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NORM ESTIMATES OF INTERPOLATION MATRICES AND THEIR INVERSES ASSOCIATED WITH STRICTLY POSITIVE DEFINITE FUNCTIONS

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Abstract. In this paper, we estimate the norms of the interpolation matrices and their inverses that arise from scattered data interpolation on spheres with strictly positive definite functions.

§1. Introduction

Let S_{m-1} denote the unit sphere in $\mathbb{R}^m$. Here $m \geq 2$ is fixed. For $x, y \in S_{m-1}$, the geodesic distance $d(x, y)$ is defined as

$$d(x, y) = \cos^{-1}(xy),$$

where xy denotes the usual inner product of vectors in $\mathbb{R}^m$. Let $L^2(S_{m-1}) = \bigoplus_{n=0}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_n$ is the space of spherical harmonics of degree n. Orthogonality here is with respect to the inner product

$$\langle f, g\rangle = \int_{S_{m-1}} f(x)g(x)\, d\omega_{m-1}(x),$$

where ω_{m-1} is the rotation invariant measure on the sphere, normalized so that $s_{m-1} := \omega_{m-1}(S_{m-1}) = 2\pi^{(m-1)/2}/\Gamma((m-1)/2)$. Let $\{Y_{n,1}, \dots, Y_{n,A_n}\}$ be an orthonormal basis for $\mathcal{H}_n$, where $A_n = \dim \mathcal{H}_n$.

A continuous function $g : [0, \pi] \to \mathbb{R}$ is said to be positive definite on S_{m-1} if for any N points $x_1, x_2, \dots, x_N \in S_{m-1}$, the matrix A having elements

$$A_{ij} = g(d(x_i, x_j))$$

is non-negative definite. If for any N distinct points $x_1, x_2, \dots, x_N \in S_{m-1}$, the matrix A is positive definite, then the function g is said to be strictly positive definite on S_{m-1}.

Schoenberg [S] characterized the class of all positive definite functions on S_{m-1} as those that have the form

$$g(t) = \sum_{n=0}^{\infty} a_n P_n^{(\lambda)}(\cos t)\ , \qquad \lambda = (m-2)/2\ , \tag{1.1}$$

where $a_n \geq 0$ and $\sum_{n=0}^{\infty} a_n < \infty$, and $P_n^{(\lambda)}$ denotes the Gegenbauer (ultraspherical) polynomials normalized so that $P_n^{(\lambda)}(1) = 1$.

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Xu and Cheney [XC] showed that if in (1.1) all the a_n's are positive, then the function g is strictly positive definite on S_{m-1}. Ron and Sun [RS] made some progress in determining the exact amount of positive coefficients in (1.1) so that the function g is strictly positive definite on S_{m-1}.

The Gegenbauer polynomial $P_n^{(\lambda)}$ arises naturally in harmonic analysis on the sphere as it is the kernel of the orthogonal projection onto $\mathcal{H}_n$ via the following formula (see Müller [M]):

$$P_n^{(\lambda)}(xy) = \gamma_n \sum_{j=1}^{A_n} Y_{n,j}(x) Y_{n,j}(y). \tag{1.2}$$

The precise value of $\gamma_n > 0$ is not important here. The above formula may be used to show that $P_n^{(\lambda)}$ is positive definite, and it is this fact (see the proof of Theorem 3.2) that is the basis for the analysis in [S], [XC], and [RS].

The motivation to study strictly positive definite functions is the following. With a strictly positive definite function g, one can interpolate arbitrary given data at any N distinct points $x_1, x_2, \ldots, x_N$ (called nodes) on S_{m-1} by a unique function in the linear span of the N functions $g(d(x, x_j))$, $j = 1, \ldots, N$. Furthermore, the interpolation matrix $(g(d(x_i, x_j)))$ is positive definite, and therefore can be handled by many efficient numerical algorithms. Since this interpolation scheme does not require the node set to have any structure, it is often called "scattered interpolation" in many practical problems.

In implementing the above interpolation scheme, it is important to estimate the norms of the interpolation matrices and their inverses. This is the sole purpose of this paper. The general idea used in this paper is similar to that of Narcowich and Ward [NW1], [NW2] who estimated the norms of interpolation matrices and their inverses arising from radial functions in Euclidean spaces. In a practical sense, the main result of this paper is a quantitative version of that of Xu and Cheney [XC].

§2. Preliminaries

In this section, we first discuss some rudimentary Fourier analysis on S_{m-1}. We then use the results to study some locally supported functions which will be used to obtain our estimates for the interpolation matrices and their inverses.

A function of the form $F(x) = f(xy)$, $f : [-1, 1] \to \mathbb{R}$, $y \in S_{m-1}$ is termed y–zonal since F is invariant under any rotation which fixes y. Equipped with $f \in L^2[-1, 1]$ we define the spherical convolution with $G \in L^2(S_{m-1})$ by $(f * G)(x) = \langle f(x\bullet), G\rangle$.

It is straightforward (see e.g. [M, p.19]) to show that the convolution of y–zonal functions is itself y–zonal. In fact, for $f, g \in L^2[-1, 1]$, there exists $h \in L^2[-1, 1]$ such that

$$h(xy) = \int_{S_{m-1}} f(xz) g(yz)\, d\omega_{m-1}(z)\ . \tag{2.2}$$

We write $h = f * g$, and it is easy to see that the convolution in (2.2) is commutative, i.e., $f * g = g * f$.

Define the Fourier coefficients $\hat{f}(n)$ of $f \in L^2[-1, 1]$ by

$$\hat{f}(n) = \int_{-1}^{1} f(t) P_n^{(\lambda)}(t)(1 - t^2)^{\lambda - 1/2}\, dt\ .$$

Note that since the orthogonal system $\{P_n^{(\lambda)}\}$ is not orthonormalized, the Fourier expansion of f is actually

$$f(t) = \sum_{n=0}^{\infty} \hat{f}(n)\beta_n^{-1} P_n^{(\lambda)}(t),$$

where

$$\beta_n = \int_{-1}^{1} [P_n^{(\lambda)}(t)]^2 (1-t^2)^{\lambda-1/2} dt > 0.$$

The following lemma is the famous Funk-Hecke formula, a proof of which may be found in [M, p.18].

Lemma 2.1. *For any $Y_n \in H_n$ and $f \in L^2[-1,1]$, the following identity is true*

$$\int_{S_{m-1}} f(xy) Y_n(y)\, d\omega_{m-1}(y) = s_{m-2} \hat{f}(n) Y_n(x) \ .$$

Lemma 2.2. *Let $f, g \in L^2[-1,1]$, and set $r = f * g$. Then $\hat{r}(n) = s_{m-2}\hat{f}(n)\hat{g}(n)$.*

Proof. Choose a $Y_n \in H_n$ and $x \in S_{m-1}$ so that $Y_n(x) \neq 0$. Applying the Funk-Hecke formula first to h and then to g and f respectively, we have

$$\begin{aligned}
s_{m-2}\, \hat{r}(n) Y_n(x) &= \int_{S_{m-1}} r(xy) Y_n(y)\, d\omega_{m-1}(y) \\
&= \int_{S_{m-1}} \left(\int_{S_{m-1}} f(xz) g(yz)\, d\omega_{m-1}(z) \right) Y_n(y)\, d\omega_{m-1}(y) \\
&= \int_{S_{m-1}} f(xz) \left(\int_{S_{m-1}} g(yz) Y_n(y)\, d\omega_{m-1}(y) \right) d\omega_{m-1}(z) \\
&= s_{m-2}\, \hat{g}(n) \int_{S_{m-1}} f(xz) Y_n(z)\, d\omega_{m-1}(z) \\
&= s_{m-2}^2\, \hat{g}(n) \hat{f}(n) Y_n(x) \ .
\end{aligned}$$

□

Let $h \in (0,1)$. We define the functions $B_h^{(k)} \in L^2[-1,1]$ in the following way. Firstly,

$$B_h^{(1)}(t) = \begin{cases} 0 \ , & \text{if } -1 \le t \le h \\ 1 \ , & \text{if } h < t \le 1 \ , \end{cases}$$

and then we define $B_h^{(k)} = B_h^{(k-1)} * B_h^{(1)}$, $k = 2, 3, \cdots$. These functions have been studied by Schreiner [Sc]. Here we need more information about these functions.

Lemma 2.3. *The functions $B_h^{(k)}$ satisfy $B_h^{(k)}(t) \ge 0$ for all $t \in [-1,1]$. Furthermore, $B_h^{(k)}(t) = 0$ if $-1 \le t \le T_k(h)$, where T_k denotes the first kind Tchebyshev polynomial normalized so that $T_k(1) = 1$.*

Proof. It is obvious from the definition of $B_h^{(k)}$ that $B_h^{(k)}(t) \ge 0$ for all $t \in [-1,1]$. Also, we have $B_h^{(1)}(t) = 0$ if $-1 \le t \le h$. That is, $B_h^{(1)}(xy) = 0$ if $d(x,y) \ge \cos^{-1} h$. It is easy to show that the radius of the support of the convolution of two functions is less than or equal to the sum of the radii of the supports of the individual functions. Therefore $B_h^{(k)}(xy) = 0$ if $d(x,y) \ge k \cos^{-1} h$. In other words, $B_h^{(k)}(t) = 0$ if $-1 \le t \le \cos(k \cos^{-1} h) = T_k(h)$. □

Lemma 2.4. *The Fourier coefficients of the function $B_h^{(1)}$ satisfy the inequality*

$$\widehat{B_h^{(1)}}(n) \le \frac{2(1-h^2)^{\lambda-1/2}}{(n+\lambda)}, \quad n \neq 0,$$

and $\widehat{B_h^{(1)}}(0) = 1$.

Proof. Proving $\widehat{B_h^{(1)}}(0) = 1$ is trivial. We concentrate on the case when $n \neq 0$. When $m = 2$, the result is obvious, so we assume $m > 2$ in the remainder of the proof.

Using the Second Mean Value Theorem for Integral, (see [W, p. 138]), we have

$$\widehat{B_h^{(1)}}(n) = \int_h^1 P_n^{(\lambda)}(t)(1-t^2)^{\lambda-1/2}\,dt = (1-h^2)^{\lambda-1/2}\int_h^\xi P_n^{(\lambda)}(t)\,dt,$$

where $h \le \xi \le 1$. By the following identity (see [Sz, p.85]),

$$\frac{d}{dt}\left\{P_{n+1}^{(\lambda)}(t) - P_{n-1}^{(\lambda)}(x)\right\} = 2(n+\lambda)P_n^{(\lambda)}(t)\ ,$$

we have

$$\begin{aligned}\widehat{B_h^{(1)}}(n) &= \frac{(1-h^2)^{\lambda-1/2}}{2(n+\lambda)}\int_h^\xi \frac{d}{dt}\left\{P_{n+1}^{(\lambda)}(t) - P_{n-1}^{(\lambda)}(t)\right\}\,dt \\ &= \frac{(1-h^2)^{\lambda-1/2}}{2(n+\lambda)}\left(\left[P_{n+1}^{(\lambda)}(\xi) - P_{n-1}^{(\lambda)}(\xi)\right] - \left[P_{n+1}^{(\lambda)}(h) - P_{n-1}^{(\lambda)}(h)\right]\right) \\ &\le \frac{2(1-h^2)^{\lambda-1/2}}{(n+\lambda)}\ .\end{aligned}$$

□

By using Lemma 2.2 and Lemma 2.4, we obtain the following result.

Lemma 2.5. *The Fourier coefficients of the function $B_h^{(k)}$ satisfy the following inequality*

$$\widehat{B_h^{(k)}}(n) \le 2^k s_{m-2}^{k-1}\cdot\frac{(1-h^2)^{k(\lambda-1/2)}}{(n+\lambda)^k}\ , \quad n \neq 0,$$

and $\widehat{B_h^{(k)}}(0) = 1$.

§3. The estimates

In this section, we estimate the ℓ_2 norms of the interpolation matrices and their inverses. Recall that for a given set of N distinct points $x_1, x_2, \ldots, x_N \in S_{m-1}$, the interpolation matrix A has ij entry

$$A_{ij} = g(d(x_i, x_j))\ ,$$

where g is a strictly positive definite function on S_{m-1}. Let $\mu_1 \le \mu_2 \le \cdots \le \mu_N$ be the eigenvalues of A. Let $\|A\|$ denote the ℓ_2 norm of the matrix A. Then from elementary matrix analysis, we know that $\|A\| = \mu_N$ and $\|A^{-1}\| = \mu_1^{-1}$.

Since the trace of a real and symmetric matrix is invariant under orthogonal transformations, we have $\mu_1 + \mu_2 + \cdots + \mu_N = N\,g(0)$. The following result is immediate:

Proposition 3.1. *Let $x_1, x_2, \dots, x_N$ be N points on S_{m-1} (not necessarily distinct). Assume that g is a positive definite function on S_{m-1}. Then the matrix A with $A_{ij} = g(d(x_i, x_j))$ satisfies $\|A\| \le N\, g(0)$.*

The remainder of this section is devoted to finding a positive lower bound for the smallest eigenvalue of the interpolation matrix.

Note that if g is a function on $[0, \pi]$, then the function $f : t \mapsto g(\cos^{-1} t)$ is defined on $[-1, 1]$, and the matrix $(g(d(x_i, x_j)))$ becomes $(f(x_i x_j))$. We find this change of variables convenient. In what follows, we will work with a function f thus defined and the corresponding matrix $(f(x_i x_j))$.

Theorem 3.2. *Let $f \in L^2[-1, 1]$ have the absolutely convergent expansion*

$$f(t) = \sum_{n=0}^{\infty} \hat{f}(n) \beta_n^{-1} P_n^{(\lambda)}(t) \ .$$

Suppose for a natural number k, the inequalities

$$\hat{f}(n) \ge C(n + \lambda)^{-k}, \quad n \ne 0,$$

and $\hat{f}(0) \ge C$, are true for some positive constant C. Then, for any N distinct points $x_1, x_2, \dots, x_N \in S_{m-1}$ with $\eta = \min_{i \ne j} d(x_i, x_j)$, the least eigenvalue μ_1 of the interpolation matrix $(f(x_i x_j))$ satisfies

$$\mu_1 \ge C \left[2^k s_{m-2}^{k-1} \sin^{k(m-3)} \frac{\eta}{k} \right]^{-1} B_h^{(k)}(1) \ .$$

Proof. For $c = (c_1, c_2, \dots, c_N) \in \mathbb{R}^N$, we have, in mind of (1.2),

$$\begin{aligned}
\sum_{i,j=1}^{N} c_i c_j f(x_i x_j) &= \sum_{i,j=1}^{N} c_i c_j \sum_{n=0}^{\infty} \hat{f}(n) \beta_n^{-1} P_n^{(\lambda)}(x_i x_j) \\
&= \sum_{n=0}^{\infty} \hat{f}(n) \beta_n^{-1} \sum_{i,j=1}^{N} c_i c_j P_n^{(\lambda)}(x_i x_j) \\
&= \sum_{n=0}^{\infty} \hat{f}(n) \beta_n^{-1} \gamma_n \sum_{i,j=1}^{N} c_i c_j \sum_{k=1}^{A_n} Y_{n,k}(x_i) Y_{n,k}(x_j) \\
&= \sum_{n=0}^{\infty} \hat{f}(n) \beta_n^{-1} \gamma_n \sum_{k=1}^{A_n} \left\{ \sum_{i=1}^{N} c_i Y_{n,k}(x_i) \right\}^2 . \qquad (3.1)
\end{aligned}$$

Since $\beta_n, \gamma_n > 0$, the above quadratic form is non–negative if $\hat{f}(n) \ge 0, n = 0, 1, \dots$, proving the result of Schoenberg [S]. Replacing $\hat{f}(n)$ with the lower bounds from

the statement of the theorem, and using Lemma 2.5, we have

$$\begin{aligned}\sum_{i,j=1}^{N} c_i c_j f(x_i x_j) &\geq C\beta_0^{-1}\gamma_0 \left\{\sum_{i=1}^{N} c_i Y_{0,0}(x_i)\right\}^2 \\ &\quad + C\sum_{n=1}^{\infty}\beta_n^{-1}\gamma_n (n+\lambda)^{-k}\sum_{k=1}^{A_n}\left\{\sum_{i=1}^{N} c_i Y_{n,k}(x_i)\right\}^2 \\ &\geq C\left[2^k s_{m-2}^{k-1}\sin^{k(m-3)}\frac{\eta}{k}\right]^{-1}\sum_{n=0}^{\infty}\widehat{B_h^{(k)}}(n)\beta_n^{-1}\gamma_n\sum_{k=1}^{A_n}\left\{\sum_{i=1}^{N} c_i Y_{n,k}(x_i)\right\}^2 \\ &= C\left[2^k s_{m-2}^{k-1}\sin^{k(m-3)}\frac{\eta}{k}\right]^{-1}\sum_{i,j=1}^{N} c_i c_j \sum_{n=0}^{\infty}\widehat{B_h^{(k)}}(n)\beta_n^{-1}P_n^{(\lambda)}(x_i x_j),\end{aligned}$$

where we have set $h = \cos\frac{\eta}{k}$, and to obtain the last equation we have simply reversed the argument used to obtain (3.1). Since the series

$$\sum_{n=0}^{\infty}\widehat{B_h^{(k)}}(n)\beta_n^{-1}P_n^{(\lambda)}(t)$$

converges to $B_h^{(k)}$ uniformly on $[-1,1]$ (see [Sc]), we have

$$\sum_{i,j=1}^{N} c_i c_j f(x_i x_j) \geq C\left[2^k s_{m-2}^{k-1}\sin^{k(m-3)}\frac{\eta}{k}\right]^{-1}\sum_{i,j=1}^{N} c_i c_j B_h^{(k)}(x_i x_j). \tag{3.2}$$

For $h = \cos\frac{\eta}{k}$, by Lemma 2.3, $B_h^{(k)}(xy) = 0$ if $-1 \leq xy \leq T_k(h) = \cos\eta$. Now $\eta = \min_{i\neq j} d(x_i, x_j)$. Thus, $x_i x_j \leq \cos\eta$ if $i \neq j$. Hence $B_h^{(k)}(x_i x_j) = 0$ if $i \neq j$. Thus, from (3.2) we see that

$$\sum_{i,j=1}^{N} c_i c_j f(x_i x_j) \geq C\left[2^k s_{m-2}^{k-1}\sin^{k(m-3)}\frac{\eta}{k}\right]^{-1} B_h^{(k)}(1)\sum_{i=1}^{N} c_i^2 \ .$$

Therefore the least eigenvalue μ_1 of the matrix $(f(x_i x_j))$ satisfies

$$\mu_1 \geq C\left[2^k s_{m-2}^{k-1}\sin^{k(m-3)}\frac{\eta}{k}\right]^{-1} B_h^{(k)}(1) \ .$$

□

Example 3.3. Let $m = 3$, and

$$f(t) = 1 - \sqrt{\frac{1-t}{2}}.$$

Then (see [L]), $\hat{f}(0) = 1/3$, and

$$\hat{f}(n) = \frac{4}{(4n^2-1)(2n+3)} \geq \frac{1}{12n^3}, \quad n \neq 0.$$

Therefore, by Theorem 3.2,

$$\mu_1 \geq \frac{1}{96}\frac{B_h^{(3)}(1)}{(2\pi)^2} \ .$$

But, for any fixed $\xi \in S_2$,

$$\begin{aligned} B_h^{(3)}(1) &= \int_{S^2} B_h^{(2)}(\xi\eta) B_h^{(1)}(\xi\eta)\, d\eta \\ &= 2\pi \int_0^{\cos^{-1} h} B_h^{(2)}(\cos\theta) \sin\theta\, d\theta. \end{aligned}$$

However, for $\xi, \zeta \in S_2$,

$$\begin{aligned} B_h^{(2)}(\xi\zeta) &= \int_{S^2} B_h^{(1)}(\xi\eta) B_h^{(1)}(\zeta\eta)\, d\eta \\ &= \int_{\xi_h \cap \zeta_h} d\eta, \end{aligned}$$

where, for $\mu \in S_2$, the spherical disk $\mu_h = \{\nu \in S_2 : d(\nu, \mu) \leq \cos^{-1} h\}$. Now, contained in $\xi_h \bigcap \zeta_h$ is a spherical disk $D_{\xi,\zeta}$ of radius $\cos^{-1} h - 1/2 \cos^{-1}(\xi\zeta)$. Thus, setting $\xi\eta = \cos\theta$,

$$\begin{aligned} B_h^{(2)}(\cos\theta) &\geq \int_{D_{\xi,\zeta}} d\eta \\ &= 2\pi \int_0^{\cos^{-1} h - \theta/2} \sin\phi\, d\phi \\ &= 2\pi \left(1 - \cos\left(\cos^{-1} h - \frac{\theta}{2}\right)\right) \\ &= 2\pi \left(1 - h\cos\frac{\theta}{2} - \sqrt{1-h^2}\sin\frac{\theta}{2}\right) \\ &\geq 2\pi \left(1 - h\cos\frac{\theta}{2} - \sqrt{1-h}\sin\frac{\theta}{2}\right), \end{aligned}$$

as long as $1 - h$ is sufficiently small. Therefore,

$$\begin{aligned} B_h^{(3)}(1) &\geq 4\pi^2 \int_0^{\cos^{-1} h} \left(1 - h\cos\frac{\theta}{2} - \sqrt{1-h}\sin\frac{\theta}{2}\right) \sin\theta\, d\theta \\ &= 4\pi^2 \left[-\cos\theta + \frac{4}{3}h\cos^3\frac{\theta}{2} - \frac{4}{3}\sqrt{1-h}\sin^3\frac{\theta}{2}\right]_0^{\cos^{-1} h} \\ &= 4\pi^2 \left((1-h) - \frac{4}{3}h\left(1 - \left(1 - \frac{1-h}{2}\right)^{\frac{3}{2}}\right) - \frac{4}{3}\sqrt{1-h}\sqrt{\frac{1-h}{2}}^3\right) \\ &\geq 4\pi^2 \left((1-h) - \frac{4}{3}h\frac{3(1-h)}{4} - \frac{\sqrt{2}}{3}(1-h)^2\right) \\ &= 4\pi^2 \left(1 - \frac{\sqrt{2}}{3}\right)(1-h)^2. \end{aligned}$$

Thus we have

$$\begin{aligned}\mu_1 &\geq \left(1-\frac{\sqrt{2}}{3}\right)\frac{(1-h)^2}{96}\\ &= \left(1-\frac{\sqrt{2}}{3}\right)\frac{1}{96}\left(1-\cos\frac{\eta}{3}\right)^2,\end{aligned}$$

where η is the minimum separation distance among the points $x_1, x_2, \ldots, x_N$.

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