THE WEIERSTRASS APPROXIMATION THEOREM
AND A CHARACTERIZATION OF THE UNIT CIRCLE

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Abstract. We study real algebraic morphisms from nonsingular real algebraic
varieties $X$ with $\dim X \geq 1$ into nonsingular real algebraic curves $C$. We show,
among other things, that the set of real algebraic morphisms from $X$ into $C$ is
never dense in the space of all $C^\infty$ maps from $X$ into $C$, unless $C$ is biregularly
isomorphic to a Zariski open subset of the unit circle.

An affine real algebraic variety is a locally ringed space isomorphic to an algebraic
subset of $\mathbb{R}^n$ (for some $n$) endowed with the Zariski topology and the sheaf of $\mathbb{R}$-
valued regular functions. For basic information on real algebraic varieties the reader
may refer to [2]. Recall that every quasiprojective real algebraic variety is in fact
affine [2, Proposition 3.2.10, Theorem 3.4.4]. Given two affine real algebraic varieties
$X$ and $Y$, we denote by $R(X, Y)$ the set of all regular maps (that is, morphisms of
locally ringed spaces) from $X$ into $Y$. We assume that $X$ and $Y$ are nonsingular
and regard $R(X, Y)$ as a subset of the space $C^\infty(X, Y)$ of all $C^\infty$ maps from $X$
into $Y$ endowed with the $C^\infty$ compact-open topology (the weak $C^\infty$ topology in the
terminology used in [7]). It is natural to study the size of $R(X, Y)$ in $C^\infty(X, Y)$
and several papers have already been devoted to this problem [2], [3], [4], [5]. In
the present note we prove the following result, which can be viewed as a version,
in a new setting, of the classical Weierstrass approximation theorem.

Theorem 1. Given an affine nonsingular irreducible real algebraic curve $C$, the
following conditions are equivalent:

(a) The curve $C$ is biregularly isomorphic to a Zariski open subset of the unit
circle $S^1 \equiv \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$;
(b) The set $R(V, C)$ is dense in $C^\infty(V, C)$ for every compact affine nonsingular
real algebraic curve $V$;
(c) There exists an affine nonsingular real algebraic variety $X$ such that $\dim X \geq 1$
and $R(X, C)$ is dense in $C^\infty(X, C)$.

To prevent possible confusion let us say explicitly that throughout this note the
adjective compact always refers to the Euclidean topology on the real algebraic
varieties.

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Before giving a proof of Theorem 1 we make a few remarks. For every affine nonsingular irreducible real algebraic curve \( V \) there exists a unique (up to biregular isomorphism over \( \mathbb{R} \)) projective nonsingular irreducible complex algebraic curve \( V_\mathbb{C} \) defined over \( \mathbb{R} \) such that \( V \) is biregularly isomorphic to a Zariski open subset of the set of real points \( V_\mathbb{C}(\mathbb{R}) \) of \( V_\mathbb{C} \) (here \( V_\mathbb{C}(\mathbb{R}) \) is regarded as a projective real algebraic variety). We denote by \( g(V) \) the genus of \( V_\mathbb{C} \). Every regular map \( f : V \to W \) of affine nonsingular irreducible real algebraic curves extends in a unique way to a (complex) regular map \( f_\mathbb{C} : V_\mathbb{C} \to W_\mathbb{C} \). Let us denote by \( \mathcal{R}^*(V,W) \) the set of all nonconstant regular maps from \( V \) into \( W \). By applying the remarks above and some classical theorems on projective complex algebraic curves we obtain information on the set \( \mathcal{R}^*(V,W) \). Namely:

(i) By the theorem of de Franchis [8, p. 227], if \( g(W) \geq 2 \), then the set \( \mathcal{R}^*(V,W) \) is finite.

(ii) The Hurwitz-Riemann formula [6, p. 140] implies that if \( g(W) = 1 \), then each map in \( \mathcal{R}^*(V,W) \) has at most \( 2g(V) - 2 \) critical points.

**Proof of Theorem 1.** Assume that (a) holds. It is known that the set \( \mathcal{R}(V, S^1) \) is dense in \( C^\infty(V, S^1) \) for every compact affine nonsingular real algebraic curve \( V \) (cf. [4, Proposition 6.1, Theorem 1.6], where the problem of determining the size of \( \mathcal{R}(X, S^1) \) in \( C^\infty(X, S^1) \) is completely solved for an arbitrary compact affine nonsingular real algebraic variety \( X \)). If \( C \) is biregularly isomorphic to a Zariski open subset of \( S^1 \), different from \( S^1 \), then \( C \) is biregularly isomorphic to a Zariski open subset of \( \mathbb{R} \) and hence the set \( \mathcal{R}(X,C) \) is dense in \( C^\infty(V, C) \) for every compact affine nonsingular real algebraic variety \( X \). Thus (b) is proved.

It is obvious that (b) implies (c). We now prove that (c) implies (a). Suppose that the set \( \mathcal{R}(X,C) \) is dense in \( C^\infty(X,C) \) for some affine nonsingular real algebraic variety \( X \) with \( \dim X \geq 1 \). We shall show \( g(C) = 0 \), which is equivalent to (a). To this end let us choose a nonsingular irreducible real algebraic curve \( Z \) in \( X \).

We have \( g(C) \leq 1 \) since otherwise the set \( \mathcal{R}^*(Z,C) \) would be finite (cf. (i)) and therefore no map \( f \) in \( C^\infty(X,C) \) whose restriction \( f|Z \) is not in \( \mathcal{R}(Z,C) \) could be approximated by regular maps from \( X \) into \( C \).

Hence it remains to exclude the case \( g(C) = 1 \). Choose a \( C^\infty \) map \( h : X \to C \) such that \( h|Z \) has at least \( 2g(Z) - 1 \) critical points, each of multiplicity exactly 2 (in particular, \( h|Z \) has in local coordinates either a local maximum or a local minimum at each critical point). If \( g(C) = 1 \), then it follows from (ii) that \( h|Z \) (and hence \( h \)) cannot be approximated by regular maps with values in \( C \). Thus \( g(C) = 0 \) is proved.

\[ \square \]

The following characterization of the unit circle is an immediate consequence of Theorem 1.

**Corollary 2.** Let \( C \) be a compact affine nonsingular irreducible real algebraic curve. Then the following conditions are equivalent:

(a) The curve \( C \) is biregularly isomorphic to \( S^1 \);

(b) The set \( \mathcal{R}(V,C) \) is dense in \( C^\infty(V,C) \) for every compact affine nonsingular real algebraic curve \( V \);

(c) There exists an affine nonsingular real algebraic variety \( X \) such that \( \dim X \geq 1 \) and \( \mathcal{R}(X,C) \) is dense in \( C^\infty(X,C) \). \[ \square \]
It would be interesting, but probably hard, to find a generalization of Theorem 1 and Corollary 2 with $C$ replaced by a higher dimensional real algebraic variety. In this direction we only have the following result.

**Theorem 3.** Let $M$ be a closed connected $C^\infty$ manifold with $\dim M \geq 1$. Then there exists an affine nonsingular real algebraic variety $Y$ such that $M$ is diffeomorphic to $Y$ and the set $R(X, Y)$ is never dense in $C^\infty(X, Y)$, where $X$ is an arbitrary affine nonsingular real algebraic variety with $\dim X \geq 1$.

We shall need the following observation.

**Lemma 4.** Let $Y$ be an affine nonsingular irreducible real algebraic variety. Assume that there exists a nonconstant regular map $\varphi : Y \rightarrow C$ into an affine nonsingular irreducible real algebraic curve $C$ with $g(C) \geq 2$. Then the set $R(X, Y)$ is never dense in $C^\infty(X, Y)$, where $X$ is an arbitrary affine nonsingular real algebraic variety with $\dim X \geq 1$.

**Proof.** Since $Y$ is irreducible and $\varphi : Y \rightarrow C$ is nonconstant, we can choose a sequence $\{U_n\}$ of nonempty open (in the Euclidean topology) subsets of $Y$ such that the subsets $\varphi(U_n)$ of $C$ are pairwise disjoint.

Suppose now that $R(X, Y)$ is dense in $C^\infty(X, Y)$ for some affine nonsingular real algebraic set $X$ with $\dim X \geq 1$. Let $Z$ be a nonsingular irreducible algebraic curve in $X$ and let $z$ be a point in $Z$. Since $R(X, Y)$ is dense in $C^\infty(X, Y)$, we can find a sequence $\{\varphi_n\}$ of regular maps $\varphi_n : Z \rightarrow Y$ such that $\varphi_n(z)$ is in $U_n$ and the composition $\varphi \circ \varphi_n$ is nonconstant. Then $\{\varphi \circ \varphi_n\}$ is an infinite family of distinct elements of $R^*(Z, C)$, which is impossible in view of (i). This contradiction completes the proof. \hfill $\square$

**Proof of Theorem 3.** Let $C$ be a compact connected affine nonsingular real algebraic curve with $g(C) \geq 2$. Let $f : M \rightarrow C$ be a nonconstant $C^\infty$ map. Then given a neighborhood $U$ of $f$ in $C^\infty(M, C)$, one can find an affine nonsingular real algebraic variety $Y$, a $C^\infty$ diffeomorphism $\sigma : M \rightarrow Y$, and a regular map $\varphi : Y \rightarrow C$ such that $\varphi \circ \sigma$ is in $U$ (one embeds $M$ in $\mathbb{R}^n$, for some $n$, and applies [1, Theorem 2.8.4], observing first that the bordism class of $(f, M)$ in the unoriented bordism group of $C$ is algebraic [1, Lemma 2.7.1]). We may assume that $\varphi$ is nonconstant by taking $U$ sufficiently small. Moreover, $Y$ must be irreducible since $M$ is connected. We now complete the proof of Theorem 3 by applying Lemma 4. \hfill $\square$

We say that a closed $C^\infty$ manifold $M$ admits a **Weierstrass algebraic model** if there exists an affine nonsingular real algebraic variety $Y$ such that $M$ is diffeomorphic to $Y$ and the set $R(X, Y)$ is dense in $C^\infty(X, Y)$ for some affine nonsingular real algebraic variety $X$ with $\dim X \geq 1$. It is likely that only very few closed $C^\infty$ manifolds admit Weierstrass algebraic models. We conjecture that among closed connected orientable $C^\infty$ surfaces only the 2-sphere $S^2$ and the torus $S^1 \times S^1$ admit Weierstrass algebraic models.

**References**


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