FINITE-DIMENSIONAL RIGHT IDEALS IN SOME ALGEBRAS ASSOCIATED WITH A LOCALLY COMPACT GROUP

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Abstract. Let \( G \) be a discrete group, a commutative discrete cancellative semigroup or a locally compact abelian group. Let \( UC(G) \) be the space of bounded, uniformly continuous, complex-valued functions on \( G \). With an Arens-type product, the conjugate \( UC(G)^* \) becomes a Banach algebra. We prove, that unlike left ideals, finite-dimensional right ideals exist in \( UC(G)^* \) if and only if \( G \) is compact.

INTRODUCTION AND PRELIMINARIES

Let \( G \) be a locally compact group, \( C(G) \) the space of bounded, continuous, complex-valued functions on \( G \), and \( UC(G) \) the subspace of \( C(G) \) which consists of those functions which are left uniformly continuous, i.e.,

\[
UC(G) = \{ f \in C(G) : s \mapsto f_s : G \to C(G) \text{ is norm continuous} \},
\]

where \( f_s \) is the left translate of \( f \) by \( s \) defined by \( f_s(t) = f(st) \) for all \( t \in G \). Then \( UC(G)^* \) is a Banach algebra under the product

\[
(\mu \nu)(f) = \mu(f\nu) \quad \text{for all} \quad f \in UC(G), \quad \text{where} \quad f\nu(s) = \nu(f_s) \quad \text{for all} \quad s \in G.
\]

In [7], we have dealt with a number of algebras including \( UC(G)^* \), and we have determined all the finite-dimensional left ideals of these algebras. We then deduced that this type of ideals exists in \( UC(G)^* \) if and only if \( G \) is amenable, i.e., there is \( \mu \in UC(G)^* \) such that \( \mu \neq 0 \) and \( \mu(f_s) = \mu(f) \) for all \( f \in UC(G) \) and \( s \in G \). As already remarked in [1, Section 4] and [7], the finite-dimensional right ideals are determined in the same way when the two Arens products coincide in the algebra; for example, in \( WAP(G)^* \), where \( WAP(G) \) is the space of weakly almost periodic functions (see [2, Section 4.2]), or in the group algebra \( L^1(G) \) and the measure algebra \( M(G) \) when \( G \) is compact. However, in [1, Section 4], we have given a class of locally compact abelian groups for which the non-trivial right ideals in \( UC(G)^* \) are all of infinite dimension. In this paper, we let \( G \) be either a discrete group, a commutative discrete cancellative semigroup or a locally compact abelian group, and we show that, in fact, finite-dimensional right ideals exist in \( UC(G)^* \) if and only if \( G \) is compact. This is achieved by using the algebraic structure of...
the uniform compactification $UG$ of $G$. We recall that $UG$ may be regarded as the spectrum of $UC(G)$ equipped with the relative weak*-topology inherited from $UC(G)^*$. By the spectrum of $UC(G)$, we mean the set of all nonzero multiplicative elements $x$ of $UC(G)^*$, i.e., $x(fg) = x(f)x(g)$ for all $f, g \in UC(G)$. It is known that the restriction of the operation of $UC(G)^*$ to $UG$ makes $UG$ into a compact right topological semigroup. This means that the operation is defined for $x$ and $y$ in $UG$ by

$$xy(f) = x(f)y$$

for all $f \in UC(G)$.

This operation is of course associative and is such that the mappings $x \mapsto xy$ and $x \mapsto sx$ from $UG$ into $UG$ are continuous for each $y \in UG$ and $s \in G$. Note that when $G$ is discrete, the space $UC(G)$ and the space of all bounded complex-valued functions on $G$ (usually denoted by $\ell^\infty(G)$) are identical, and so $UG$ and the Stone-Čech compactification $\beta G$ of $G$ are identical.

Recall that the Gelfand mapping $f \mapsto \tilde{f}$, where $\tilde{f}(x) = x(f)$ for $x \in UG$ and $f \in UC(G)$, identifies $UC(G)$ with $C(UG)$ (see [2, Theorem 3.1.7]). Hence the Banach spaces $UC(G)^*$ and $C(UG)^*$ may also be identified by the mapping $\mu \mapsto \tilde{\mu}$, where $\tilde{\mu}(f) = \mu(f)$ for $\mu \in UC(G)^*$ and $f \in UC(G)$.

The closure in $UG$ of a subset $A$ of $UG$ is denoted by $\overline{A}$. If $A$ is a subset of $G$, then $\overline{A}^*$ will denote $\overline{A} \setminus A$. For more information on $UG$, see [2] and [4].

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**Right ideals in $UC(G)^*$**

We begin with some results concerning the algebraic structure of $UG$. The correspondence between $UG$ and $UC(G)^*$ enables us then to prove our main theorem.

**Definition 1.** A subset $V$ of $G$ is said to be **sparse** if it is countably infinite and $sV \cap tV$ is finite whenever $s$ and $t$ are distinct elements of $G$.

These sets exist and were used in [3], [5] and [6] to show the following results.

**Theorem 1.** Let $G$ be either a discrete group or a commutative cancellative discrete semigroup. Let $V$ be a sparse subset of $G$. Then

1. each $x \in V^*$ satisfies $yx \neq zx$ whenever $y \neq z$ in $\beta G$, i.e., $x$ is right cancellative in $\beta G$; and
2. $(\beta G)x_1 \cap (\beta G)x_2 = \emptyset$ whenever $x_1$ and $x_2$ are distinct elements in $V^*$.

As the theorem below shows, these results are also valid in $UG$ when $G$ is a non-compact, locally compact abelian group. For the proof, we need to recall the following facts used in [4] to transfer properties from $\beta G$ when $G$ is discrete to $UG$ when $G$ is not discrete. Write, by [9, Theorem 24.30], $G = \mathbb{R}^n \times H$, where $n \in \mathbb{N}$ and $H$ is a locally compact abelian group containing a compact open subgroup $K$. Let $\phi : H \to H/K$ be the quotient mapping, and $\psi : \mathbb{Z}^n \times H \to \mathbb{Z}^n \times H/K$ be the mapping defined by $\psi(m, h) = (m, \phi(h))$. By [2, Theorem 4.4.4], let $\psi : U(\mathbb{Z}^n \times H) \to \beta(\mathbb{Z}^n \times H/K)$ be the continuous homomorphism which extends $\psi$ to $U(\mathbb{Z}^n \times H)$. We recall also from [4] that $U(\mathbb{Z}^n \times H) = \overline{\mathbb{Z}^n \times H}$ (the closure is taken in $UG$), and that each $x$ in $UG$ can be written as $x = (s, e)\bar{x}$ where $s \in [0, 1]^n$, $e$ is the identity in $H$ and $\bar{x} \in U(\mathbb{Z}^n \times H)$. 


Lemma 1. Let $y$ and $z$ be elements of $U(\mathbb{Z}^n \times H)$, and suppose that $(k,e) y \neq z$ for all $k \in \{0,1\}^n$. Then $(s,e) y \neq z$ in $U(\mathbb{R}^n \times H)$ for all $s \in [0,1]^n$.

Proof. The case of $n = 0$ is trivial, so we start with $n = 1$, and suppose that $y \neq z$ and $(1,e) y \neq z$. We pick a function $f \in UC(\mathbb{Z} \times H)$ such that

$$\hat{f}(y) = \hat{f}((1,e) y) = 0 \quad \text{and} \quad \hat{f}(z) = 1.$$ 

We extend $f$ to a function $g$ which is defined on $\mathbb{R} \times H$ in the following way. We write each $u \in \mathbb{R}$ as $u = m + s$, where $m \in \mathbb{Z}$ and $s \in [0,1]$, and let

$$g(u,h) = \begin{cases} f(m + s, h) & \text{for all } (u,h) \in \mathbb{R} \times H. 
\end{cases}$$ 

This means that, for each fixed $h \in H$, the function $g_h$ defined on $\mathbb{R}$ by $g_h(u) = g(u,h)$ is linear in the interval $[m,m+1]$, $m \in \mathbb{Z}$. Then it is not difficult to verify that the function $g$ is uniformly continuous on $\mathbb{R} \times H$. Let $\tilde{g}$ be the continuous extension of $g$ to $U(\mathbb{R} \times H)$, and let $(y_\alpha, h_\alpha)$ be a net in $(\mathbb{Z} \times H)$ which converges to $y$ in $U(\mathbb{R} \times H)$. Then we have, for each $s \in [0,1],

$$\tilde{g}(s,e) y = \lim_{\alpha} g(y_\alpha + s, h_\alpha)$$ 

$$= \lim_{\alpha} (f(y_\alpha + 1, h_\alpha) - f(y_\alpha, h_\alpha)) s + \lim_{\alpha} f(y_\alpha, h_\alpha) = 0,$$

whereas it is clear that $\tilde{g}(z) = \hat{f}(z) = 1$. Thus $(s,e) y \neq z$.

We come now to the general case. Suppose that $(k,e) y \neq z$ for all $k \in \{0,1\}^n$, and let $s = (s_1, s_2, ..., s_n) \in [0,1]^n$. Then the proof given for the case $n = 1$ implies that $(s_1, k,e) y \neq z$ for all $k \in \{0,1\}^{n-1}$ and $s_1 \in [0,1]$. Then we consider $\mathbb{R} \times H$ instead of $H$. The same argument shows again that $(s_1, s_2, k,e) y \neq z$ for all $k \in \{0,1\}^{n-2}$. Inductively, this leads to the desired result.

\[\square\]

Theorem 2. Let $G$ be a non-compact, locally compact abelian group. Then there are points $x_1$ and $x_2$ in $UG \setminus G$ such that

1. $x_1$ and $x_2$ are right cancellative in $UG$, and
2. $(UG) x_1 \cap (UG) x_2 = \emptyset$.

Proof. Recall that $G = \mathbb{R}^n \times H$. Let $V$ be a sparse subset of $\mathbb{Z}^n \times H/K$. Then, by Theorem 1, each point of $V^*$ is right cancellative in $\beta(\mathbb{Z}^n \times H/K)$. From [4, Theorem 5.4], it follows that every point of $(\tilde{\psi})^{-1}(V^*)$ belongs to $UG \setminus G$ and is right cancellative in $UG$. So Statement (1) follows.

For Statement (2), let $a_1$ and $a_2$ be two distinct elements in $V^*$, and let $x_1$ and $x_2$ be in $U(\mathbb{Z}^n \times H)$ such that $\tilde{\psi}(x_1) = a_1$ and $\tilde{\psi}(x_2) = a_2$. Let $y$ and $z$ be arbitrary elements in $UG$. We claim that $yx_1 \neq zx_2$. We write $y = (s,e) \tilde{y}$ and $z = (t,e) \tilde{z}$, where $s, t \in [0,1]^n$ and $\tilde{y}, \tilde{z} \in U(\mathbb{Z}^n \times H)$. Then, by Theorem 1,

$$\beta(\mathbb{Z}^n \times H/K) a_1 \cap \beta(\mathbb{Z}^n \times H/K) a_2 = \emptyset.$$ 

It follows that, for all $k \in \mathbb{Z}^n$,

$$\tilde{\psi}((k,e) \tilde{y}) \tilde{\psi}(x_1) = \tilde{\psi}((k,e) \tilde{y}) a_1 \neq \tilde{\psi}(\tilde{z}) a_2 = \tilde{\psi}(\tilde{z}) \tilde{\psi}(x_2).$$ 

Since $\tilde{\psi}$ is a homomorphism, this implies that

$$\tilde{\psi}((k,e) \tilde{y} x_1) \neq \tilde{\psi}(\tilde{z} x_2)$$
Lemma 2. Let $UC$.

Then it is easy to check that

If one takes the corresponding $\tilde{\mu}$ extended in the usual way to an element of $UC(G)^\ast$. The lemma above leads to the desired conclusion.

We come now to the correspondence between $UG$ and $UC(G)^\ast$. We need to recall the following definitions.

Definition 2. The total variation of an element $\mu$ of $UC(G)^\ast$ is denoted by $|\mu|$ and defined first for $f \in UC(G)$, $f \geq 0$ by

$$|\mu|(f) = \sup\{|\mu(h)| : h \in UC(G) \text{ and } |h| \leq f\},$$

then extended in the usual way to an element of $UC(G)^\ast$.

The support of an element $\mu$ of $UC(G)^\ast (= C(UG)^\ast)$ is denoted by $\text{supp}(\mu)$ and defined by

$$\text{supp}(\mu) = \{x \in UG : |\mu|(f) \neq 0 \text{ whenever } f \in UC(G), f \geq 0 \text{ and } \tilde{f}(x) \neq 0\}.$$  

Remark. If one takes the corresponding $\tilde{\mu}$ in $C(UG)^\ast$, regards by the Riesz representation theorem (see, e.g., [9, Theorem 14.10]), $\tilde{\mu}$ as a bounded, regular, Borel measure on $UG$, and defines (as usual) the support of $\tilde{\mu}$ by

$$\text{Supp}(\tilde{\mu}) = UG \setminus \bigcup\{U : U \text{ open in } UG \text{ and } |\tilde{\mu}|U = 0\},$$

then it is easy to check that $\text{supp}(\mu) = \text{Supp}(\tilde{\mu})$.

Lemma 2. Let $x$ be a right cancellative element in $UG$, and let $\mu$ be a nonzero element of $UC(G)^\ast$. Then

1. $C = \{f_x : f \in UC(G)\}$ is norm-dense in $UC(G)$ (and so $\mu x \neq 0$),
2. $|\mu x| = |\mu|x$,
3. $\text{supp}(\mu x) = \text{supp}(\mu)x$.

Proof. Clearly, $C$ is a subalgebra of $UC(G)$ since $x$ is multiplicative. Furthermore, since $x$ is right cancellative, we have $yx \neq zx$ in $UG$ whenever $y \neq z$ in $UG$. So there is $f \in UC(G)$ such that $\tilde{f}(yx) \neq \tilde{f}(zx)$. It follows that

$$\tilde{f}_x(y) = y(f_x) = (yx)(f) = \tilde{f}(yx) \neq \tilde{f}(zx) = (zx)(f) = z(f_x) = \tilde{f}_x(z).$$

Therefore $\tilde{C} = \{\tilde{f}_x : f \in UC(G)\}$ separates the points in $UG$, which implies that $\tilde{C}$ is norm-dense in $C(UG)$. Equivalently, $C$ is norm-dense in $UC(G)$. This implies that for a nonzero $\mu$ in $UC(G)^\ast$, there must be a function $f \in UC(G)$ with $\mu x(f) = \mu(f_x) \neq 0$, and so Statement (1) follows.

For simplicity of notation we assume now that $\|\mu\| = 1$. It is known and not difficult to check that $|\mu\nu| \leq |\mu||\nu|$ for all $\mu$ and $\nu$ in $UC(G)^\ast$, in particular $|\mu x| \leq |\mu|x$. So we only need to show that $|\mu|x \geq |\mu|x$. Let $f$ be a nonnegative function in $UC(G)$, and let $\epsilon > 0$ be fixed. Then there exists $h \in UC(G)$ such that $|h| \leq f_x$ and $|\mu(h)| \geq |\mu|(f_x) - \frac{\epsilon}{2}$. Since $\{g_x : g \in UC(G)\}$ is norm-dense in $UC(G)$, we pick $g \in UC(G)$ such that $\|g_x - h\| < \frac{\epsilon}{2}$, and so

$$|\mu g_x| = |\mu(g_x)| \geq |\mu|(h) - \frac{\epsilon}{2} \geq |\mu|(f_x) - \epsilon.$$
Then \( f \) with \( \tilde{\mu} \)ative, i.e., \( \tilde{\mu} \)ingly, \( UG \), we define the following function on \( y \)

\[
\tilde{g}'(x) = \begin{cases} 
(\tilde{f}(y) + \frac{\varepsilon}{2}) \tilde{\mu}(y), & \text{if } |\tilde{f}(y)| \geq \tilde{f}(y) + \frac{\varepsilon}{2} \\
\tilde{g}(y), & \text{if } |\tilde{g}(y)| < \tilde{f}(y) + \frac{\varepsilon}{2}.
\end{cases}
\]

Then \( \tilde{g}' \) is continuous on \( UG \), \( |g'| \leq f + \frac{\varepsilon}{2} \), and

\[
|\tilde{g}(sx)| = |g_x(s)| < |h(s)| + \frac{\varepsilon}{2} \leq f_x(s) + \frac{\varepsilon}{2} = \tilde{f}(sx) + \frac{\varepsilon}{2} \text{ for all } s \in G.
\]

Therefore \( g'_x(s) = \tilde{g}'(sx) = \tilde{g}(sx) = g_x(s) \) for all \( s \) in \( G \), and so the functions \( g'_x \) and \( g_x \) are equal. It follows that

\[
|\mu x(f + \varepsilon)| \geq |\mu x(g')| = |\mu(g'_x)| = |\mu(g_x)| \geq |\mu(f_x)| - \varepsilon = |\mu(x(f)) - \varepsilon|.
\]

Thus \( |\mu x(f)| \geq |\mu x(f)| \), which completes the proof of Statement (2).

Because of Statement (2) we may assume in this last statement that \( \mu \) is nonnegative, i.e., \( \mu = |\mu| \). Let \( y \in supp(\mu) x \), and let \( f \) be a nonnegative function in \( UC(G) \) with \( \tilde{f}(y) \neq 0 \). We need to verify that \( \mu x(f) \neq 0 \). Write \( y = z x \) with \( z \in supp(\mu) \). Then \( f_x \) is clearly nonnegative and \( \tilde{f}_x(z) = z(f_x) = (z x)(f) = \tilde{f}(z x) = \tilde{f}(y) \neq 0 \). Since \( z \in supp(\mu) \), this implies that \( (\mu x)(f) = \mu(f_x) = |\mu|(f_x) \neq 0 \), and so \( y \in supp(\mu x) \).

Conversely, we regard (by the Riesz representation theorem) \( \tilde{\mu} \) as a bounded, regular, Borel measure on \( UG \), and let \( y \) be a point not in \( supp(\mu) x \). Then \( supp(\mu) x \) is compact since it is the continuous image of a compact set, and so it is closed. Therefore there is \( f \in UC(G) \) such that \( \tilde{f}(y) \neq 0 \) and \( \tilde{f}(supp(\mu) x) = \{0\} \). Accordingly,

\[
\mu x(f) = \mu(f_x) = \tilde{\mu}(\tilde{f}_x) = \int_{UG} \tilde{f}_x(z) d\tilde{\mu}(z) = \int_{supp(\mu)} \tilde{f}_x(z) d\tilde{\mu}(z)
\]

\[
= \left( \int_{supp(\mu)} \tilde{f}(z x) d\tilde{\mu}(z) \right) = \int_{supp(\mu)} 0 d\tilde{\mu}(z) = 0.
\]

This means that \( y \) does not belong to \( supp(\mu x) \), and so the proof is complete.

We are now ready to give the main result of the paper.

**Theorem 3.** Let \( G \) be either a discrete group, a commutative cancellative discrete semigroup, or a locally compact abelian group. Then finite-dimensional right ideals exist in \( UC(G)^* \) if and only if \( G \) is compact (and so \( G \) is finite in the first two cases).

**Proof.** The sufficiency is straightforward. In fact, if \( G \) is compact then \( UC(G) = C(G) \), so \( UC(G)^* = M(G) \) (the algebra of bounded, regular, Borel measures on \( G \)), and so the finite-dimensional right ideals are determined in a similar fashion as the left ones, see [7].

For the necessity we suppose that \( G \) is not compact. Let \( R \) be a right ideal of \( UC(G)^* \), and let \( \mu \) be a nonzero element of \( R \). By Theorems 1 and 2 we can take two elements \( x_1 \) and \( x_2 \) in \( UG \setminus G \) which are right cancellative in \( UG \) and with disjoint principal left ideals, i.e., \( (UG) x_1 \cap (UG) x_2 = \emptyset \). Then, by Lemma 2, \( \mu x_1 \)
and $\mu x_2$ are nonzero elements of $R$. Furthermore, 
\[\text{supp}(\mu x_1) \subseteq \text{supp}(\mu) x_1 \subseteq (UG)x_1 \quad \text{and} \quad \text{supp}(\mu x_2) \subseteq \text{supp}(\mu) x_2 \subseteq (UG)x_2.\]
Accordingly, $\text{supp}(\mu x_1) \cap \text{supp}(\mu x_2) = \emptyset$. Suppose now that $R$ is of dimension $n$, say. Then, for some complex scalars $a_1, a_2, ..., a_n$, we must have
\[a_1 \mu x_1 + a_2 \mu x_2 + ... + a_n \mu x_1 x_2^{n-1} = 0.\]
Such an identity is clearly true if and only if $a_1 = a_2 = ... = a_n = 0$, and so the proof is complete. \hfill \Box

\textbf{Remark.} Let $G$ be a nondiscrete, locally compact, abelian group. As it is well known, $L^\infty(G)^*$ is a Banach algebra with the (first) Arens product. The product of $\mu$ and $\nu$ in $L^\infty(G)^*$ may be described by $(\mu \circ \nu)(f) = \mu(f_\nu)$, where $f_\nu(\phi) = \nu(\hat{\phi} * f)$ for $f \in L^\infty(G)$ and $\phi \in L^1(G)$ and where $\hat{\phi}(s) = \phi(s^{-1})$ for $s \in G$. In this algebra, we can only answer partially the question of whether finite-dimensional right ideals exist. In fact, let $R$ be a right ideal, and suppose that $\mu(f) \neq 0$ for some $\mu$ in $R$ and $f \in UC(G)$. It can be checked directly that the restriction of the product of $L^\infty(G)^*$ to $UC(G)$ coincides with that of $UC(G)^*$, i.e., $\mu \circ \nu(f) = \mu \nu(f)$ for all $f \in UC(G)$. Accordingly, we may regard $R$ as a nonzero right ideal of $UC(G)^*$, and so Theorem 3 says that $R$ cannot be of finite dimension unless $G$ is compact. However, it may happen that $\mu(f) = 0$ for all $f \in UC(G)$ and for all $\mu \in R$. Such elements satisfy $L^\infty(G)^* \circ \mu = \{0\}$, and they exist because $L^\infty(G) \setminus UC(G)$ is even nonseparable, see [8]. We still do not know whether in this situation $\mu \circ L^\infty(G)^*$ can be of finite dimension.

\textbf{References}


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