A NOTE ON HOLOMORPHIC MAPS WITH UNIPOTENT JACOBIAN MATRICES

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(Communicated by Steven R. Bell)

Abstract. We prove that a holomorphic map $H : \mathbb{C}^2 \to \mathbb{C}^2$ is invertible if its Jacobian matrix $JH$ is unipotent.

1. Introduction

Let $\mathbb{C}$ be the complex number field. Given a polynomial map $F : \mathbb{C}^n \to \mathbb{C}^n$ with $F = (f_1, f_2, \cdots, f_n)$, where $f_i \in \mathbb{C}[z_1, z_2, \cdots, z_n]$, a simple algebraic argument tells us that

$$J_F = \det \left[ \frac{\partial f_i}{\partial z_j} \right] \in \mathbb{C}^* = \mathbb{C}\{0\}$$

whenever $F$ is invertible. The Jacobian Conjecture asserts that the converse is true also.

The Jacobian Conjecture is false for holomorphic maps. An easy example is

$F : \mathbb{C}^2 \to \mathbb{C}^2$,

$f_1 = e^{z_1}$,

$f_2 = z_2 e^{-z_1}$.

In [BCW] the Jacobian Conjecture has been reduced to the Unipotent Jacobian Conjecture, which states

The Unipotent Jacobian Conjecture. If the Jacobian matrix $JF$ of $F$ is a unipotent matrix, then $F$ is invertible.

We suspect that this conjecture could also be true for holomorphic maps. In this note, we give a proof of the Unipotent Jacobian Conjecture for holomorphic maps with $n = 2$.

2. Main results

In this section, we first prove a theorem concerning holomorphic maps $F : \mathbb{C}^n \to \mathbb{C}^n$ with $JF^2 = 0$. The idea of the proof is obtained from [CSW]. We then apply the theorem to the case $n = 2$ and $F(z) = H(z) - z$, where $H(z)$ is an arbitrary holomorphic map $H : \mathbb{C}^2 \to \mathbb{C}^2$ with unipotent Jacobian matrix. This yields that $H$ is invertible, i.e. Corollary 2.3.

Received by the editors September 26, 1997.

1991 Mathematics Subject Classification. Primary 32H99.
Theorem 2.1. Let \( F : \mathbb{C}^n \to \mathbb{C}^n \) be a holomorphic map. The following statements are equivalent:

1. \( JF(z)^2 = 0 \) for all \( z \in \mathbb{C}^n \),
2. \( F(z + JF(z)z') = F(z) \) for all \( z, z' \in \mathbb{C}^n \), and
3. \( JF(z + JF(z)z')JF(z) = 0 \) for all \( z, z' \in \mathbb{C}^n \).

Proof. (1)\( \Rightarrow \) (2). Using Taylor expansion

\[
f(z + y) = f(z) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k = 1}^{n} \frac{\partial^k f(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \prod_{s=1}^{k} y_{i_s},
\]

one has

\[
f_i(z + JF(z)z') = f_i(z) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k = 1}^{n} \frac{\partial^k f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \prod_{s=1}^{k} \frac{\partial f_i(z)}{\partial z_{j_s}} z_{j_s} = f_i(z) + \sum_{k=1}^{\infty} \frac{1}{k!} \sum_{i_1, i_2, \ldots, i_k = 1}^{n} \frac{\partial^k f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \prod_{s=1}^{k} \frac{\partial f_i(z)}{\partial z_{j_s}} \prod_{s=1}^{k} z_{j_s}.
\]

Define

\[
D^{[i]}_{j_1, j_2, \ldots, j_k} = \sum_{i_1, i_2, \ldots, i_k = 1}^{n} \frac{\partial^k f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \prod_{s=1}^{k} \frac{\partial f_i(z)}{\partial z_{j_s}}.
\]

We now show that \( D^{[i]}_{j_1, j_2, \ldots, j_k} = 0 \) for all \( k \geq 1 \) and all \( 1 \leq j_1, j_2, \ldots, j_k, i \leq n \) by induction on \( k \). When \( k = 1 \), the \( D^{[i]}_{j_1} \)'s are the entries of \( JF(z)^2 \), and therefore are equal to 0. Suppose \( D^{[i]}_{j_1, j_2, \ldots, j_k} = 0 \) for all \( 1 \leq j_1, j_2, \ldots, j_k, i \leq n \). Then

\[
D^{[i]}_{j_1, j_2, \ldots, j_k+1} = \sum_{i_1, i_2, \ldots, i_{k+1} = 1}^{n} \frac{\partial^{k+1} f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_{k+1}}} \prod_{s=1}^{k+1} \frac{\partial f_i(z)}{\partial z_{j_s}}
\]

\[
+ \sum_{i_{k+1} = 1}^{n} \sum_{i_1, i_2, \ldots, i_k = 1}^{n} \frac{\partial^k f_i(z)}{\partial z_{i_1} \partial z_{i_2} \cdots \partial z_{i_k}} \sum_{s=1}^{k} \frac{\partial^2 f_i(z)}{\partial z_{j_s} \partial z_{i_{k+1}}} \frac{\partial f_{i_{k+1}}(z)}{\partial z_{j_{l}}},
\]

Since

\[
\sum_{i_{k+1} = 1}^{n} \frac{\partial f_{i_{k+1}}(z)}{\partial z_{j_{k+1}}} \frac{\partial f_{i_{k+1}}(z)}{\partial z_{j_{k+1}}} = 0,
\]

we have

\[
\sum_{i_{k+1} = 1}^{n} \frac{\partial^2 f_i(z)}{\partial z_{j_s} \partial z_{i_{k+1}}} \frac{\partial f_{i_{k+1}}(z)}{\partial z_{j_{k+1}}} = - \sum_{i_{k+1} = 1}^{n} \frac{\partial f_i(z)}{\partial z_{i_{k+1}}} \frac{\partial^2 f_{i_{k+1}}(z)}{\partial z_{j_s} \partial z_{j_{k+1}}}.
\]
Thus
\[
D^{[i]}_{J_1, J_2, \ldots, J_{k+1}} = \sum_{s=1}^{k} \sum_{t=1}^{n} D^{[i]}_{J_1, J_2, \ldots, J_{s-1}, J_{s+1}, \ldots, J_{k+1}} \frac{\partial f_i(z)}{\partial z_{i_1}} \frac{\partial f_k(z)}{\partial z_{i_{k+1}}} = 0.
\]

(2)⇒(3). We fix \( z \) and consider \( z' \) as variable and compute the Jacobian matrix with respect to \( z' \) in the left-hand side of (2). (3)⇒(1). Set \( z' = 0 \) in (3).

\[ \]
satisfies $JH(z)^2 = 0$ and its second row equals 0. This implies

$$H : \mathbb{C}^2 \rightarrow \mathbb{C}^2,$$

$$h_1 = h(z_2),$$

$$h_2 = \gamma,$$

where $h$ is an entire function and $\gamma$ is a constant. If we denote $\det JG(z)^{-1}$ by $d$, $dh$ by $f$, $-db'\gamma$ by $c_1$ and $da'\gamma$ by $c_2$, then

$$F : \mathbb{C}^2 \rightarrow \mathbb{C}^2,$$

$$f_1 = bf(a_1 + bz_2) + c_1,$$

$$f_2 = -af(a_1 + bz_2) + c_2.$$

The converse is obvious and the proof is completed.

**Corollary 2.3.** If the Jacobian matrix $JH$ of a holomorphic map $H : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ is a unipotent matrix, then $H$ is invertible.

**Proof.** Let $H(z) = z + F(z)$, where $F$ is a holomorphic map with nilpotent Jacobian matrix. By Theorem 2.2, it is easy to check that $G(z) = z - F(z - F(z))$ is the inverse of $H$.

**Remark 1.** If $F$ is a polynomial map, Theorem 2.2 can be derived from a result of [BCW].

**Remark 2.** Given a holomorphic function $f : \mathbb{C}^2 \rightarrow \mathbb{C}$, there is a holomorphic function $g : \mathbb{C}^2 \rightarrow \mathbb{C}$ such that the holomorphic map $F = (f, g)$ or $(g, f) : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ has nilpotent Jacobian matrix $JF$ if and only if $f$ satisfies the partial differential equation

\[(\frac{\partial f}{\partial z_1})^2 \frac{\partial^2 f}{\partial z_2^2} + (\frac{\partial f}{\partial z_2})^2 \frac{\partial^2 f}{\partial z_1^2} = 2 \frac{\partial f}{\partial z_1} \frac{\partial f}{\partial z_2} \frac{\partial^2 f}{\partial z_1 \partial z_2}.
\]

This follows since $\det(JF) = 0$ and $\text{Tr}(JF) = 0$ and we can then eliminate the function $g$ by using the mixed second derivative of $g$. Thus Theorem 2.2 is equivalent to the following, which can also be proved in the same way as the combination of proofs of Theorem 2.1 and Theorem 2.2:

**Theorem 2.4.** Let $f$ be a holomorphic function on $\mathbb{C}^2$. Then $f$ satisfies the differential equation (*) if and only if $f = h(a_1 + b_2)$, where $h$ is an analytic function on $\mathbb{C}$ and $a$ and $b$ are constants.

**Acknowledgment**

The author would like to thank Professor Henry H. Glover for his guidance and encouragement.

**References**


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