ACCELERATIONS OF RIEMANNIAN QUADRATICS

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Abstract. A Riemannian corner-cutting algorithm generalizing a classical construction for quadratics was previously shown by the author to produce a \( C^1 \) curve \( p_\infty \) whose derivative is Lipschitz. The present paper takes the analysis of \( p_\infty \) a step further by proving that it possesses left and right accelerations everywhere. Two-sided accelerations are shown to exist on the complement of a countable dense subset \( D \) of the domain. The results are shown to be sharp in the following sense. For almost any scaled triple in Euclidean space there is a Riemannian perturbation of the Euclidean metric such that the two-sided accelerations of the resulting curve \( p_\infty \) exist nowhere in \( D \).

1. Background in brief

A very detailed description of the construction of the Riemannian quadratics is given in [11], but the following summary is enough for the present paper to be read independently. Let \( \langle \cdot, \cdot \rangle \) be a Riemannian metric on an open subset \( V \) of \( \mathbb{R}^n \), possibly realised as a coordinate chart of a more general Riemannian manifold. Let \( V \) be geodesically convex in the sense that any two points in \( V \) are joined by a geodesic segment, unique up to reparameterization and minimal. Let \( U \) be another open subset of \( \mathbb{R}^n \) whose closure is compact and contained in \( V \).

Let \( d \) be the metric on \( U \) defined by the Riemannian distance. Then \( d(x_a, x_b) \) is bounded for all \( x_a, x_b \in U \). For \( a < b \in \mathbb{R} \) let \( C[a, b] \) be the complete metric space of continuous curves \( \omega: [a, b] \to U \) with respect to the uniform metric \( d_U \) where

\[
d_U(\omega, \omega') = \max_{t \in [a, b]} d(\omega(t), \omega'(t)).
\]

The Christoffel transformations

\[
\Gamma_x: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n
\]

are also bounded for \( x \in U \).

Because \( U \) is convex we can define for any \( x_a, x_b \in U \) the midpoint \( M(x_a, x_b) \in U \) to be \( \omega((a + b)/2) \) where \( \omega: [a, b] \to U \) is the minimal geodesic from \( x_a \) to \( x_b \). A Riemannian scaled triple is a quadruple \( Y = (y_0, y_1, y_2, h) \in U^3 \times \mathbb{R}_+ \) where \( y_0, y_1, y_2 \) are the vertexes of \( Y \) and \( h \) is the scale. The fundamental polygon \( p: [0, 2h] \to U \) of \( Y \) is the track sum of the geodesic segments joining \( y_0, y_1 \) and \( y_1, y_2 \), parameterized by \( [0, h] \) and \( [h, 2h] \) respectively. Using the midpoint map
$M: U \times U \to U$, a splitting of $Y$ into its left triple $Y^L$ and right triple $Y^R$ are defined as follows.

**Definition 1.** Let $y_1 = M(y_0, y_1), y_2 = M(y_1, y_2)$ and $y_5 = M(y_3, y_4)$. Then $Y^L$ is the scaled triple $(y_1, y_3, y_5, h/2)$ and $Y^R = (y_5, y_4, y_2, h/2)$.

Splitting can also be applied to $Y^L$ and then $Y^R$, producing scaled triples

$$Y^{LL}, Y^{LR}, Y^{RL}, Y^{RR}$$

of scales $h/4$. Continuing, after $m$ iterations we obtain, for every word $w$ of length $m$ in the symbols $L, R$, a scaled triple $Y^w$ of scale $h/2^m$ called descendants of $Y$ in generation $m$. Writing the $Y^w$ in dictionary order, the track-sum of scales $h$ turn out to have a smoother appearance than the fundamental polygon $p$. Indeed the main result of [11] is

**Theorem 1.** The sequence $\{p_m: m \geq 1\}$ converges uniformly in $C[0, 2h]$ to a curve $p_\infty \in C[0, 2h]$ with the properties

1. $p_\infty$ is differentiable on $(0, 2h)$,
2. $p_\infty$ is right-differentiable at 0,
3. $p_\infty$ is left-differentiable at 2$h$,
4. $p_\infty$ is Lipschitz.

For the classical quadratic algorithm, $(\cdot, \cdot)$ is the Euclidean inner product and $p_\infty$ is well-known to be a quadratic polynomial curve. However, other generalizations of the classical algorithm produce curves with pathological properties [12], [4], [5], [6], [7], [8], [1]. The present Riemannian generalization turns out to be both regular and pathological.

Let $D \subset [0, 2h]$ be the countable dense subset consisting of multiples of $h$ by dyadic rationals. Except in the classical case it is rare for $p_\infty$ to be twice differentiable at points in $D$. Interestingly, although Theorem 1 says $p_\infty$ is $C^1$ everywhere, it is really only at points in $D$ that this seems plausible. So we might expect higher derivatives of $p_\infty$ to be better behaved on $D$ than elsewhere; exactly the opposite is true. In fact the main result of the present paper is

**Theorem 2.** 1. $\hat{p}_\infty$ is left-differentiable on $(0, 2h)$.
2. The left-acceleration $\hat{p}_\infty^-$ is left-continuous.
3. $\hat{p}_\infty$ is right-differentiable on $(0, 2h)$.
4. The right-acceleration $\hat{p}_\infty^+$ is right-continuous.
5. $\hat{p}_\infty$ is differentiable on the complement of $D$ in $[0, 2h]$.

The proof of Theorem 2 is carried out in two stages. First, candidates for the one-sided covariant derivatives of the velocity field $\hat{p}_\infty$ are constructed as limits of sequences of functions $\alpha_{m, \pm}: [0, 2h] \to \mathbb{R}^n$. The $\alpha_{m, \pm}$ are themselves constructed from accelerations of descendants of the scaled triple $Y$ together with the geometric operation of parallel translation. Inheritance properties of accelerations of scaled triples lead to analytic results concerning the $\alpha_{m, \pm}$ and their limits. The second step is to prove that the limits are in fact one-sided accelerations of $p_\infty$. The main ingredients are an inheritance property for accelerations of scaled triples, and the well-known relationship between geodesics and parallel translation.

The question of whether $\hat{p}_\infty$ has a two-sided derivative at points in $D$ may now be considered. For any Riemannian manifold there will always be special configurations of $y_0, y_1, y_2$ for which the answer is “yes”.
Example 1. Suppose that $y_1$ lies on the minimal geodesic $\omega: [0, 2h]$ from $y_0$ to $y_2$. Each $p_m$ is obtained by preceding $\omega$ with a piecewise-linear function

$$q_m: [0, 2h] \rightarrow [0, 2h],$$

and the sequence $\{q_m : m \geq 1\}$ converges uniformly to a quadratic function of the form

$$q_\infty(t) = at + (1 - a)t^2/(2h)$$

where $\omega(ah) = y_1$. Then $p_\infty = \omega \circ q_\infty$ and is therefore $C^\infty$.

The answer is also “yes” when $U = \mathbb{R}^n$ with the Euclidean inner product, regardless of the scaled triple $Y$, because then $p_\infty$ is a quadratic polynomial curve. However, a small Riemannian perturbation can completely change this, as the following result shows.

Theorem 3. Let $y_0, y_1, y_2$ be non-colinear points in $\mathbb{R}^n$ with the Euclidean metric. Fix $h > 0$. Then there is a Riemannian metric $\langle \cdot, \cdot \rangle$, $C^\infty$-close to the Euclidean inner product, with the following property. For the Riemannian scaled triple $Y = (y_0, y_1, y_2, h)$ of $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$, $p_\infty$ has a two-sided derivative nowhere in $D$.

The proof is by direct construction and is given in §4.

2. Accelerations of Triples

The mesh $\mu(Y)$ of a scaled triple $Y = (y_0, y_1, y_2, h)$ is defined to be the larger of $d(y_0, y_1), d(y_1, y_2)$. Recall from [11] that $\mu(Y^w) \leq \mu(Y)/2^m$ where $w$ is a word of length $m$ in the symbols $L, R$. The acceleration $\alpha(Y)$ of a scaled triple $Y$ is defined in [11] as

$$\frac{(\omega_{12}(h) - \omega_{01}(h))}{(2h)}$$

where, for $i = 0, 1$, $\omega_{ii+1}: [ih, (i + 1)h] \rightarrow U$ is the minimal geodesic from $y_i$ to $y_{i+1}$. Recall from [11] the following inheritance property of accelerations of scaled triples.

Lemma 2.1.

$$\alpha(Y^L) = \alpha(Y) + O(1)\mu(Y) = \alpha(Y^R).$$

In a Riemannian setting the parallel translation of a vector based at $x_a$ to a vector based at $x_b$ depends on a choice of curve joining $a, b$. The curve needs to be $C^1$ and the velocity vector should satisfy a Lipschitz condition. Parallel translation is also defined along a continuous track-sum of such curves with finitely many summands. Parallel translation is transitive and preserves Riemannian inner products. More details can be found in introductory books on Riemannian geometry, for example [2]. For $m \geq 1$ accelerations of scaled triples can be used to define functions

$$\alpha_{m-1}, \alpha_{m+}: [0, 2h] \rightarrow \mathbb{R}^n.$$

Roughly speaking, $\alpha_{m-}(s)$ is the parallel translation along $p_m$ to $p_m(s)$ of the acceleration of the nearest scaled triple to the left of $s$ in generation $m$. Replacing “left” by “right” describes $\alpha_{m+}(s)$. The formal definitions are as follows.

Definition A. 1. For $t \in (2(i-1)h/2^m, 2ih/2^m]$ let $w$ be the $i$th word of length $m$ in the symbols $L, R$. Then $\alpha_{m-}(t)$ is the parallel translation of $\alpha(Y^w)$ along $p_m$ from $p_m((2i-1)h/2^m)$ to $p_m(t)$.
Lemma 2.4. If \( s \notin D_m \), then
\[
\alpha_{\infty+}(s) = \alpha_{\infty-}(s) + O(1)\mu(Y)/2^m.
\]

Proof. Because \( s \notin D_m \), \( s \in (2(i-1)h/2^m, 2ih/2^m) \) for some \( 1 \leq i \leq 2^m \). For \( r \geq 1 \), \( \alpha_{m+r-}(s) = \alpha(Z_-) + O(1)\mu(Y)/2^m \) and \( \alpha_{m+r+}(s) = \alpha(Z_+) + O(1)\mu(Y)/2^m \) where \( Z_+, Z_- \) are descendants in generation \( r \) of \( Y^w \) (possibly the same descendant). Here \( w \) is the ith word in the symbols \( L, R \). By Lemma 2.1
\[
\alpha(Z_+) = \alpha(Y^w) + O(1)\mu(Y)/2^m = \alpha(Z_-)
\]
and this proves the lemma.

Let \( D_\infty = \bigcup_{m \geq 1} D_m \). By Lemma 2.4, \( \alpha_{\infty-} \) and \( \alpha_{\infty+} \) agree on the complement of \( D_\infty \).

Lemma 2.5. \( \alpha_{\infty-} \) and \( \alpha_{\infty+} \) are continuous at any point in the complement of \( D_\infty \).

Proof. Because \( s \notin D_\infty \), given any \( m \geq 1 \) and any \( t \) sufficiently near \( s \), we have \( t \notin D_m \) and therefore \( \alpha_{m-}(t) = \alpha_{m+}(t) \). As in the proof of Lemma 2.4,
\[
\alpha_{m-}(t) - \alpha_{\infty-}(t) = O(1)\mu(Y)/2^m = \alpha_{m+}(t) - \alpha_{\infty+}(t)
\]
and therefore
\[ \alpha_{\infty-}(t) - \alpha_{\infty+}(t) = O(1)\mu(Y)/2^m. \]
Similarly, \( \alpha_{\infty-}(s) = \alpha_{\infty+}(s) \). Because \( \alpha_{\infty-}, \alpha_{\infty+} \) are left and right-continuous this proves the lemma. \( \square \)

Next we establish relationships between the functions
\[ \alpha_{\infty-}, \alpha_{\infty+} : [0, 2h] \to \mathbb{R}^n \]
and the left and right covariant accelerations of the curve \( p_\infty \) resulting from the Riemannian quadratic construction.

3. Proof of Theorem 2

Let \( q : [a, b] \to U \) be \( C^1 \) where \( \dot{q} \) is Lipschitz, and let \( X_t \in \mathbb{R}^n \) be a vector based at \( q(t) \). The parallel translation of \( X_t \) along \( q \) from \( q(t) \) to \( q(s) \) is denoted by \( X_{t \to s} \).

In particular, \( \dot{q} \) may be regarded as a vector field defined along the curve \( q \), so that \( \dot{q}(t) \) is based at \( q(t) \). We have two kinds of cases in mind. Firstly, \( q \) might be \( p_\infty \) and then \( \dot{q} \) is Lipschitz according to Theorem 1. Secondly, \( q \) might be a geodesic segment. Parallel translation along a track-sum of geodesics is defined by parallelly translating along successive summands.

**Definition C.** 1. The left covariant acceleration \( \nabla_{\dot{q}(s)} - \dot{q} \) of \( q \) at time \( s \in (0, 2h] \) is defined to be
\[ \lim_{t \to s-} \frac{(\dot{q}(t)_{t \to s} - \dot{q}(s))/(t - s) \dot{q}(s)} \]

Whenever the limit exists.

2. The right covariant acceleration \( \nabla_{\dot{q}(s)} + \dot{q} \) of \( q \) at time \( s \in [0, 2h) \) is
\[ \lim_{t \to s+} \frac{(\dot{q}(t)_{t \to s} - \dot{q}(s))/(t - s) \dot{q}(s)} \]

When the limit exists.

A necessary and sufficient condition for \( \dot{q} \) to be left- (respectively right-) differentiable at \( s \) is that the left (respectively right) covariant acceleration of \( q \) should exist at time \( s \). A necessary and sufficient condition for the two-sided acceleration \( \ddot{q}(s) \) to exist is that
\[ \nabla_{\dot{q}(s)} - \dot{q} = \nabla_{\dot{q}(s)} + \dot{q}. \]

The reason for considering covariant accelerations instead of \( \ddot{q} \) is that geometric constructions are most easily investigated using covariant objects. When \( q = p_\infty \) it is by no means clear whether either of the covariant accelerations exist. A large part of the answer provided by Theorem 2 comes from the following result.

**Lemma 3.1.** 1. For \( s \in (0, 2h] \)
\[ \nabla_{\dot{p}(s)} - \dot{p} = \alpha_{\infty-}(s). \]

2. For \( s \in [0, 2h) \)
\[ \nabla_{\dot{p}(s)} + \dot{p} = \alpha_{\infty+}(s). \]
Proof. Given \( s \in (0,2h) \) and \( l \geq 1 \) we have \( s \in [2(k-1)h/2^l, 2kh/2^l) \) for some \( k \). Let \( t > s \) lie in the same subinterval. Given \( m > l \) let

\[
s \in (2(i-1)h/2^m, 2ih/2^m] \quad \text{and} \quad t \in (2(j-1)h/2^m, 2jh/2^m).
\]

Then \( i \leq j \). Because \( s, t \) are separated by at least \( j-i \) subintervals of length \( 2h/2^m \)

\[
(\text{SEP}) \quad 2(j-i)h/2^m \leq t-s.
\]

Let \( X_- \) and \( X_+ \) be the piecewise-continuous left and right velocity vector fields \( \dot{p}_{m-} \) and \( \dot{p}_{m+} \), defined along the piecewise geodesic curve \( p_m \). When \( u \) is an integer each restriction of \( p_m \) to a subinterval of \([0,2h]\) of the form \([2(u-1)h/2^m, (2u+1)h/2^m] \) is a geodesic. Because velocities of geodesics are translated parallely,

\[
X_{-(2u+1)h/2^m} - (2u-1)h/2^m = X_{+(2u-1)h/2^m}.
\]

Therefore \( X_{-t\rightarrow s} - \dot{p}_{m+}(s) \) can be written in the following form:

\[
(X_{+(2j-1)h/2^m} - X_{-(2j-1)h/2^m})(2j-1)h/2^m \rightarrow s + \\
(X_{+(2j-3)h/2^m} - X_{-(2j-3)h/2^m})(2j-3)h/2^m \rightarrow s + \\
\ldots \\
(X_{+(2u+1)h/2^m} - X_{-(2u+1)h/2^m})(2u+1)h/2^m \rightarrow s + \\
(X_{+(2u-1)h/2^m} - X_{-(2u-1)h/2^m})(2u-1)h/2^m \rightarrow s + \\
\ldots \\
(X_{+(2i+1)h/2^m} - X_{-(2i+1)h/2^m})(2i+1)h/2^m \rightarrow s + \\
(X_{+(2i-1)h/2^m \rightarrow s} - \dot{p}_{m+}(s)).
\]

Here every term is a parallel translation to \( p_{m}(s) \) of a difference of translated velocities parameterized within a subinterval of width \( 2h/2^m \).

All but the first and last terms are scalar multiples by \( 2h/2^m \) of parallel translations to \( p_{m}(s) \) of accelerations of the descendants

\[
Y^1, Y^2, \ldots, Y^{2^m}
\]

of the scaled triple \( Y \). The sum of these intermediate terms is

\[
2h/2^m \sum_{i<u<j} \alpha(Y^u)(2u-1)h/2^m \rightarrow s.
\]

The first and last terms are small multiples of parallel translations \( \alpha(Y^j) \) and \( \alpha(Y^i) \), depending on the precise locations of \( s \) and \( t \). In any case

\[
X_{+t \rightarrow s} - \dot{p}_{m+}(s) - 2h/2^m \sum_{i<u<j} \alpha(Y^u)(2u-1)h/2^m \rightarrow s
\]

is bounded in norm by \( O(1)\mu(Y)/2^m \) according to Lemma 2.1.

Recall that \( t \) was chosen to lie in the subinterval \([2(k-1)h/2^l, 2kh/2^l)\) containing \( s \) where \( l \) was given. Let \( Z \) be the \( k \)th descendant of \( Y \) in generation \( l \). Then the scaled triples

\[
Y^i, Y^{i+1}, \ldots, Y^j
\]

are descendants of \( Z \) in generation \( m-l \). So be Lemma 2.1 the accelerations

\[
\alpha(Y^j), \alpha(Y^{j-1}), \ldots, \alpha(Y^{i+1}), \alpha(Y^i)
\]
Proof. and midpoints are calculated using the Riemannian metric $Y$
the descendants $\langle \rangle$ which does not intersect the segment
is close to $1$ is very close to $(t-s)$ does not intersect the segment
is orthogonal to the segment
is Euclidean on the convex hull of $Y$ is close to $(t-s)O(1)\mu(Y)/2^l$. To complete the proof of part 2 of the lemma let $l \to \infty$. Part 1 follows from part 2 applied to the scaled triple $(y_2, y_1, y_0, h)$.

So the left and right covariant derivatives exist on $(0, 2h]$ and on $[0, 2h)$, respectively. The left covariant derivative is left-continuous and the right covariant derivative is right-continuous by Lemma 2.3. They also agree on the complement of $D_\infty$ by Lemma 2.4. Theorem 2 is proved.

4. Proof of Theorem 3

Let $Y$ be a scaled triple. We first deform the Euclidean metric so as to change the descendants $Y^L, Y^R$ of $Y$ in generation 1 while retaining the Euclidean metric on the convex hull of the vertices of $Y^L, Y^R$. Only the convex hull is relevant to further corner-cutting and so this kind of perturbation can be carried out with $Y^L, Y^R$ in place of $Y$, as in Lemma 4.2. The perturbation in Lemma 4.1 is chosen to create differences between the accelerations of $Y, Y^L, Y^R$.

Lemma 4.1. Let $Y = \langle y_0, y_1, y_2, h \rangle$ be a Riemannian scaled triple for the Euclidean metric on $U \subseteq \mathbb{R}^n$ where $y_0, y_1, y_2$ are not colinear. Then there is a perturbation of the Euclidean metric to a Riemannian metric $\langle \rangle$ with the following properties:

(a) $\langle \rangle$ is $C^\infty$-close to Euclidean,

(b) $\langle \rangle$ is Euclidean on the convex hull of $y_0, y_3, y_4, y_2$ where

\[ y_3 = M(y_0, y_1), \quad y_4 = M(y_1, y_2) \]

and midpoints are calculated using the Riemannian metric $\langle \rangle$,

(c) neither $y_0, y_3, y_5$ nor $y_5, y_4, y_2$ are colinear,

(d) $\alpha(Y), \alpha(Y^L), \alpha(Y^R)$ are distinct, where $Y^L, Y^R$ and their accelerations are calculated using $\langle \rangle$.

Proof. Let $c = (3y_1 + y_2)/4$. Because $y_0, y_1, y_2$ are not colinear there is an open ball $B(c, r)$ which does not intersect the segment $y_0y_1$. Without loss $0 < r \leq \|y_1 - y_2\|/8$. Let $r$ be so small that $B(x, r)$ does not intersect the segment whose endpoints are $(y_0 + y_1)/2$ and $(y_1 + y_2)/2$.

Modify the Euclidean metric on $\mathbb{R}^n$ by inserting a ridge within $B(x, r/2)$ whose axis $A$ is orthogonal to the segment $y_1y_2$. Flatten the ends of $A$ within $B(x, r) - B(x, r/2)$ so that the resulting Riemannian metric is Euclidean outside $B(x, r)$. Then

\[ y_4 = ay_1 + (1 - a)y_2 \quad \text{where} \quad a \in (1/2, 1) \]

and $a$ is close to $1/2$ when the ridge has small height $\rho$. Choose $\rho > 0$ so small that $y_4$ is very close to $(y_1 + y_2)/2$, namely so close that the segment $y_3y_4$ does not
intersect $B(x, r)$. Here $y_3 = M(y_0, y_1) = (y_0 + y_1)/2$ as with the Euclidean metric. Then (a), (b) are satisfied.

Since $y_0, y_1, y_2$ are not colinear neither are
\[ y_0(y_0 + y_1)/2, \quad (y_0 + 2y_1 + y_2)/4 \]
nor
\[ (y_0 + 2y_1 + y_2)/4, \quad (y_1 = y_2)/2, y_2. \]

Now $y_3 = (y_0 + y_1)/2$ and
\[ y_4 \approx (y_1 + y_2)/2, \quad y_5 \approx (y_0 + 2y_1 + y_2)/4. \]
If $\rho > 0$ is small enough these approximations ensure (c). To prove (d) note that
\[ y_5 = M(y_3, y_4) = (y_0 + (1 + 2\alpha)y_1 + 2(1 - \alpha)y_2)/4 \]
and then
\[ \alpha(Y^R) - \alpha(Y^L) = 2(1 - 2\alpha)(y_1 - y_2)/h^2. \]
Similarly, $\alpha(Y^L) \neq \alpha(Y) \neq \alpha(Y^R)$.

Next Lemma 4.1 is used to generate a sequence of perturbations of the Euclidean metric. Perturbations in generation $m+1$ are negligible in comparison with those in generation $m$, so that differences in accelerations in generation $m$ are not wiped out by subsequent perturbations. Then the main differences between accelerations of triples in generation $m+1$ are attributable to differences in accelerations of parents in generation $m$.

**Lemma 4.2.** Let $Y = (y_0, y_1, y_2, h)$ be a Riemannian scaled triple for the Euclidean metric on $\mathbb{R}^n$ where $y_0, y_1, y_2$ are not colinear. Then there is a sequence $\{\beta_m > 0 : m \geq 1\} \subset \mathbb{R}$ and, for each $m \geq 1$, a perturbation of the Euclidean metric to a Riemannian metric $\langle , \rangle_m$ on $\mathbb{R}^n$ with the following properties, where $Y^1, Y^2, \ldots, Y^{2^m}$ are the descendants of the scaled triple $Y$ of $\langle \mathbb{R}^n, \langle , \rangle \rangle$ in generation $m$.

(a) $\langle , \rangle_m$ is $C^\infty$-close to Euclidean.
(b) $\langle , \rangle_m$ is Euclidean on the convex hull of the vertices of any $Y^i$ where $i = 1, 2, \ldots, 2^m$.
(c) The vertices of $Y^i$ are not colinear for any $i = 1, 2, \ldots, 2^m$.
(d) Let $\alpha(Y^j)_l$ denote the parallel translation
\[ \alpha(Y^j)_{(2j+1)h/2^m - (2j+3)h/2^m} \]
of $\alpha(Y^j)$ along $p_m$ all calculated with respect to the Riemannian metric $\langle , \rangle_l$ where $l \geq m$. Then
\[ \|\alpha(Y^j)_l - \alpha(Y^j+1)_l\| > \beta_m \]
for $j = 1, 2, \ldots, 2^m - 1$ and all $l \geq m$.
(e) If $Y^j, Y^j$ have a common ancestor in generation $r < m$, then the norms of differences, after parallel translation along $p_m$, of $\alpha(Y^i), \alpha(Y^j)$ are smaller than $\beta_r/4$ for the Riemannian metric $\langle , \rangle_l$ and any $l \geq m$.
(f) The sequence $\{\langle , \rangle_m : m \geq 1\}$ converges as a sequence of $C^\infty$ Riemannian metrics to a $C^\infty$ Riemannian metric $\langle , \rangle_\infty$. 

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Proof. In Lemma 4.1 write $\beta_1 = \|\alpha(Y^l) - \alpha(Y^R)\|/2$. Set $\langle , \rangle_1 = \langle , \rangle$. Then since $\langle , \rangle_1$ is Euclidean on the convex hull of the vertices of $Y^l, Y^R$ condition (e) holds when $m = 1$. The other conclusions depend on the $\beta_m$ where $m > 1$ and we define these inductively as follows.

Suppose that Lemma 4.2 holds for $m < k$ and let $Z^1, Z^2, \ldots, Z^{2^{k-1}}$ be the descendants of $Y$ in generation $k - 1$. Apply Lemma 4.1 to each $Z^j$, choosing perturbations so small that for any $j = 1, 2, \ldots, 2^{k-1}$ the difference in norms of accelerations of any pair from $Z^j, (Z^j)^L, (Z^j)^R$ is less than

$$\left(\min_{m=1,2,\ldots,k-1} \beta_m\right) / 2^{k+2}.$$ 

Take care also that the perturbations are so small that they do not undo the previous inequalities (d) for $m \leq k - 1$. Now let $W^1, W^2, \ldots, W^{2^k}$ be the immediate descendants of the $Z^j$ and set

$$\beta_k = \min_{i=1,2,\ldots,2^k-1} \|\alpha(W^i) - \alpha(W^{i+1})\|/2.$$ 

To ensure the convergence in (f) make each perturbation so much smaller than the last that $\{\langle , \rangle_m : m \geq 1\}$ is Cauchy. As in [3], Theorem 1.1.11, the $C^\infty$ Riemannian metrics comprise a complete metric space, which proves (f).

To prove Theorem 3 consider the perturbation $\langle , \rangle_\infty$ of the Euclidean metric and the associated $Y^w$ for words $w$ in $L, R$. If $s = 2ih/2^m \in D_m$, then

$$\|\alpha_m - (s) - \alpha_m + (s)\| > \beta_m$$

by Lemma 4.2(d). By Lemma 4.2(e)

$$\|\alpha_m (s) - \alpha_m (s)\| \leq \beta_m / 4$$

and therefore

$$\|\alpha_m (s) - \alpha_m (s)\| \geq \beta_m / 2.$$ 

Theorem 3 now follows from Lemma 3.1.

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