PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 127, Number 5, Pages 1563–1565
S 0002-9939(99)05245-4
Article electronically published on January 29, 1999

HOMOTOPICALLY NON-TRIVIAL ADDITIVE SUBGROUPS OF HILBERT SPACES

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(Communicated by Dale Alspach)

Abstract. We prove that for a line-free closed additive subgroup of a Hilbert space certain orthogonal projections lead to coverings of this group. This makes it possible to obtain additive subgroups which are homotopically non-trivial.

1. Introduction

An example of a closed additive subgroup in a Hilbert space which is LC^0 but not LC^1 was constructed by Robert Cauty in [Ca]. This solves a problem from the list of J. West [We].

The main step in this construction is to find a connected and arcwise-connected subgroup Γ in a Hilbert space with non-trivial fundamental group: $\pi_1(\Gamma) \neq \{0\}$.

Our aim in this note is to prove a general result in this direction which substantially simplifies the proof that this construction really works.

2. Subgroups with non-trivial fundamental group

Since we are interested in the additive structure in Hilbert spaces, we assume all the vector spaces to be real. We start with the following observation.

Theorem 2.1. *Let Γ be a closed additive subgroup in a Hilbert space H, and let L be a one-dimensional subspace of H. If $L \cap \Gamma \neq \{0\}$, then the orthogonal projection p on $H_0 = L^\perp$ is a continuous and open homomorphism of the group Γ onto its image $\Gamma_0 = p(\Gamma)$. The kernel of this homomorphism is isomorphic to $\mathbb{R}$ or to the group $\mathbb{Z}$ of integers.*

Proof. It is obvious that p is a continuous homomorphism of the group Γ onto Γ_0. We shall show that this mapping is open. Since $K = L \cap \Gamma$ is a closed non-trivial subgroup of the real line, it is isomorphic to $\mathbb{R}$ or $\mathbb{Z}$. In the case $K = \mathbb{R}$ the result is obvious and let us assume that K is isomorphic to $\mathbb{Z}$, so generated by an element $e \in \Gamma \cap L$. Without loss of generality we can also assume that $\|e\| = 1$. Put

$$U = \{g \in \Gamma : |\langle e, g\rangle| \leq \frac{1}{2}\}, \tag{2.1}$$

where $\langle\ ,\ \rangle$ stands for the scalar product.

Received by the editors August 1, 1998 and, in revised form, August 28, 1998.
1991 *Mathematics Subject Classification.* Primary 46B25, 54F45, 20F26, 22A05.
Supported by KBN, grant No. 2 PO3A 042 10.

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It is easy to see that every coset from Γ/K has a representative in U, i.e. $p(U) = p(\Gamma)$. First, we shall show that there is $0 < M < +\infty$ such that

$$\|g\| \leq M\|p(g)\| \tag{2.2}$$

for all $g \in U$.

Assume the contrary. Then, for each natural n there is $g_n \in U$, $g_n = a_n e + p(g_n)$, such that $n\|p(g_n)\| \leq \|g_n\|$. It follows that $n^2\|p(g_n)\|^2 \leq a_n^2 + \|p(g_n)\|^2$ and

$$\|p(g_n)\|^2 \leq \frac{a_n^2}{n^2-1} \leq \frac{1}{4(n^2-1)} \to 0. \tag{2.3}$$

Passing to a subsequence we can assume that (a_n) is convergent, say $a_n \to a$. Hence, $|a| \leq \frac{1}{2}$ and $g_n \to ae \in \Gamma$, so that $a = 0$. But then $a_n \to 0$ and

$$\left[\frac{1}{2a_n}\right] g_n = \left[\frac{1}{2a_n}\right] a_n e + \left[\frac{1}{2a_n}\right] p(g_n) \ \to \ \frac{e}{2} \in \Gamma, \tag{2.4}$$

since, due to (2.3),

$$\frac{\|p(g_n)\|^2}{a_n^2} \leq \frac{1}{n^2-1} \to 0; \tag{2.5}$$

a contradiction. Here $[\alpha]$ denotes the integer part of α.

To show now that $p : \Gamma \to p(\Gamma)$ is open, it suffices to prove that $p(V)$ is open in Γ_0 for small open neighborhoods V of 0 in Γ. Let us take such a neighborhood $V \subset U$. Let $x = p(g)$, $g \in V$. There is $r > 0$ such that $g + g_0 \in V$ for $g_0 \in \Gamma$, $\|g_0\| < r$. Take $x' \in \Gamma_0$, $\|x - x'\| < \frac{r}{M}$. Since $x' - x$ has a representative in U, i.e. there is $g_0 \in U$ such that $x' - x = p(g_0)$, we deduce from (2.2) that $\|g_0\| \leq M\|x' - x\| < r$. Hence $g + g_0 \in V$ and $x' = p(g + g_0)$, so $p(V)$ is open. □

The additive subgroups we will consider should not contain non-trivial linear subspaces (we call them *line-free*), since linear subspaces are uninteresting subgroups from the homotopical point of view and we can always divide them out from the subgroup and the space (cf. [DG]).

Corollary 2.1. *If Γ is a locally arcwise-connected and 1-connected closed line-free subgroup of a Hilbert space H and $e \in \Gamma$, $e \neq 0$, then Γ is a universal covering of its image Γ_0 under the orthogonal projection on $H_0 = e^\perp$ with the kernel isomorphic to $\mathbb{Z}$. In particular, $\pi_1(\Gamma_0) = \mathbb{Z}$.*

Proof. We apply Theorem 2.1 to the line $L = \mathbb{R}e$ and use the universal property of coverings of locally arcwise-connected spaces. □

Example 2.1. In the Hilbert space $H = L^2[0,1]$ consider the subgroup $\Gamma = L^2_{\mathbb{Z}}[0,1]$ consisting of square-integrable functions taking integer values. The subgroup Γ is closed line-free and contractible, since the continuous restriction of the support gives a contraction (cf. [DG]). This subgroup is the closure of the group G generated by the path $g(t) = \chi_{[0,t]}$ of characteristic functions in $L^2[0,1]$. Taking $e = \chi_{[0,1]}$, we get H_0 to be the subspace of $L^2[0,1]$ of mean-zero functions. The orthogonal projection p of $L^2_{\mathbb{Z}}[0,1]$ onto H_0 gives us a universal covering of $p(\Gamma)$, $\pi_1(p(\Gamma)) = \mathbb{Z}$. Elements of $p(\Gamma)$ are mean-zero functions which differ from integer-valued functions by constants.

3. Cauty's example

Cauty's example in [Ca] is the following. Let I be the closed interval $[0,1]$ and E the boundary of the square I^2. For $x = (x^1, x^2) \in E$ let us put

$$R(x) = \{(y^1, y^2) \in I^2 : 0 \le y^i \le x^i, \quad i = 1, 2\} \tag{3.1}$$

and let $F(x)$ be the characteristic function of the rectangle $R(x)$. Let G_2 be the subgroup in the Hilbert space $L^2(I^2)$ generated by $\{F(x) : x \in E\}$ and let Γ_2 be its closure. It is proved in [Ca] in a rather complicated way that both of them have non-trivial fundamental groups.

We shall show it easily using the result from the previous section. First of all, we interpret $L^2(I^2)$ as the tensor product of Hilbert spaces $H \otimes H$, where $H = L^2[0,1]$. Then, $F(x) = 0$ if $x_1 = 0$ or $x_2 = 0$ and $F(1,t) = e \otimes \chi_{[0,t]}$, $F(t,1) = \chi_{[0,t]} \otimes e$, where $e = \chi_{[0,1]}$. This means that $G_2 = G \otimes e + e \otimes G \subset H \otimes H$ and $\Gamma_2 = \Gamma \otimes e + e \otimes \Gamma \subset H \otimes H$, for G and $\Gamma = L^2_{\mathbb{Z}}[0,1]$ from the previous example. In other words, if we regard the product $\Gamma \times \Gamma$ as a subgroup of $H \oplus H$, then Γ_2 is the image of $\Gamma \times \Gamma$ under the continuous group homomorphism

$$\Phi : \Gamma \times \Gamma \ni (g_1, g_2) \mapsto g_1 \otimes e + e \otimes g_2 \in H \otimes H. \tag{3.2}$$

The kernel of the mapping Φ on the level of $H \oplus H$ is $\mathbb{R}(e, -e)$, so that the mapping Φ factors through the orthogonal projection on $H_1 = (e, -e)^\perp$. We know already that the image Γ_1 of $\Gamma \times \Gamma$ under this projection has non-trivial fundamental group (we use $\Gamma \times \Gamma$ instead of Γ, $H \oplus H$ instead of H and $(e, -e)$ instead of e in Corollary 2.1): $\pi_1(\Gamma_1) = \mathbb{Z}$. But the mapping Φ on the level of Γ_1 is a homeomorphism of Γ_1 onto Γ_2. Indeed, for $(g_1, g_2) \in H_1$ we have $\langle g_1, e\rangle = \langle g_2, e\rangle$, where $\langle\ ,\ \rangle$ is the scalar product in H, so

$$\begin{aligned} \|\Phi(g_1, g_2)\|^2 &= \|g_1 \otimes e + e \otimes g_2\|^2 = \|g_1\|^2 + \|g_2\|^2 + 2\langle g_1, e\rangle\langle g_2, e\rangle \\ &\ge \|g_1\|^2 + \|g_2\|^2 \end{aligned} \tag{3.3}$$

which means that $\|\Phi(g_1, g_2)\| \ge \|(g_1, g_2)\|$, whence $\Phi : \Gamma_1 \to \Gamma_2$ is open. Thus the group Γ_2 of Cauty is topologically isomorphic to a certain group our Corollary is speaking about.

Acknowledgment

I am grateful to T. Dobrowolski who drew my attention to the paper [Ca].

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