega |\dot{y}(t)|\,\mathrm{d}t < 2\bar{b}\omega. \tag{2.6}$$

Moreover, (2.4) implies that there exists a point $\xi_1 \in [0,\omega]$ such that

$$x(\xi_1 - \sigma(\xi_1)) = \log \frac{b(\xi_1)}{\beta(\xi_1)} \le \log \left(\frac{b}{\beta}\right)_M$$

hence

$$|x(\xi_1 - \sigma(\xi_1))| \le \max_{t\in[0,\omega]} \left|\log \frac{b(t)}{\beta(t)}\right| \stackrel{\text{def}}{=} M_1.$$

Denote $\xi_1 + \sigma(\xi_1) = t_1 + n_1\omega$, $t_1 \in [0,\omega]$, and n_1 is an integer; then

$$x(t_1) \le \log \left(\frac{b}{\beta}\right)_M \quad \text{and} \quad |x(t_1)| \le M_1.$$

In view of this and (2.5), we have

$$\begin{aligned} x(t) &\le x(t_1) + \int_0^\omega |\dot{x}(t)|\,\mathrm{d}t \\ &\le \log \left(\frac{b}{\beta}\right)_M + 2\bar{r}\omega \end{aligned} \tag{2.7}$$

and

$$\begin{aligned} |x(t)| &\le |x(t_1)| + \int_0^\omega |\dot{x}(t)|\,\mathrm{d}t \\ &\le M_1 + 2\bar{r}\omega \stackrel{\text{def}}{=} M_2. \end{aligned}$$

By (2.3), (2.7) and assumption (i), we find that there exists a point $\xi_2 \in [0,\omega]$ such that

$$\frac{r(\xi_2)e^{x(\xi_2-\tau(\xi_2))}}{K(\xi_2)} + \alpha(\xi_2)e^{y(\xi_2)} = r(\xi_2),$$

which implies that

$$e^{y(\xi_2)} < \frac{r(\xi_2)}{\alpha(\xi_2)} \le \left(\frac{r}{\alpha}\right)_M$$

and

$$\begin{aligned} e^{y(\xi_2)} &= \frac{r(\xi_2)}{\alpha(\xi_2)}\left[1 - \frac{e^{x(\xi_2-\tau(\xi_2))}}{K(\xi_2)}\right] \\ &\ge \frac{r(\xi_2)}{\alpha(\xi_2)}\left[1 - \frac{(b/\beta)_M e^{2\bar{r}\omega}}{K(\xi_2)}\right] \\ &\ge \left(\frac{r}{\alpha}\right)_m \left[1 - \frac{(b/\beta)_M e^{2\bar{r}\omega}}{(K)_m}\right] \stackrel{\text{def}}{=} M_3 > 0. \end{aligned}$$

Thus,

$$|y(\xi_2)| < \max\left\{\left|\log\left(\frac{r}{\alpha}\right)_M\right|,\ |\log M_3|\right\} \stackrel{\text{def}}{=} M_4.$$

In view of this and (2.6), we obtain that

$$|y(t)| \le y(\xi_2)| + \int_0^\omega |\dot{y}(t)|\,\mathrm{d}t < M_4 + 2\bar{b}\omega \stackrel{\text{def}}{=} M_5.$$

Clearly, M_i $(i = 1, 2, 3, 4, 5)$ are independent of λ, and under the assumption (ii) of the theorem, the system of algebraic equations

$$\begin{cases} \bar{r} - \overline{\left(\dfrac{r}{K}\right)} u - \bar{\alpha} v = 0, \\ -\bar{b} + \bar{\beta} u = 0 \end{cases} \tag{2.8}$$

has a unique solution (u^*, v^*) which satisfies $u^* > 0$ and $v^* > 0$. Denote $M = M_2 + M_5 + C$, where $C > 0$ is taken sufficiently large so that the unique solution of system (2.8) satisfies $\|(u^*, v^*)^T\| = |u^*| + |v^*| < M$. Now we take $\Omega = \{(x(t), y(t))^T \in X \colon \|(x, y)^T\| < M\}$. This satisfies condition (a) of Lemma 1.1. When $(x, y)^T \in \partial\Omega \cap \operatorname{Ker} L = \partial\Omega \cap R^2$, $(x, y)^T$ is a constant vector in R^2 with $|x| + |y| = M$. Then

$$QN \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \bar{r} - \overline{\left(\frac{r}{K}\right)} e^x - \bar{\alpha} e^y \\ -\bar{b} + \bar{\beta} e^x \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Furthermore, it can easily be seen that

$$\deg\{QN(x, y)^T,\ \Omega \cap \operatorname{Ker} L,\ (0, 0)^T\} = \operatorname{sign}[\bar{\alpha}\bar{\beta} u^* v^*] \neq 0.$$

By now we know that Ω verifies all the requirements of Lemma 1.1 and then (2.1) has at least one ω-periodic solution. This completes the proof. $\square$

Acknowledgment

The author is grateful to the referee for his/her careful reading of the manuscript and his/her suggestions for improving the presentation of its contents.

References

1. H. I. Freedman and J. Wu, *Periodic solutions of single-species models with periodic delay*, SIAM J. Math. Anal. **23** (1992), 689–701. MR **93e:**92012
2. Y. Kuang, *Delay Differential Equations with Applications in Population Dynamics*, Academic Press, New York, 1993. MR **94f:**34001
3. R. M. May, *Stability and Complexity in Model Ecosystems*, Princeton Univ. Press, Princeton, NJ, 1974.
4. R. E. Gaines and J. L. Mawhin, *Coincidence Degree and and Non-linear Differential Equations*, Springer, Berlin, 1977. MR **58:**30551

Department of Mathematics, Yunnan University, Kunming, Yunnan 650091, People's Republic of China

E-mail address: yklie@ynu.edu.cn