COMPOSITION OPERATORS: HYPERINVARIANT SUBSPACES, QUASI-NORMALS AND ISOMETRIES

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Abstract. We exhibit hyperinvariant subspaces of some composition operators. We also consider quasi-normal composition operators and discuss the commutant of isometric composition operators.

1. Introduction

Let $D$ be the unit disc in the complex plane. The Hardy space on $D$, $H^2(D)$, is defined to be the set of analytic functions on $D$ which have square summable power series coefficients. Given an analytic self map of the disc, $\phi$, we may define a composition operator $C_\phi$ on $H^2$ by $C_\phi(f) = f \circ \phi$, for all $f$ in $H^2$. These bounded operators have been studied extensively (see [18] or [10]).

We say an operator $B$ commutes with an operator $A$ if $AB = BA$. Recently, we have been interested in which operators $B \in B(H^2)$ commute with a given composition operator, $C_\phi$ (see [6] and [7]). In the second section we address the question of what are the hyperinvariant subspaces for a composition operator, $C_\phi$, induced by a particular type of function, $\phi$. The question was of interest in that a solution may provide a tool for classifying the commutant of $C_\phi$; that is, the algebra of all operators which commute with $C_\phi$. In particular, in the second section, we show (Corollary 2) that if $C_\phi$ is Riesz, it has a triangularizing chain of hyperinvariant subspaces. In the third section, we develop some tools for classifying quasi-normal composition operators, while in the fourth section, we discuss the commutant of composition operators which are also isometries and pose some research questions.

2. Hyperinvariant subspaces

Let $\phi$ be an analytic self map of the disc and define

$$\phi^{[n]} = \phi \circ \phi \circ \ldots \circ \phi,$$

$n$ times
the \( n^{th} \) iterate of \( \phi \) under composition. Also, suppose \( \phi(0) = 0 \) and \( 0 < |\phi'(0)| < 1 \).

In 1884, Koenigs showed that the sequence \( \{\sigma_k\} \) with

\[
\sigma_k(z) = \frac{\phi^{|k|}(z)}{(\phi'(0))^k}
\]

converges uniformly on compact subsets of \( D \) to a non-constant function \( \sigma \), which is known as the Koenigs' function for \( \phi \) (see [18] or [10]). Paul Bourbon proved the following theorem when \( \phi \) was univalent, and shortly afterwards Pietro Poggi-Corradini was able to remove that hypothesis (see [1] or [14]).

**Theorem 1.** Let \( \phi \) be an analytic self map of the disc with \( \phi(0) = 0 \) and \( 0 < |\phi'(0)| < 1 \). Let \( \sigma \) be the Koenigs' function of \( \phi \) and \( q \) a natural number. If \( (\sigma)^q \) is in \( H^2 \), then the sequence \( \{(\sigma_k)^q\} \) converges to \((\sigma)^q \) in the \( H^2 \) norm.

In [2], Bourdon and Shapiro proved a sufficient condition for the Koenigs' function to belong to \( H^p \) and showed the condition to be necessary in the case that the function \( \phi \) is analytic on the closed unit disc. In [15], Pietro Poggi-Corradini was able to prove the necessity of the condition without any additional conditions on \( \phi \). These results together lead to the following theorem.

**Theorem 2.** Let \( \phi \) be an analytic self map of the disc with \( \phi(0) = 0 \) and \( 0 < |\phi'(0)| < 1 \). Let \( \sigma \) be the Koenigs function. Then \((\sigma)^q \) is in \( H^2 \) if and only if \( |\phi'(0)|^q \) exceeds the essential spectral radius of \( C_\phi \).

A subspace \( M \) is hyperinvariant for an operator \( A \) if it is invariant for every operator which commutes with \( A \).

**Theorem 3.** Let \( \phi \) be an analytic self map of the disc with \( \phi(0) = 0 \) and \( 0 < |\phi'(0)| < 1 \). Suppose \((\sigma)^q \) is in \( H^2 \) where \( \sigma \) is the Koenigs' function of \( \phi \). Then for each natural number \( k \), \( 1 \leq k \leq q \), the subspaces \( z^k H^2 \) are hyperinvariant for \( C_\phi \).

**Proof.** In [7], we showed that \( z H^2 \) is a hyperinvariant subspace for such a \( C_\phi \). This covers the case \( k = 1 \). Proceeding by induction on \( n < q \), we will assume, for \( k \leq n \), that \( z^k H^2 \) is hyperinvariant for \( C_\phi \). Let \( A \) be an operator which commutes with \( C_\phi \) and let \( z^{n+1} p \) be a function in \( z^{n+1} H^2 \) where \( p \) is a polynomial. We wish to show

\[
\langle A(z^{n+1} p), z^l \rangle = 0
\]

for \( l \leq n \). Since \( z^{n+1} p \) is in \( z^n H^2 \), we have by the induction hypothesis that

\[
\langle A(z^{n+1} p), z^l \rangle = 0
\]

for \( l < n \). Now

\[
\langle AC^*_\phi(z^{n+1} p), z^n \rangle = \langle A(z^{n+1} p), (C^*_\phi)^m(z^n) \rangle
\]

\[
= \langle A(z^{n+1} p), (\phi'(0))^{mn} z^n \rangle.
\]

The last equality follows from the induction hypothesis and the fact that \( C^*_\phi \) is upper triangular as a matrix when represented in the standard basis. Dividing both sides of the equation by \((\phi'(0))^{mn}\), it follows that

\[
\langle A((\sigma_m)^n \phi^{|m|}(p \circ \phi^{|m|})), z^n \rangle = \langle A(z^{n+1} p), z^n \rangle.
\]

The left-hand side of this equation is equal to

\[
\langle (\sigma_m)^n \phi^{|m|}(p \circ \phi^{|m|}), A^*(z^n) \rangle.
\]
Since \( H^{2q} \subset H^{2n} \) when \( q \geq n \), \((\sigma)^n\) is in \( H^2 \). By Theorem 1, \((\sigma_m)^n\) is a bounded sequence in the \( H^2 \) norm. Since \( p \) is in \( H^\infty \),
\[
\{(\sigma_m)^n \phi[Lp](p \circ \phi[Lm])\}
\]
is also a bounded sequence in the \( H^2 \) norm. This sequence converges pointwise to 0 as \( m \) goes to \( \infty \), and thus it converges weakly to 0. It follows that \( \langle A(z^{n+1}p), z^n \rangle = 0 \).

If \( z^{n+1}f \) is in \( z^{n+1}H^2 \), we may find a sequence of polynomials \( \{p_l\} \) which converges strongly to \( f \) as \( l \) goes to \( \infty \). In particular, \( \langle A(z^{n+1}p_l), z^n \rangle \) will converge to \( \langle A(z^{n+1}f), z^n \rangle \) as \( l \) goes to infinity. Hence the result holds.

Let \( \|A\|_e \) denote the essential norm of an operator \( A \). An operator \( A \) is a Riesz operator if \( \|A^n\| \) tends to 0 as \( n \) tends to \( \infty \). This implies that \( A \) is Riesz if and only if the essential spectrum of \( A \) is \( \{0\} \). In [3], Bourdon and Shapiro show that a Riesz composition operator must be induced by a function which has an interior fixed point in \( D \). For basic details of Riesz operators see [11].

**Corollary 1.** Let \( \phi \) be an analytic self map of \( D \) with \( \phi(b) = b \) where \( b \) is in \( D \) and \( 0 < |\phi'(b)| < 1 \). If \( C_\phi \) is a Riesz operator, then \( C_\phi \) has a triangularizing chain of hyperinvariant subspaces.

**Proof.** Let \( \alpha(z) = \frac{b-z}{1-bz} \). Define \( \psi = \alpha \circ \phi \circ \alpha \). Note \( \psi(0) = 0 \) and \( 0 < |\psi'(0)| < 1 \). Since \( C_\psi = C_\alpha C_\phi C_\alpha \), it follows that \( C_\psi \) is a Riesz operator. Now by Theorem 2 and Theorem 3, \( C_\psi \) has the \( z^nH^2 \) as hyperinvariant subspaces for all \( q = 0, 1, 2, \ldots \). Thus \( C_\phi \) has \((\alpha)^qH^2, q = 0, 1, 2, \ldots \) as a chain of hyperinvariant subspaces. Moreover, any operator which holds these subspaces invariant must be lower triangular with respect to the orthonormal basis \( \{((\alpha)^n \frac{1-|b|^2}{1-b^2} : n = 0, 1, 2, \ldots\} \).

Let \( f^{(k)} \) be the \( k \)th derivative of \( f \).

**Corollary 2.** Let \( \phi \) be an analytic self map of the disc with \( \phi(0) = 0 \) and \( 0 < |\phi'(0)| < 1 \). Let \( k \) be a natural number greater than or equal to 1. Suppose \((\sigma)^{k+1}\) is in \( H^2 \) and let \( A \) commute with \( C_\phi \). Then, for all \( f \) in \( H^2 \),
\[
\frac{d^{k}}{dz^k}(A(f))(0) = k! \sum_{n=1}^{k} \frac{f^{(n)}(0)}{n!}(A(z^n), z^k).
\]

**Proof.** Let \( f = \sum_{n=0}^{\infty} a_n z^n \) be the power series expansion of \( f \). In [7], we showed that if \( A \) commutes with \( C_\phi \), then \( \langle A(f), z^n \rangle = A(1)f(0) \) where \( A(1) \) is a constant. This implies that \( \langle A(a_0), z^k \rangle = 0 \). Now
\[
\langle A(f), z^k \rangle = \langle A(a_0), z^k \rangle + \sum_{n=1}^{k} a_n \langle A(z^n), z^k \rangle + \langle A(\sum_{n=k+1}^{\infty} a_n z^n), z^k \rangle.
\]
The last term is 0 since by Theorem 3, \( z^{k+1}H^2 \) is hyperinvariant for \( C_\phi \). Since \( a_n = \frac{f^{(n)}(0)}{n!} \), the result follows.

The next example shows that there are analytic self maps of \( D \) with \( \phi(0) = 0 \) and \( \phi'(0) = 0 \) such that the \( z^nH^2 \) are not hyperinvariant for \( C_\phi \) for \( q \geq 2 \).

**Example 1.** Let \( n \) be a natural number greater than 1. For \( C_{z^n} \), the subspaces \( z^nH^2 \) for \( q \geq 2 \) are not hyperinvariant.
Proof. In [6], we discuss the commutant of $C_z$ in more detail. Let $q$ be a natural number greater than or equal to 2. If we define an operator $A$ on $H^2$ by $A(z^{q_n}) = z^{n'}$ and $A(z^k) = 0$ when $k \neq q \cdot n'$, then $A$ commutes with $C_z$. Now $A(z^q) = z$ and hence $z^q H^2$ is not a hyperinvariant subspace for $C_z$. □

3. QUASI-NORMALS

We now turn to quasi-normal operators. An operator $A$ is quasi-normal if $A$ commutes with $A^* A$. For more information, see [8].

Lemma 1. If $C_z$ is quasi-normal, then $\phi(0) = 0$.

Proof. $C_z C_z C_z C_z(1) = C_z C_z C_z C_z(1)$ implies that

$$\frac{1}{1 - \phi(0) \phi(z)} = \frac{1}{1 - \phi(0) z}.$$ 

It follows that $\phi(0) = 0$. □

Lemma 2. Let $\phi$ belong to $L^2(\partial D)$ with $\phi$ non-zero and not a characteristic function of a proper subset of the unit circle. Also suppose that $\|\phi\|_{L^2} = \mu$ and $\|\phi(\cdot t)\|_{L^2} = \mu$ where $k_t$ is a sequence of natural numbers which diverges to $\infty$ and $\mu$ is a non-zero real number. Then $|\phi(e^{i\theta})| = 1$ almost everywhere on $\partial D$.

Proof. Let $\lambda = \mu^2$. Let $m(A)$ be the normalized standard Lebesgue measure of a set $A \subset \partial D$. Suppose that $|\phi(e^{i\theta})| > 1$ on a set of positive measure of $\partial D$. Then, in particular, there exists $\epsilon > 0$ such that the set $A_\epsilon = \{ e^{i\theta} : |\phi(e^{i\theta})| > 1 + \epsilon \}$ has positive measure. It follows that

$$\lambda = \|\phi(k_t)\|_{L^2}^2 \geq \frac{1}{2\pi} \int_{A_\epsilon} |\phi(z)|^2dz \geq (1 + \epsilon)^{2k_t} m(A_\epsilon).$$

As $k_t$ tends to infinity, the right-hand side also tends to infinity. This is a contradiction; thus $|\phi(e^{i\theta})| \leq 1$ almost everywhere.

Similarly, suppose that $|\phi(e^{i\theta})| < 1$ on a set of positive measure of $\partial D$. Then, in particular, there exists $\epsilon > 0$ such that the set $B_\epsilon = \{ e^{i\theta} : |\phi(e^{i\theta})| < 1 - \epsilon \}$ has positive measure. Then

$$\lambda = \frac{1}{2\pi} \int_{B_\epsilon} |\phi(z)|^2dz + \frac{1}{2\pi} \int_{\partial D \setminus B_\epsilon} |\phi(z)|^2dz.$$

Let $\int_{\partial D \setminus B_\epsilon} |\phi(z)|^2dz = \kappa$. We note $\kappa$ is strictly less than $\lambda$. Now

$$\frac{1}{2\pi} \int_{B_\epsilon} |\phi(z)|^2dz \leq (1 - \epsilon)^{2k_t} m(B_\epsilon).$$

Choose a natural number $N$ such that $k_t \geq N$ implies that $(1 - \epsilon)^{2k_t} m(B_\epsilon)$ is strictly less than $\lambda - \kappa$. Since $|\phi(z)| \leq 1$ almost everywhere, we note that $|\phi(z)|^{k_t} \leq |\phi(z)|$. Then with $k_t \geq N$,

$$\lambda = \|\phi(k_t)\|_{L^2}^2 = \frac{1}{2\pi} \int_{B_\epsilon} |\phi(z)|^2 dz + \frac{1}{2\pi} \int_{\partial D \setminus B_\epsilon} |\phi(z)|^2 dz$$

$$\leq \frac{1}{2\pi} \int_{B_\epsilon} |\phi(z)|^2 dz + \frac{1}{2\pi} \int_{\partial D \setminus B_\epsilon} |\phi(z)|^2 dz$$

$$\leq \lambda - \kappa.$$
Proof. If inner.

implies that \( \phi \) is an inner function.

Proof. Let the power series of \( \ker \) normal, then \( A \)

Next suppose that only a finite number of the coefficients \( a_n \) are non-zero but more than one of them is non-zero. Thus let \( \phi(z) = z^k a_k + \cdots + a_l z^l \) where \( a_l \) and \( a_k \) are non-zero and \( k \) is the smallest power of \( z \) with a non-zero coefficient and \( l \) is the greatest. Now,

\[
\langle (\phi(z))^l, (\phi(z))^k \rangle = (a_k)^l (a_l)^k,
\]

which by hypothesis must be zero. Thus if only a finite number of coefficients are non-zero, we have \( \phi(z) = cz^k \) where \( c \) is a constant of modulus less than or equal to 1. If \( k \geq 2 \) and \( c \) is less than 1 in modulus, then \( C_\phi \) is not quasi-normal, so in this case \( k = 1 \).

Suppose an infinite number of the coefficients \( a_n \) are non-zero. Then \( \phi(z) = \sum_{n=k}^{\infty} a_n z^n \) where \( a_k \) is the first non-zero coefficient. Now \( C_\phi^* C_\phi(z) = C_\phi C_\phi^*(z) \) implies that

\[
\sum_{n=k}^{\infty} \lambda_n a_n z^n = \sum_{n=k}^{\infty} \lambda_1 a_n z^n.
\]

It follows that \( \lambda_n = \lambda_1 \) for infinitely many \( n \). Since \( \lambda_1 = \langle \phi(z), \phi(z) \rangle \) and \( \lambda_n = \langle (\phi(z))^n, (\phi(z))^n \rangle \), we may apply Lemma 2 to conclude that \( \phi \) must be an inner function.

\[\square\]

Corollary 3. If \( C_\phi \) has \( z^q H^2 \), \( 1 \leq q < \infty \), as hyperinvariant subspaces, and \( C_\phi \) is quasi-normal, then either \( \phi(z) = cz \) for some \( c \) of modulus less than 1 or \( \phi \) is inner.

Proof. If \( C_\phi \) is quasi-normal, then \( C_\phi \) commutes with \( C_\phi^* C_\phi \), as does \( C_\phi^* \). Thus \( C_\phi^* C_\phi \) is both upper and lower triangular with respect to the standard basis and thus diagonal. Apply Theorem 4.

\[\square\]

We could apply Corollary 1 to conclude that if \( C_\phi \) is quasi-normal and Riesz, then \( \phi(z) = cz \) for \( ||c|| < 1 \). This would say that if \( C_\phi \) is Riesz and quasi-normal, then it is normal. As the next theorem shows, this is true in greater generality.

Theorem 5. If \( A \) is an operator on a Hilbert space \( H \), and \( A \) is Riesz and quasi-normal, then \( A \) is normal.

Proof. If \( A \) is identically 0, then we are done. Assume \( A \) is not equal to 0. By [4], \( ker(A) \) is reducing, so we may decompose \( A \) as \( A_1 \oplus 0 \) with \( A_1 \) quasi-normal and injective. Moreover, we may decompose \( A_1 \) as \( BV \) with \( B \) self-adjoint and injective, \( V \) isometric, and \( B \) commuting with \( V \). Since \( A_1^n = V^n B^n \), we have \( \|(V^*)^n A_1^n\| = \|B^n\| \). Thus

\[
\|B^n\|_{\mathcal{L}}^{\frac{1}{2}} \leq \|(V^*)^n\|_{\mathcal{L}}^{\frac{1}{2}} \|A_1^n\|_{\mathcal{L}}^{\frac{1}{2}} \leq \|A_1^n\|_{\mathcal{L}}^{\frac{1}{2}}.
\]
The last term goes to 0 as $n$ goes to $\infty$, and thus $B$ is also Riesz. By Theorem 3.7 in [19], since $B$ is self-adjoint and Riesz, it must be compact. Hence by the spectral theorem, $B$ is unitarily equivalent to $\sum \oplus \lambda_k I_k$ where $\lambda_k$ are the non-zero eigenvalues and the $I_k$ are identity operators on finite-dimensional spaces. Since eigenspaces are hyperinvariant, and $V$ commutes with $B$, $V$ is unitarily equivalent to $\sum \oplus V_k$, where the $V_k$ are isometries on finite-dimensional spaces and hence unitary. Thus $V$ is a unitary operator and $A_1$ is a product of commuting normals and thus normal. \qed

4. Isometries

An operator $A$ on $H^2$ is an isometry if $A^* A = I$. In [17], Schwarz proves the following theorem.

**Theorem 6.** $C_\phi$ is an isometry on $H^2$ if and only if $\phi(0) = 0$ and $\phi$ is an inner function.

If $\phi$ is an elliptic disc automorphism, then the commutant of $C_\phi$ is well understood (see [6] or [7]). If $\phi$ is not an elliptic disc automorphism and $C_\phi$ is an isometry, Nordgren ([13]) shows that $C_\phi$, restricted to the constants, and $C_\phi$, restricted to $zH^2$, are the unitary and purely isometric parts, respectively. If $A$ commutes with $C_\phi$, then the constants are a reducing subspace for $A$ (see [7]). Hence we may consider the commutant of $C_\phi$ as the direct sum of the commutant of the unitary part and the commutant of the purely isometric part. The purely isometric part is similar to a unilateral shift of infinite multiplicity on the wandering subspace, $M$, of $C_\phi$ (see [16]). The commutant of such a unilateral shift is given in terms of multiplication operators on $H^2(M)$ (see [16]).

**Lemma 3.** Let $\phi = zf$ where $f$ is a nonconstant inner function. Let $\{g_n : n = 0, 1, \ldots\}$ be an orthogonal basis for $(fH^2)^\perp$. Then the following set, $S$,

$$\{z(\phi)^k g_n : n = 0, 1, 2, \ldots, k = 0, 1, 2, \ldots\},$$

is an orthogonal basis for the wandering subspace of $C_\phi$ on $H^2$.

**Proof.** It is easy to see that the vectors in $S$ are orthogonal. The wandering subspace is given by the perp of $C_\phi(zH^2)$ in $zH^2$ (see [12]). We want to show that the vectors in $S$ span this space. We note that the set of vectors $T$, given by $\{(\phi)^k : k = 1, 2, \ldots\}$ span $C_\phi(zH^2)$. Thus it is sufficient to show that the vectors in $T$ and $S$, along with the set containing the constant function 1, span all of $H^2$ since these three sets are mutually orthogonal.

Suppose that $h$ is orthogonal to the vectors in these three sets. We claim that $h$ is 0. First of all $\langle h, 1 \rangle = 0$ implies that $h = zk_1$ for some $k_1$ in $H^2$. Now $\langle zk_1, zg_n \rangle = \langle k_1, g_n \rangle = 0$ for all $n$ implies that $k_1$ belongs to $fH^2$. Thus $h = zfh_1$ for some $h_1$ in $H^2$. Thus $h = \phi h_1$. Now if we proceed by induction and assume that $h = (\phi)^l h_l$ for some $h_l$ in $H^2$, we have that $\langle (\phi)^l h_1, (\phi)^l \rangle = 0$ implies that $h_l = zk_{l+1}$ for some $k_{l+1}$ in $H^2$. Thus $\langle (\phi)^l z k_{l+1}, (\phi)^l g_n \rangle = \langle k_{l+1}, g_n \rangle = 0$ for all $n$ implies that $k_{l+1} = fh_{l+1}$ for some $h_{l+1}$ in $H^2$. Thus $h = (\phi)^{l+1} h_{l+1}$ and we have by induction that $(\phi)^l$ divides $h$ for arbitrary powers of $l$ and hence $h = 0$. \qed
In particular, let $\phi = zB$ where $B$ is a Blaschke product with zero set $\{a_n : n = 0, 1, 2, \ldots\}$. Let $k_\lambda$ be the reproducing kernel function for $\lambda$ in $D$ and let $B_l$ be the Blaschke product with zero set $\{a_n : n = 0, 1, \ldots, l\}$. Then the following is an orthonormal basis for $(BH^2)_⊥$: 
\[
\{g_0 = k_{a_0}, g_1 = B_0 k_{a_1}, \ldots, g_n = B_{n-1} k_{a_n}\}.
\]
This gives us a basis for the wandering subspace of $C_\phi$ and the commutant of $C_\phi$ can be explicitly interpreted in terms of this basis.

Questions

(1): If $\phi$ is an analytic self map of $D$, $\phi(0)=0$ and $0 < |\phi'(0)| < 1$, are the subspaces $\{z^qH^2\}$ hyperinvariant subspaces for all natural numbers $q$? If this is true, then the only $C_\phi$ which are quasi-normal are either isometries or are given by $\phi(z) = cz$ for some constant $c$ of absolute value less than 1.

(2): If $\phi$ is an analytic self map of $D$, $\phi(0)=0$ and $\phi'(0)=0$, what can be said about the hyperinvariant subspaces of $C_\phi$? If $C_\phi$ is quasi-normal, what form does $\phi$ have to take?

(3): Given a basis for the wandering subspace of an isometric composition operator, can the hyperinvariant subspaces or the composition operators which commute be explicitly determined?

References


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